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LP-GAMES AND COMBINATORIAL OPTIMIZATION GAMES

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ABSTRACT

A survey is given of combinatorial optimization game theory. In more detail flow games and linear production games are treated.

1. INTRODUCTION

Since the early developments of Operations Research and Game Theory there was an interesting interaction between the two fields. This is not surprising, because in both fields decision makers, guided by an objective function, want to act in an optimal way. The main difference between the fields is that in O.R. there is acting only one decision maker, while in game theory at least two decision makers are in interaction.

Game theory deals with mathematical models of competition and cooperation. Roughly speaking the theory is divided into two parts: non-cooperative theory, where the players (agents, decision makers) cannot make binding agreements and cooperative theory where binding agreements and often also sidepayments are possible.

Well-known is the interrelation between O.R. and non-cooperative game theory. We only mention:

(i) Duality results in mathematical programming theory and minimax results in zero-sum game theory.

(ii) Linear complementarity theory and the theory of bimatrix games.

(iii) Markov decision theory and the theory of stochastic games.

(iv) Optimal control theory and the theory of differential games.

Interrelation between cooperative game theory and O.R. is less known and it is the purpose of this paper to pay some attention to this interrelation. We will introduce cooperative games arising from optimization situations, in which many decision makers are involved, because they own e.g. resources or arcs in a network, or because they control pieces of networks etc. In working together, the decision makers create extra gains or save costs and the problem arises: how to share the extra rewards or how to allocate the costs? One way, to answer this question is to look at TU-games (transferable utility games) arising from such situations and to use existing solution concepts or to create new suitable solution concepts to solve the problem.

Since the beginning of the seventies many situations are considered. I mention:

(i) Minimum spanning tree games and spanning network games (Claus and Kleitman (1973), Bird (1976), Megiddo (1978), Granot and Huberman (1981), Granot and Maschler (1991), van den Nouweland, Maschler and Tijs (1992)).

(ii) Linear production games, LP-games and mathematical programming games (Owen (1975), Dubey and Shapley (1984), Granot (1986), Potters (1987), Curiel, Pederzoli and Tijs (1988)).

(iii) Flow games (Kalai and Zemel (1982), Curiel, Derks and Tijs (1989), Derks and Tijs (1985, 1986)).

(iv) Traveling salesman games (Potters, Curiel and Tijs (1992), Tamir (1989), Kuipers (1991)).

(v) Sequencing games, permutation games and assignment games (Shapley and Shubik (1972), Curiel, Pederzoli and Tijs (1989), Curiel and Tijs (1986), Potters and Tijs (1987), Hamers, Borm and Tijs (1992)).

For further reading I recommend Curiel (1987, 1988), Driessen (1988), Sharkey (1992) and Tijs and Driessen (1986).

In the next sections we introduce some of these classes of games, and indicate some main results.

Section 2 gives a short introduction into cooperative game theory. Especially, attention is paid there to the core and in examples minimum spanning tree games, and traveling salesman games are introduced.

Section 3 deals mainly with max flow games.

In section 4 linear production situations and their corresponding games are considered in detail, as well as permutation games.

An extensive list of references will guide the reader through the literature.

2. COOPERATIVE GAME THEORY

Cooperative game theory is concerned primarily with *coalitions* - groups of players - who coordinate their actions and pool their winnings. For each set S of players, $v(S)$ denotes the amount they can gain if they form a coalition, excluding the other players. One of the problems is how to divide extra earnings (or cost savings) among the members of the formed coalition.

For $N = \{1, 2, \dots, n\}$ denote the collection of subsets of N by 2^N.

Definition 2.1. A *cooperative n-person game in characteristic function form* is an ordered pair $< N, v >$, where $N := \{1, 2, \dots, n\}$ (the *set of players*) and $v : 2^N \to \mathbb{R}$ is a map, assigning to each coalition $S \in 2^N$ a real number, such that $v(\emptyset) = 0$. The function v is called the *characteristic function* of the game, $v(S)$ is called the *worth* (or *value*) of coalition S.

Example 2.2. (Three cooperating communities.) Communities 1, 2 and 3 want to be connected with a nearby power source. The possible transmission links and their costs are shown in figure 2.1.

