Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation X + A*X-1A = Q

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Necessary and Sufficient Conditions for the Existence of a Positive Definite Solution of the Matrix Equation $X + A^*X^{-1}A = Q$

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ABSTRACT

We consider the problem of when the matrix equation $X + A^*X^{-1}A = Q$ has a positive definite solution. Here Q is positive definite. We study both the real and the complex case. This equation plays a crucial role in solving a special case of the discrete-time Riccati equation. We present both necessary and sufficient conditions for its solvability. This result is obtained by using an analytic factorization approach. Moreover, we present algebraic recursive algorithms to compute the largest and smallest the solution of the equation, respectively. Finally, we discuss the number of solutions.

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1. INTRODUCTION

Recently there has been renewed interest in positive definite solutions to the matrix equation $X + A^*X^{-1}A = Q$, with $Q > 0$. In [2] this equation was studied from the point of view of shorted operators, while in [5] the real case was considered, and an application to optimal-control theory was given. The equation appears in many other applications as well; see the references given in [2].

In this paper we continue the study of this equation. In Section 2 a necessary and sufficient condition for solvability is given, as well as a description of all solutions in terms of symmetric factorizations of the rational matrix-valued function $\psi(\lambda) = Q + \lambda A + \lambda^{-1}A^*$. Also, the order structure of the set of solutions is studied. In Section 3 it is shown that the general case can be reduced to the case where $Q = I$ and A is invertible. Section 4 presents iterative procedures to approximate the largest and the smallest solution to the equation. In Section 5 the particular case where $Q = I$ and A invertible is studied. Here the following results of [3] are re-proved (using the result of Section 2): the equation $X + A^*X^{-1}A = I$ has a positive solution X if and only if the numerical range of A is contained in the closed disc of radius $\frac{1}{2}$ in the complex plane. Section 6 makes a connection to the theory of matrices in an indefinite scalar product. It describes the set of solutions of the equation $X + A^*X^{-1}A = I$ with A invertible in terms of Lagrangian subspaces invariant under the matrix

$$\begin{bmatrix} 0 & -A^{-1} \\ A^* & -A^{-1} \end{bmatrix}.$$

This enables one to make precise statements concerning the number of solutions. In Section 7 a relation to the theory of algebraic Riccati equations is outlined. Finally, in Section 8 the real case is considered.

2. NECESSARY AND SUFFICIENT CONDITIONS IN TERMS OF FACTORIZATION

In this section the equation

$$X + A^*X^{-1}A = Q \tag{2.1}$$

So $C_1^* = A^* C_0^{-1}$. Thus

$$Q = C_0^* C_0 + A^* C_0^{-1} C_0^{*-1} A = X + A^* X^{-1} A,$$

i.e., X solves Equation (2.1). ■

REMARK. "$\psi(\lambda) \geqslant 0$ on the unit circle" does not imply "$\psi(\lambda)$ regular." Consider e.g.,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Q = I.$$

Not every factorization of $\psi(\lambda)$ as in (2.3) corresponds to a solution X of the equation (2.1); the requirement $\det C_0 \neq 0$ is necessary for this. To see this consider the trivial example $A = 0$. In that case, $X = Q$. Taking a minimal factorization of $\psi(\lambda)$, we have $\psi(\lambda) \equiv Q = C_0^* C_0$, so for such factorizations we obtain the solution $X = Q$ as in the theorem. However, taking the nonminimal factorization $\psi(\lambda) = \lambda^{-1} Q^{1/2} Q^{1/2} \lambda$, we see $C_0 = 0$, and we do not obtain the solution $X = Q$ by taking $C_0^* C_0$.

The next theorem describes the order structure of the set of solutions of Equation (2.1) in terms of the factorizations of the type (2.3).