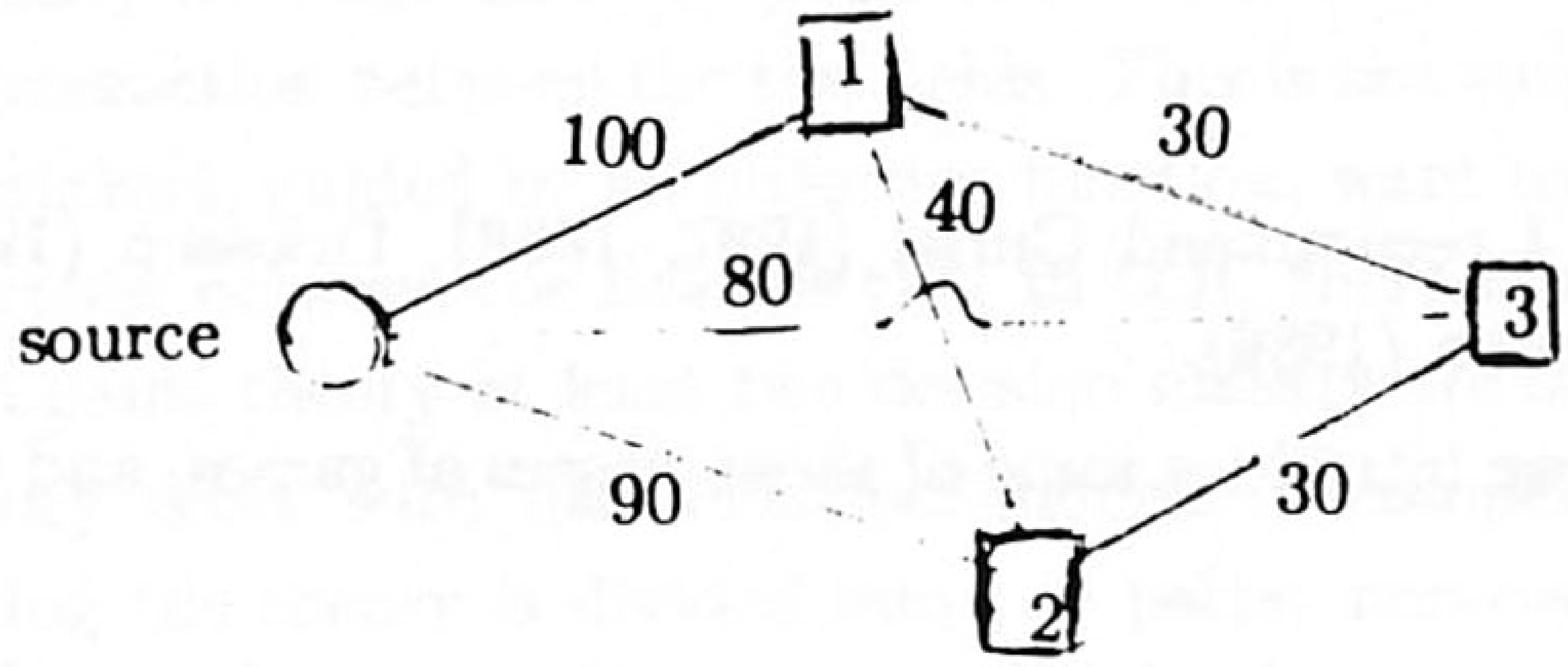

Figure 2.1.

The *cost game* $< N, c >$ associated with this situation is given by $N = \{1, 2, 3\}$ and the first two lines of the next table.

$S =$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S) =$	0	100	90	80	130	110	110	140
$v(S)$	0	0	0	0	60	70	60	130

The game $< N, v >$ in the third line of the table is the *cost savings game* corresponding to $< N, c >$, determined by

$$v(S) := \sum_{i \in S} c(i) - c(S) \text{ for each } S \in 2^N.$$

The cost savings $v(S)$ for coalition S is the difference in costs corresponding to the situation where all members of S work alone and the situation where all members of S work together.

Games arising from situations as in example 2.2 for obvious reasons are called *minimum spanning tree games.* They are extensively studied. For a survey I refer to Sharkey (1992).

Example 2.3 (Saving travel costs).

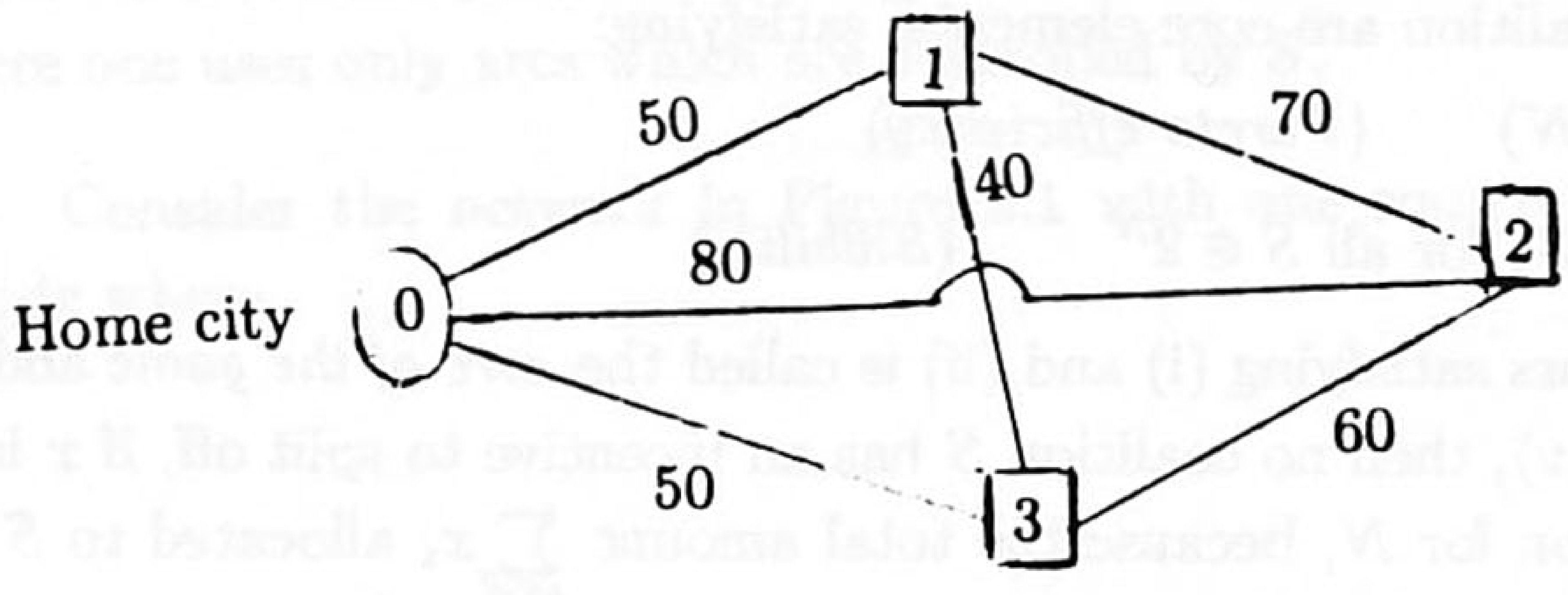

Figure 2.2.

Three university cities 1, 2 and 3 are interested in inviting a scientist living in (home) city 0. In figure 2.2 the one way travel costs are given. So going from city 0 to city 1 and back costs $2 \times 50 = 100$ units etc. In cooperating the cities can save travel costs. The problem is: "who pays what to the scientist?". In the next table you find the corresponding cost game and cost savings game.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	0	100	160	100	200	140	190	230
$v(S)$	0	0	0	0	60	60	70	130

Games arising from situations as in example 2.3 are called *traveling salesman games.* For recent results I refer to Potters and Tijs (1992), Potters (1990), Tamir (1989) and Kuipers (1991).

Example 2.4. $N = \{1, 2, , 3\}$, $v(S) = 1$ if $S \supset \{2, 3\}$, $v(S) = 0$ otherwise. This is an example of a *simple game*, where $v(S) \in \{0, 1\}$ for all S and $v(N) = 1$. Such games play a role in the description of voting situations, where $v(S) = 1 \iff S$ is winning. Also in control situations of an object simple games are useful in the description of those coalitions who can use the object i.e. those coalitions with worth 1. In the game in this example coalitions containing players 2 and 3 are powerful. Another simple game is the dictator game $< N, \delta^i >$, where $\delta^i(S) = 1$ iff $i \in S$.

For a game $< N, v >$ interesting distributions $(x_1, x_2, \cdots, x_n)$ of the worth $v(N)$ of the grand coalition are core elements, satisfying:

(i) $\sum_{i=1}^{n} x_i = v(N)$ *(Pareto efficiency)*

(ii) $\sum_{i \in S} x_i \geq v(S)$ for all $S \in 2^N$ *(Stability)*

The set of vectors satisfying (i) and (ii) is called the *core of the game* and is denoted by $C(v)$. If $x \in C(v)$, then no coalition S has an incentive to split off, if x is the proposed reward allocation for N, because the total amount $\sum_{i \in S} x_i$ allocated to S is not smaller than the amount $v(S)$ which they can obtain by forming a subcoalition.

For the game in figure 2.1 the vector $(70, 60, 0) \in C(v)$, which corresponds to the vector $(30, 30, 80)$ in the cost core, which is defined by

$$C(c) := \{x \in \mathbf{R}^N \mid \sum_{i=1}^{n} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S\}.$$

This vector $(30, 30, 80)$ is tightly related to the minimum spanning tree, corresponding to this problem.

It is well-known that all minimum cost spanning tree games have a non-empty core.

For the game $< N, v >$ in example 2.3 the vector $(40, 40, 50)$ is in the core. However, there exist traveling salesman games with an empty core (cf. Potters and Tijs (1992)).

In view of a famous theorem concerning necessary and sufficient conditions for the non-emptiness of the core (Bondareva (1963) and Shapley (1967)), games with a non-empty core are also called *balanced games.* The core is one of the most interesting solution concepts. Others are e.g. the Shapley value (1953), the nucleolus (Schmeidler, 1969), the τ-value (Tijs, 1981).

In Curiel, Pederzoli and Tijs (1989) for sequencing games an interesting rule, the Equal Gain Splitting Rule, was introduced, which fits nicely with these class of games.

In the following sections only the core will be considered, and most of the considered games will be balanced.

3. FLOW GAMES

Nice examples of balanced games are those arising from flow situations with veto control. In the flow situation there is one source and one sink and on the arcs there are capacity restrictions. Furthermore, with the aid of a simple game for each arc, one can describe which coalitions are allowed to use the arc. These are the coalitions which are winning in the simple game. In this context such games are called *control games.* The value of a coalition S is the maximal flow through the network from source to sink (per time unit), where one uses only arcs which are controlled by S.

Example 3.1. Consider the network in Figure 3.1 with one source, one sink, one intermediate node where

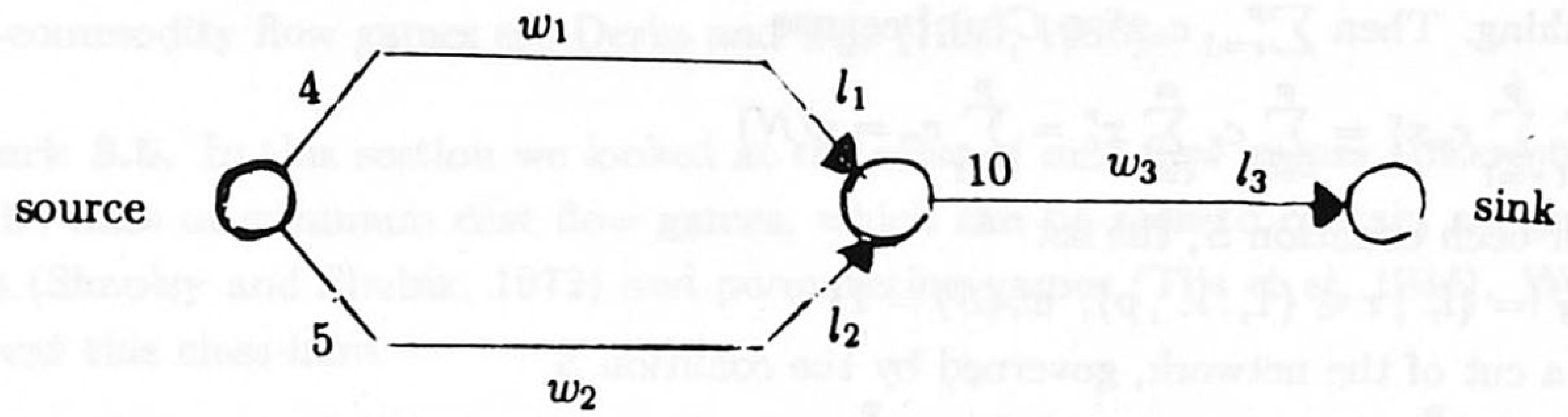

Figure 3.1.

the three arcs l_1, l_2, l_3 have capacities 4, 5 and 10 respectively.