THEOREM 2.2. *Let X_1 and X_2 be positive definite solutions of Equation (2.1), and let $\varphi_i(\lambda) = C_{0i} + \lambda C_{1i}$ $(i = 1, 2)$ be such that $\psi(\lambda) = \varphi_i(\bar{\lambda}^{-1})^* \varphi_i(\lambda)$ and $\det C_{0i} \neq 0$ and $C_{0i}^* C_{0i} = X_i$. Suppose $\varphi_2(\lambda)\varphi_1(\lambda)^{-1}$ is analytic in the open unit disc D. Then $X_2 \leqslant X_1$. In particular, if X_L denotes the solution corresponding to the factorization (2.3) of $\psi(\lambda)$ such that $\det(C_0 + \lambda C_1) \neq 0$ for $|\lambda| < 1$, then X_L is the largest solution of (2.1). Moreover, X_L is the unique solution for which $X + \lambda A$ is invertible for all $\lambda \in D$.*

Proof. Put $U(\lambda) = \varphi_2(\lambda)\varphi_1(\lambda)^{-1}$. Then $U(\bar{\lambda}^{-1})^* U(\lambda) \equiv I$, i.e., $U(\lambda)$ is a unitary rational matrix function. Such a function has no poles on the unit circle; see, e.g., [1, 7]. As $\varphi_2 \varphi_1^{-1}$ is analytic on D by assumption, it is analytic on $\overline{D}$ (i.e. the closure of the unit disc). As U is rational, it is actually analytic on a disc of radius $R > 1$. Write $U(\lambda) = \sum_{j=0}^{\infty} U_j \lambda^j$ for $|\lambda| < R$. Then $U(\bar{\lambda}^{-1})^* = \sum_{j=0}^{\infty} U_j^* \lambda^{-j}$. So

$$I = U(\bar{\lambda}^{-1})^* U(\lambda) = \left(\sum_{j=0}^{\infty} U_j^* \lambda^{-j} \right) \left(\sum_{j=0}^{\infty} U_j \lambda^j \right) = \sum_{j=0}^{\infty} U_j^* U_j.$$

is studied in terms of properties of the corresponding rational matrix-valued function

$$\psi(\lambda) = Q + \lambda A + \lambda^{-1}A^*. \tag{2.2}$$

Here Q is assumed throughout to be positive definite and we are looking for a positive definite solution X. The function ψ is called regular if $\det \psi(\lambda)$ is not identically zero, i.e., if there exists at least one point where $\det \psi(\lambda) \neq 0$. As $\det \psi(\lambda)$ is itself a rational (scalar) function, there are only a finite number of points for which $\det \psi(\lambda) = 0$ in case ψ is regular.

THEOREM 2.1. *Suppose Q is positive definite. Then the equation $X + A^*X^{-1}A = Q$ has a positive definite solution X if and only if ψ is regular and $\psi(\lambda) \geqslant 0$ for all λ on the unit circle.*

In that case $\psi(\lambda)$ factors as

$$\psi(\lambda) = (C_0^* + \lambda^{-1}C_1^*)(C_0 + \lambda C_1) \tag{2.3}$$

with $\det C_0 \neq 0$, and $X = C_0^* C_0$ is a solution of (2.1). Every positive definite solution is obtained in this way.

Proof. Suppose $X > 0$ is a solution. Put $C_0 = X^{1/2}$, $C_1 = X^{-1/2}A$. Then

$$\begin{aligned}\psi(\lambda) &= (I + \lambda^{-1}A^*X^{-1})X(I + \lambda X^{-1}A)\\ &= (C_0^* + \lambda^{-1}C_1^*)(C_0 + \lambda C_1),\end{aligned}$$

so $\psi(\lambda)$ is positive semidefinite for $|\lambda| = 1$. Since X is invertible, we have $\det(C_0 + \lambda C_1) \neq 0$ for $|\lambda|$ small; hence ψ is regular.

Conversely, suppose ψ is regular, and positive semidefinite for $|\lambda| = 1$. Then it is well known that there exists a factorization as in (2.3) (see, e.g., Section 6.6 in [15] and the references given there). Moreover, the factor $C_0 + \lambda C_1$ can actually be chosen such that it is invertible for $|\lambda| < 1$, i.e., $\det(C_0 + \lambda C_1) \neq 0$ for $|\lambda| < 1$ (also see, e.g., Section 6.6 in [15]). Put $X = C_0^* C_0$, where C_0 comes from this particular factorization. As $\det C_0 \neq 0$ in this case, $X > 0$. From (2.3) one sees

$$Q = C_0^* C_0 + C_1^* C_1, \qquad A^* = C_1^* C_0, \qquad A = C_0^* C_1.$$

that in [2] it is allowed that $X \geqslant 0$, the inverse in the equation (2.1) being interpreted as a generalized inverse. This explains the differences between our results and those in [2].