The control games are w_1, w_2, w_3 with

$w_1(S) = 1$ if $S \in \{\{1,2\}, N\}$ and $w_1(S) = 0$ otherwise
$w_2(S) = 1$ if $S \in \{\{1,3\}, N\}$ and $w_2(S) = 0$ otherwise
$w_3 = \delta_1$, player 1 owns arc l_3

The coalition $\{1,2\}$ can only use the arcs l_1 and l_3, so the maximal flow (per time unit) for $\{1,2\}$ is 4. This results in $v(\{1,2\}) = 4$ for the corresponding flow game $< N, v >$, whose characteristic function is given by

$v(\{i\}) = 0$ if $i \in N$, $v(\{1,2\}) = 4$, $v(\{1,3\}) = 5$
$v(\{2,3\}) = 0$ and $v(N) = 9$.

A minimum cut for the flow situation corresponding to the grand coalition is $\{l_1, l_2\}$. By the max-flow min-cut theorem of Ford and Fulkerson (1962) the sum of the capacities of l_1 and $l_2 (4+5)$ is equal to $v(N)$. Divide $v(N)$ as follows. Divide 4 equally among the veto players of w_1, and 5 equally among the veto players of w_2. The result for the players is the payoff vector $(4\frac{1}{2}, 2, 2\frac{1}{2})$. Note that this vector is in $C(v)$.

The next theorem shows that the non-emptiness of the core of the control games is inherited by the flow game.

Theorem 3.2. (Cf. Curiel, Derks and Tijs (1989)) Suppose all control games in a controlled flow situation have veto players. Then the flow game is balanced.

Proof. Take a maximal flow for the grand coalition and a minimum cut in the network for the grand coalition consisting of the arcs

$$l_1, l_2, \ldots, l_p \text{ with capacities } c_1, c_2, \ldots, c_p$$

and control games $w_1, w_2, \ldots, w_p$, respectively. Then the theorem of Ford-Fulkerson says that $v(N) = \sum_{r=1}^{p} c_r$. For each r take $x^r \in \text{core}(w_r)$ and divide c_r according to the division key x^r (i.e. $c_r x_i^r$ is the amount for player i). Note that non-veto players get nothing. Then $\sum_{r=1}^{p} c_r x^r \in C(v)$ because

(i) $\sum_{i=1}^{n} \sum_{r=1}^{p} c_r x_i^r = \sum_{r=1}^{p} c_r \sum_{i=1}^{n} x_i^r = \sum_{r=1}^{p} c_r = v(N)$

(ii) For each coalition S, the set
$L_S := \{l_r \mid r \in \{1, \ldots, p\},\ w_r(S) = 1\}$
is a cut of the network, governed by the coalition S.

Hence, $\sum_{i \in S} (\sum_{r=1}^{p} c_r x_i^r) = \sum_{r=1} c_r \sum_{i \in S} x_i^r \geq \sum_{r=1}^{p} c_r w_r(S) = \sum_{l_r \in L_S} c_r = \text{capacity } (L_S) \geq v(S)$
where the last inequality follows from the Ford-Fulkerson theorem. ∎

The question arises whether any non-negative game can be obtained from a flow situation with veto control (i.e. all control games possess veto players). The affirmative answer is given in

Theorem 3.3. Any non-negative balanced game arises from a flow situation with veto control.

Proof. Let v be such a game. By a theorem of Derks (1987) there are positive numbers $c_1, c_2, \ldots, c_k$ and balanced simple games $w_1, w_2, \ldots, w_k$ such that $v = \sum_{r=1}^{k} c_r w_r$. Take the flow situation as

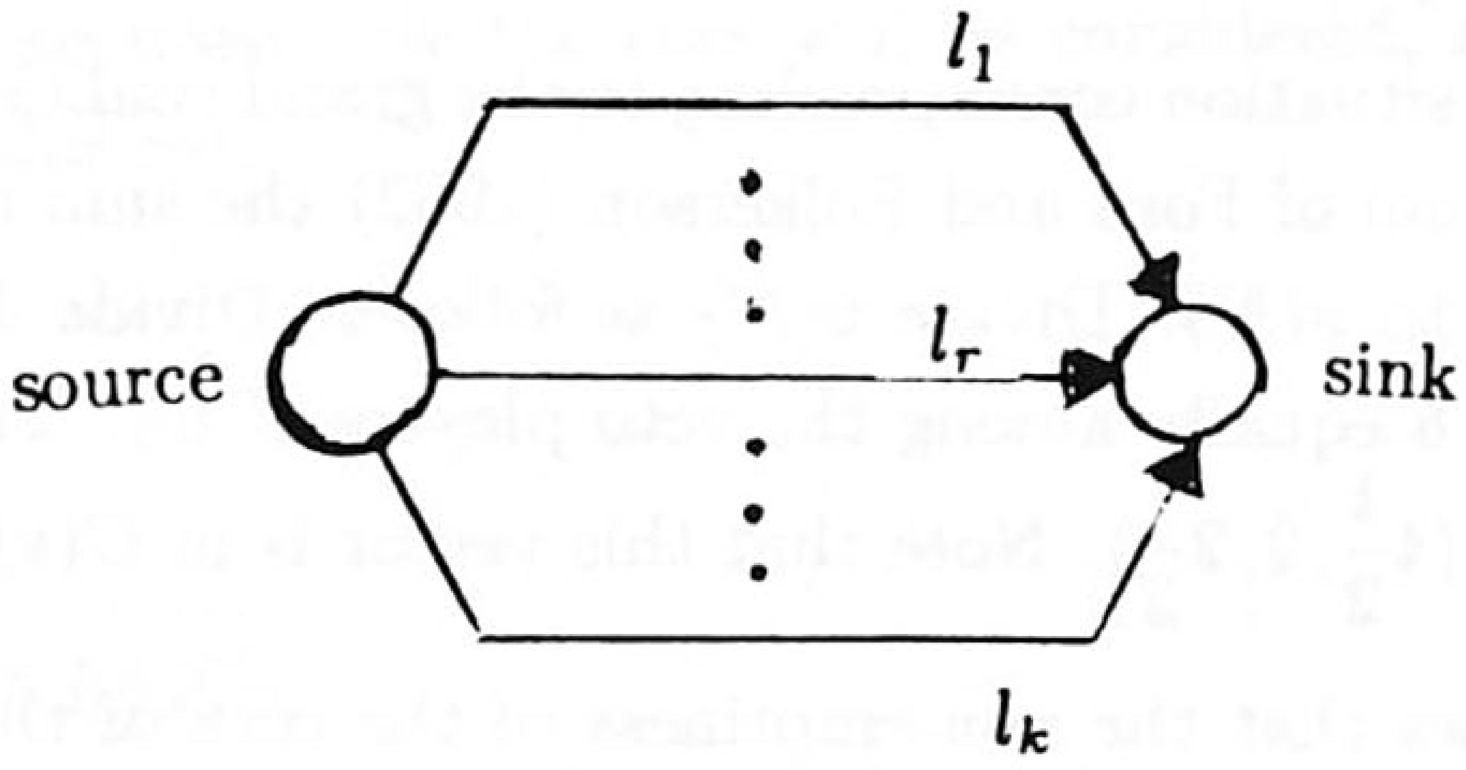

Figure 3.2.

in figure 3.2 with no intermediate nodes and arcs $l_1, l_2, \ldots, l_k$ from source to sink with capacities $c_1, c_2, \ldots, c_k$ and control games $w_1, w_2, \ldots, w_k$ respectively. Then, one can easily prove that the corresponding flow game coincides with $< N, v >$. ■

Remark 3.4. (Cf. Kalai and Zemel (1982a, 1982b).) Let $< N, v >$ be the flow game corresponding to a flow situation where all control games are dictatorial games. Then it is easy to prove that each subgame $< S, v_s >$ (where v_s is the restriction of v to 2^S) has a non-empty core. (Such games are called *totally balanced*). For extensions to multi-commodity flow games see Derks and Tijs (1985, 1986).

Remark 3.5. In this section we looked at the class of max flow games. Interesting is also the class of minimum cost flow games, which can be seen to contain assignment games (Shapley and Shubik, 1972) and permutation games (Tijs et al, 1984). We will not treat this class here.

4. LINEAR PRODUCT GAMES

Linear production games are introduced by Owen (1975). Generalizations of Owen's results are given in Dubey and Shapley (1984), Granot (1986) and Curiel, Derks and Tijs (1989).

Consider the linear production situation where products $P_1, P_2, \ldots, P_j, \ldots, P_m$ can be made using resources $G_1, G_2, \ldots, G_k, \ldots, G_q$. Suppose that for the production of α units of $P_j (\alpha \geq 0) \alpha a_{j1}$ units of G_1, αa_{j2} units of $G_2, \ldots$, are required. Furthermore, the price per unit of product P_j is c_j. Let $A := [a_{jk}]_{j=1,k=1}^{m\ \ q}$ be the corresponding *production matrix*. We suppose that $A \geq 0$ and that each row of A has a positive coordinate. If a bundle of $b \in \mathbb{R}_+^q$ of resources is available, then feasible production plans can be described as vectors $x \in \mathbb{R}_+^m$ with $xA \leq b$, where x corresponds to the plan: make for each $j \in \{1, 2, \ldots, m\}$ x_j units of product P_j. The value of the products made by such a plan at price c is equal to the inner product $x \cdot c$ of x and c.
Let

$$\text{val}(b) := \max\{x \cdot c \mid x \geq 0,\ xA \leq b\}$$

be the value corresponding to an optimal production plan, given the resource bundle b. That val(b) is well-defined follows from the fact that $\{x \geq 0,\ xA \leq b\}$ is a compact set, in view of the conditions put on A.

Now suppose that a group $N = \{1, 2, \ldots, n\}$ of agents controls in some way the resources and that $b(S)$ is the resource bundle which is available to a coalition $S \in 2^N$, provided they cooperate. Then the above situation corresponds to a linear production game $< N, v >$, where

$$v(S) := \text{val}(b(S)) \text{ for each } S \in 2^N \backslash \{\emptyset\}.$$

The papers of Owen (1975), Granot (1986) and Curiel, Derks and Tijs (1989) give sufficient conditions for the map $S \to b(S)$ to guarantee the non-emptiness of the core of the corresponding linear production game.

Example 4.1. Consider the linear production situation

$$\begin{array}{c} \\ P_1 \\ P_2 \end{array} \begin{array}{c} \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \begin{pmatrix} 5 \\ 7 \end{pmatrix} \end{array}$$

$$b^1 = (5,8)$$
$$b^2 = (5,2)$$
$$b^3 = (0,2)$$

Table 4.1.

as given in table 4.1, where 2 resources G_1 and G_2 are involved and 2 products P_1 and P_2, with prices per unit 5 and 7. Suppose that 10 units of G_1 and 12 units of G_2 are available: player 1, 2 and 3 own, respectively, resource bundles (5,8), (5,2) and (0,2). This situation corresponds to a 3-person linear production game $< N, v >$ with

$$v(S) := \max\{5x_1 + 7x_2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + 2x_2 \leq b_1(S), 2x_1 + x_2 \leq b_2(S)\} \quad (4.1)$$

where $b_k(S) = (\sum_{i \in S} b^i)_k$ for $k \in \{1,2\}$ denotes the total amount of resource G_k owned by coalition S.
If one wishes to calculate v, rather than (4.1) it is easier to use formula (4.2), which is derived from the duality theorem of linear programming theory. The reason is that for all dual programs corresponding to the $2^n - 1 = 7$ coalitions the feasible region is the same. We have

$$v(S) = \min\{b_1(S)y_1 + b_2(S)y_2 \mid y_1 \geq 0, y_2 \geq 0, y_1 + 2y_2 \geq 5, 2y_1 + y_2 \geq 7\} \quad (4.2)$$

The feasible region of the dual problem has 3 extreme points, namely $\hat{y} = (3,1)$, $\hat{y}' = (5,0)$, $\hat{y}'' = (0,7)$. For each of the 7 problems in (4.2) the minimum is attained at one of these extreme points. The next table 4.2 describes where the minimum is attained and gives also the characteristic function v.