3. REDUCTION TO A SPECIAL CASE

In this section the general equation (2.1) will be reduced to the special case where $Q = I$ and A is invertible. This reduction is a repeated application of two steps. The first step is the following simple observation.

PROPOSITION 3.1. *Let Q be positive definite. Then X is a solution of the equation*

$$X + A^*X^{-1}A = Q$$

if and only if $Y = Q^{-1/2}XQ^{-1/2}$ is a solution of the equation

$$Y + \hat{A}^*Y^{-1}\hat{A} = I,$$

where $\hat{A} = Q^{-1/2}AQ^{-1/2}$.

For the second step let us consider the equation

$$X + A^*X^{-1}A = I, \tag{3.1}$$

with A a singular $n \times n$ matrix. If $A = 0$, the equation is trivial. Otherwise decompose $\mathbb{C}^n$ as follows: $\mathbb{C}^n = \operatorname{Ker} A \oplus \operatorname{Im} A^*$. With respect to this orthogonal decomposition write

$$A = \begin{bmatrix} 0 & A_1 \\ 0 & A_2 \end{bmatrix}, \qquad X = \begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix}.$$

(X necessarily must have this form, as $X|_{\operatorname{Ker} A} = I|_{\operatorname{Ker} A}$ and $X \leqslant I$.) Then (3.1) reduces to an equation for X_2:

$$X_2 + A_2^* X_2^{-1} A_2 = I - A_1^* A_1. \tag{3.2}$$

Thus, if there is a positive solution X of (3.1) then $I - A_1^* A_1 > 0$. Applying Proposition 3.1, we can reduce the equation (3.2) once again to one of the

In particular, $U_0^* U_0 \leqslant I$, i.e. U_0 is a contraction. From

$$U(\lambda)\varphi_1(\lambda) = \sum_{j=0}^{\infty} U_j \lambda^j (C_{01} + \lambda C_{11})$$

$$= \varphi_2(\lambda) = C_{02} + \lambda C_{12}$$

one verifies $U_0 C_{01} = C_{02}$. Therefore

$$X_2 = C_{02}^* C_{02} = C_{01}^* U_0^* U_0 C_{01} \leqslant C_{01}^* C_{01} = X_1.$$

Let X_L be the solution corresponding to the factorization (2.3) for which $\det(C_0 + \lambda C_1) \neq 0$, $|\lambda| < 1$. Denote by $\varphi_L(\lambda)$ this particular factor. Then $\varphi_L(\lambda)^{-1}$ is analytic on D, so for any solution X and the corresponding factor $\varphi(\lambda)$ we have $\varphi(\lambda)\varphi_L(\lambda)^{-1}$ is analytic on D. Thus $X \leqslant X_L$ because of what we have just proved.

Let $\varphi_L(\lambda) = C_0 + \lambda C_1$ be the factor for which $\det(C_0 + \lambda C_1) \neq 0$, $|\lambda| < 1$. Then $X_L = C_0^* C_0$, and so $C_0 = U_1 X_L^{1/2}$ for a unitary U_1. As $A = C_0^* C_1 = X_L^{1/2} U_1^* C_1$ we have $C_1 = U_1 X_L^{-1/2} A$. Thus

$$\varphi_L(\lambda) = U_1 X_L^{-1/2}(X_L + \lambda A).$$

So $\det(X_L + \lambda A) \neq 0$ for $|\lambda| < 1$. Now suppose X_1 is a solution of (2.1) such that $X_1 + \lambda A$ is invertible for $\lambda \in D$. Put $\varphi_1(\lambda) = X_1^{1/2} + \lambda X_1^{-1/2} A$. Then $\varphi_L(\lambda)\varphi_1(\lambda)^{-1}$ is analytic in D, and by the first part of the proof we have $X_L \leqslant X_1$. As $X_1 \leqslant X_L$ is already proved, we get $X_1 = X_L$. ■

The fact that the solution corresponding to the factorization of $\psi(\lambda)$ for which $\det(C_0 + \lambda C_1) \neq 0$, $\lambda \in D$, is the largest solution can also be derived quite easily from [15], Theorem A in Section 5.9.