S	$b(S)$	minimum	$v(S)$
(1)	(5,8)	$\hat{y}$	23
(2)	(5,2)	$\hat{y}''$	14
(3)	(0,2)	$\hat{y}'$	0
(1,2)	(10,10)	$\hat{y}$	40
(1,3)	(5,10)	$\hat{y}, \hat{y}'$	25
(2,3)	(5,4)	$\hat{y}$	19
(1,2,3)	(10,12)	$\hat{y}$	42

Table 4.2.

For the grand coalition N the value is equal to 42 and is attained at $\hat{y} = (3,1)$. Here, 3 can be interpreted as (shadow) price for G_1. According to these prices the bundle $b^1 = (5,8)$, owned by player 1, has value $5 \cdot 3 + 8 \cdot 1 = 23$. For players 2 and 3 the values of their bundles are 17 and 2 respectively. These values correspond to the imputation (23,17,2) of $< N, v >$. Note that $(23,17,2)$ is even a core element of $< N, v >$ and that to find this vector we only need to solve the dual linear program in (4.2) for $S = N$.

In the Owen model each player i owns a bundle $b^i = (b^i_1, b^i_2, \ldots, b^i_q)$ of resources and $b(S) := \sum_{i \in S} b^i$ for each $S \in 2^N \backslash \{\emptyset\}$. An example is given in 4.1.

In the Granot model one considers the q commodity games $< N, c_k >$ with $c_k(S) := b_k(S)$ and assumes that these games are balanced.

In the Curiel-Derks-Tijs model for each commodity G_k there are portions $\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kt(k)}$ available, where the control of these portions is described by control games $w_{k1}, w_{k2}, \ldots, w_{kt(k)}$ respectively. One assumes that the control games are simple games with veto players. Hence, $b_k(S) = \sum_{r=1}^{t(k)} \alpha_{kr} w_{kr}(S)$ for each S. (See example 4.4.)

Note that in the Owen model $b_k(S) = \sum_{i \in S} b^i_k = \sum_{i=1}^{n} b^i_k \delta_i(S)$ for all $k \in N$ and $S \in 2^N \backslash \{\emptyset\}$, where δ_i is the dictator game with dictator i. This implies that this model is a special case of the third model by taking for each $k \in \{1, \ldots, q\} : t(k) = n$, and for each $i \in N : \ \alpha_{ki} = b^i_k, \ w_{ki} = \delta_i$.

The Granot model can also be incorporated in the third model, since by a theorem of Derks (1987) each of the non-negative balanced commodity games c_k can be written in the form $c_k = \sum_{r=1} \alpha_{kr} w_{kr}$, where $\alpha_{kr} > 0$ and w_{kr} is a balanced simple game. Conversely, the third model can also be seen as a special case of the second model by noting that in the third model, for each k, $S \to b_k(S) = \sum_{r=1}^{t(k)} \alpha_{kr} w_{kr}(S)$ is a balanced game, because balanced games form a cone.

All the three models give rise to balanced linear production games. In view of the above we only need to prove this for the third model.

Theorem 4.2. (Curiel-Derks-Tijs (1989).) Consider the linear production game $< N, v >$ corresponding to the linear production situation with production matrix $A \in \mathbf{R}^{m \times q}$, price $c \in \mathbf{R}^m$ and commodity function $b : 2^N \backslash \{\emptyset\} \to \mathbf{R}^q$ such that for all $k \in \{1, 2, \ldots, q\}$ and all $S \in 2^N \backslash \{\emptyset\}$:

$$b_k(S) = \sum_{r=1}^{t(k)} \alpha_{kr} w_{kr}(S)$$

where all α_{kr} are positive real numbers and all w_{kr} are simple games with veto players. Then

(i) $< N, v >$ is a balanced game.

(ii) A core element x of $< N, v >$ can be obtained by taking

$$x = \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \alpha_{kr} \hat{y}_k |V(k,r)|^{-1} e^{V(k,r)} \tag{4.3}$$

where $V(k,r)$ is the set of veto players in w_{kr}, $\hat{y}$ is an optimal vector in the dual program 4.2 with $S = N$ and e^S is the vector in $\mathbf{R}^N$ with coordinate $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$.

Proof. We only need to show that x given by (4.3) is a core element.

(i)

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \alpha_{kr} \hat{y}_k |V(k,r)|^{-1} e_i^{V(k,r)}$$

$$= \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \hat{y}_k \alpha_{kr} \sum_{i=1}^{n} |V(k,r)|^{-1} e_i^{V(k,r)}$$

$$= \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \hat{y}_k \alpha_{kr} = \sum_{k=1}^{q} \hat{y}_k b_k(N) = \hat{y} \cdot b(N) = v(N)$$

where the last equality follows from (4.2).

(ii) Now take $S \subset N$, $S \neq \emptyset$. First note that $\hat{y}$ is feasible for the dual program in (4.2) corresponding to S. Hence,

$$v(S) \leq \sum_{k=1}^{q} b_k(S) \hat{y}_k = \sum_{k=1}^{q} \hat{y}_k (\sum_{r=1}^{t(k)} \alpha_{kr} w_{kr}(S))$$

$$\leq \sum_{k=1}^{q} \hat{y}_k \sum_{r=1}^{t(k)} \alpha_{kr} |V(k,r)|^{-1} |S \cap V_{k,r}|$$

$$= \sum_{i \in S} \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \alpha_{kr} \hat{y}_k |V(k,r)|^{-1} e_i^{V(k,r)}$$

$$= \sum_{i \in S} x_i.$$

So $x \in C(v)$. ■

Note that the core element x in theorem 4.2 is obtained as follows.

(i) First calculate shadow prices $\hat{y}_1, \ldots, \hat{y}_q$ for the commodities by solving the linear program (4.2) for the grand coalition.

(ii) With respect to the shadow price $\hat{y}_k$ the portion α_{kr} has value $\alpha_{kr}\hat{y}_k$. Divide this amount equally among the veto players in the game w_{kr} which describes the control of the portion α_{kr}.

(iii) To find x_i add the amount obtained by i over all portions.

Remark 4.3. Consider the third model where the value $\alpha_{kr}\hat{y}_k$ is divided equally among the veto players in w_{kr}. We change the situation and take as division key for $\alpha_{kr}\hat{y}_k$ a core element z^{kr} of w_{kr} i.e. player i obtains from $\alpha_{kr}\hat{y}_k$ the amount $\alpha_{kr}\hat{y}_k z_i^{kr}$. It is then easy to prove that this procedure also leads to a core element $\widetilde{x}$ of the linear production game, where

$$\widetilde{x} = \sum_{k=1}^{q} \sum_{r=1}^{t(k)} \alpha_{kr}\hat{y}_k z^{kr}.$$

Example 4.4. Consider the linear production situation given in table 4.3 where there are 2 products and 2 resources. There are 3 players involved, player 1 owns 4 units of G_1, and 6 units of G_1 are controlled by the unanimity game $u_{\{2,3\}}$. The control of the 12 available units of G_2 is described by the control game w with $w(1,2) = w(1,2,3) = 1$, $w(S) = 0$ otherwise.

$$\begin{array}{cc} & \begin{array}{cc} G_1 & G_2 \end{array} \\ \begin{array}{c} P_1 \\ P_2 \end{array} & \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 30 \\ 40 \end{pmatrix} \\ & \begin{array}{cc} (10 & 12) \\ \uparrow & \uparrow \\ 4 & \\ \delta_1 & 12 \\ 6 & w \\ u_{\{2,3\}} & \end{array} \end{array}$$

Table 4.3.

Then

$$v(\{1,2,3\}) = \min\{10y_1 + 12y_2 \mid y_1, y_2 \geq 0,\ y_1 + 3y_2 \geq 30,\ 4y_1 + 2y_2 \geq 40\}$$
$$= (10,12)(\hat{y}_1, \hat{y}_2) \text{ with } \hat{y}_1 = 6,\ \hat{y}_2 = 8$$

The first portion 4 of G_1 has (shadow)value $4 \cdot 6 = 24$ going to player 1. The second portion 6 of g_1 has value 36, where player 2 and 3 get each 18 units. The 12 units of G_2 have value $12 \cdot 8 = 96$ and are divided equally among the veto players 1 and 2. This results in the core element

$$(24,0,0) + (0,18,18) + (48,48,0) = (72,66,18)$$

of the corresponding linear production game. For this linear production game v we have

$$v(1,2) = 120, \ v(N) = 156 \text{ and } v(S) = 0 \text{ otherwise.}$$

In case we have a linear production situation with some portions of resources controlled by committees where there are no veto players, then not necessarily the corresponding linear production game is balanced. One can stabilize the situation e.g. by replacing the control games with empty cores by so called corresponding tax games and then use a core element of these tax games as a division key for the value of such a portion. This is done in Curiel, Pederzoli and Tijs (1988).

Linear production games can be seen as special cases of linear programming games (cf. Dubey and Shapley (1984), Granot (1986)) which arise from linear programming situations described by an $m \times q$-matrix $A = [a_{ij}]_{i=1,j=1}^{m\ \ ,q}$, $c \in \mathbb{R}^m$ and a function $b : 2^N \backslash \{\emptyset\} \to \mathbb{R}^q$ which are such that for all $S \in 2^N \backslash \{\emptyset\}$ the set $\{xA \leq b(S), \ x \geq 0\}$ is non-empty and $x \to x \cdot c$ is bounded from above on this set. The corresponding LP-game $< N, v >$ is given by

$$\begin{aligned} v(S) : &= \max\{x \cdot c \mid xA \leq b(S), \ x \geq 0\} \\ &= \min\{b(S) \cdot y \mid Ay \geq c, \ y \geq 0\} \end{aligned}$$

Theorem 4.5. Let A, b, c be as above. Suppose that, for each $j \in \{1, \ldots, q\}$, the core of $< N, \beta_j >$ is non-empty, where $\beta_j(\emptyset) = 0$ and

$$\beta_j(S) = (b(S))_j \text{ for each } S \in 2^N \backslash \{\emptyset\} \tag{4.4}$$

Then the LP-game $< N, v >$ has a non-empty core. Moreover, if $z^j \in C(\beta_j)$ for each j and $\hat{y} \in \mathbb{R}^q$ is such that

$$b(N) \cdot \hat{y} = v(N), \ A\hat{y} \geq c, \ \hat{y} \geq 0$$

then $x := \sum_{j=1}^{q} \hat{y}_j z^j \in C(v)$.

Proof. Note that $\hat{y}$ is feasible in the dual of the LP-problem determining $v(S)$ and optimal in the dual LP-program determining $v(N)$. This implies that

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\sum_{j=1}^{q} \hat{y}_j z^j = \sum_{j=1}^{q} \hat{y}_j \beta_j(N) = \sum_{j=1}^{q} \hat{y}_j b_j(N) = v(N)$$

$$\sum_{i\in S} x_i = \sum_{j=1}^{q} \hat{y}_j \sum_{i\in S} z^j \geq \sum_{j=1}^{q} \hat{y}_j \beta_j(S) = b(S)\cdot \hat{y} \geq v(S).$$

Hence, x is a core element of $< N, v >$. ■

Remark 4.6. Let A, b, c be as above and let D be an $m \times r$-matrix. Consider LP-games $< N, v >$ with

$$v(S) := \max\{x \cdot c \mid xA \leq b(S),\ xD = 0,\ x \geq 0\}$$

(i) It is easy to prove that such LP-games have a non-empty core if the corresponding games $\beta_1, \beta_2, \ldots, \beta_q$, as defined in (4.4), have a non-empty core.

(ii) Flow games arising from flow situations with veto control, as in section 3, can be seen as LP-games of the above type. This leads to another proof of theorem 3.2.

Now we introduce permutation games and show that these games are balanced by showing that they can be seen as LP-games. For that purpose we need the theorem of Birkhoff (1948) - von Neumann (1953) on the extreme points of doubly stochastic matrices.

Theorem 4.7. Let

$$DS^n := \{D = [d_{ij}]_{i=1,j=1}^{n\ \ n} \in \mathbb{R}_+^{n\times n} \mid e^N D = e^N,\ De^N = e^N\}$$

be the convex set of *doubly stochastic* $n \times n$-matrices. Let

$$\text{PERM}^n := \{D \in DS^n \mid d_{ij} \in \{0,1\} \text{ for all } i,j \in N\}$$

be the set of permutation matrices. Then DS^n is the convex hull of PERM^n.

Consider the following situation (cf. Tijs, Parthasarathy, Potters and Rajendra Prasad (1984).)

(i) There are n persons $1, 2, \ldots, n$ and each person i, $1 \leq i \leq n$, possesses a machine M_i and has a job J_i to be processed.

(ii) Each machine M_j can process any job J_i, but no machine is allowed to process more than one job.

(iii) Coalition formation and sidepayments are allowed.

(iv) If a person does not cooperate, his job has to be processed on his own machine.

(v) The cost of processing job J_i on machine M_j equals k_{ij}, where $1 \leq i, j \leq n$.

This situation can be reduced to the n-person cost game $< N, c >$ where for each coalition $S \in 2^N \backslash \{\emptyset\}$:

$$c(S) = \min_{\sigma} \sum_{i \in S} k_{i\sigma(i)} \tag{4.5}$$

with the minimum taken over all S-permutations $\sigma : S \to S$. Such a σ corresponds to a plan where job J_i of player $i \in S$ is processed on machine $M_{\sigma(i)}$ of player $\sigma(i)$. The game defined by (4.5) is called the ***permutation game***, corresponding to the ***cost matrix*** $K = [k_{ij}]_{i=1,j=1}^{n}{}^{n}$.

We want to prove that the core

$$C(c) := \{x \in \mathbf{R}^n \mid \sum_{i=1}^{n} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for all } S \in 2^N\}$$

of such permutation game is non-empty by reducing such a game to an LP-game as in theorem 4.5. But let us first give an example.

Example 4.8. Consider the 3-person permutation game $< N, c >$ with cost matrix

$$K = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{bmatrix}.$$

Then $< N, c >$ and the corresponding cost savings game $< N, w >$ are given in table 4.4. It follows e.g. that $c(1,2) = \min\{1+6, 3+2\} = 5$, corresponding to the costs incurred when job J_1 (J_2) is processed on machine M_2 (M_1). Note that $(-2, 6, 9) \in C(c)$ and $(3, 0, 3) \in C(w)$.

S	$\emptyset$	(1)	(2)	(3)	(1, 2)	(1, 3)	(2, 3)	(1, 2, 3)
$c(S)$	0	1	6	12	5	7	17	13
$w(S)$	0	0	0	0	2	6	1	6

Table 4.4

Denote the standard inner product of 2 $n \times n$-matrices $A = [a_{ij}]_{i=1}^{n}{}_{,j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{n}{}_{,j=1}^{n}$ by $A * B$. Hence, $A * B := \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}$. For $S \in 2^N \backslash \{\emptyset\}$ let

$$\text{PERM}_s^n := \{P = [p_{ij}]_{i=1}^{n}{}_{,j=1}^{n} \in \mathbf{R}^{n \times n} \mid p_{ij} \in \{0,1\} \text{ for all } i,j,\ e^N P = e^S,\ \ Pe^N = e^S\}.$$

Then (4.5) can be rewritten as

$$c(S) = \min\{K * P \mid P \in \ \text{PERM}_S^n\}.$$

To prove that $C(c) \neq \emptyset$, it is sufficient to show that for the reward game $< N, v >$, where $v = -c$, the core is non-empty. We can prove this last fact by showing that $< N, v >$ is an LP-game as in theorem 4.5.
Note that for each $S \in 2^N \backslash \{\emptyset\}$:

$$\begin{aligned} v(S) &= \max\{(-K) * P \mid P \in \ \text{PERM}_S^n\} \\ &= \max\{(-K) * D \mid D \geq 0, e^N D \leq e^S, -e^N D \leq -e^S, De^N \leq e^S, -De^N \leq -e^S\} \end{aligned}$$

where the second equality follows from the Birkhoff - von Neumann theorem. But then one easily shows that $< N, v >$ is an LP-game (with $m = q = n^2$). Finally, theorem 4.9 follows from theorem 4.5 by noting that the games $< N, \beta_j >$ are additive.

Theorem 4.9. Permutation games are (totally) balanced games.

For further information on permutation games and related assignment games we refer to Shapley and Shubik (1972), Curiel-Tijs (1986) and Potters-Tijs (1987).

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