The function $\psi(\lambda)$ can also be viewed as the symbol of the Toeplitz operator

$$T = \begin{bmatrix} Q & A^* & & 0 \\ A & Q & A^* & \\ & A & Q & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

Positive semidefiniteness of T is equivalent with $\psi(\lambda)$ being positive semidefinite on the unit circle. This provides a link to [2, Section 4]. Observe form (3.1) but now in lower dimensions. Continuing this process, one ends with either one of the next two possibilities: an equation of the form (3.1) with $A = 0$, or an equation of the form (3.1) with A nonsingular. (In the former case, necessarily the original A must have been nilpotent to start with.) In fact, a combination of the two reduction steps applied repeatedly proves the following theorem.

THEOREM 3.2. *Suppose $Q > 0$. Then, in case the equation $X + A^*X^{-1}A = Q$ has a positive solution, either it has precisely one such solution or there are nonsingular matrices W, and $\tilde{A}$ completely determined by A and Q, such that any solution X is of the form*

$$X = W^*\begin{bmatrix} I & 0 \\ 0 & \tilde{X} \end{bmatrix} W$$

for a positive solution $\tilde{X}$ of the equation

$$\tilde{X} + \tilde{A}^*\tilde{X}^{-1}\tilde{A} = I. \tag{3.3}$$

Proof. After applying Proposition 3.1 and the reduction that leads from equation (3.1) to (3.2), it is seen that any solution X of $X + A^*X^{-1}A = Q$ is of the form

$$Q^{1/2}\begin{bmatrix} I & 0 \\ 0 & X_2 \end{bmatrix} Q^{1/2}$$

for a solution X_2 of (3.2). Apply again Proposition 3.1: let $Q_1 = I - A_1^* A_1$. Then $X_2 = Q_1^{1/2} Y_2 Q_1^{1/2}$ for a solution Y_2 of

$$Y_2 + \hat{A}_2^* Y_2^{-1} \hat{A}_2 = I,$$

where $\hat{A}_2 = Q_1^{-1/2} A_2 Q_1^{-1/2}$. If $\hat{A}_2$ is nonsingular or zero, we are done. Otherwise decompose the space again and repeat the argument. ■

Note that this reduction process respects the order structure on the set of solutions. In other words, if X_1 and X_2 are two positive solutions of $X + A^*X^{-1}A = Q$, and

$$X_i = W^*\begin{bmatrix} I & 0 \\ 0 & \tilde{X}_i \end{bmatrix} W,$$

where W is as in the theorem and $\tilde{X}_1$, $\tilde{X}_2$ are positive solutions of (3.3), then $X_1 \leqslant X_2$ if and only if $\tilde{X}_1 \leqslant \tilde{X}_2$.

THEOREM 3.3. *Let A be invertible. Then X solves the equation* (3.1), *i.e.*,

$$X + A^*X^{-1}A = I$$

if and only if $Y = I - X$ solves

$$Y + AY^{-1}A^* = I. \tag{3.4}$$

In particular, if Y_L is the maximal solution of (3.4) *then $X_s = I - Y_L$ is the minimal solution of* (3.1). *Moreover, X_s is the unique positive solution for which $X + \lambda A^*$ is invertible for $|\lambda| > 1$.*

Proof. Let X be a solution of (3.1). Then $A^*X^{-1}A = I - X$. Hence $X^{-1} = A^{*-1}(I - X)A^{-1}$. Taking inverses yields $X = A(I - X)^{-1}A^*$, so $Y = I - X$ solves (3.4). The converse is seen in the same way.

Note that $X_1 \leqslant X_2$ if and only if $Y_1 \geqslant Y_2$. Hence the relation between X_s and Y_L. By Theorem 2.2, X_s is the unique solution for which $Y_L + \lambda A^* = I - X_s + \lambda A^*$ is invertible for all $\lambda \in D$. Now by (3.1),

$$I - X_s + \lambda A^* = A^*X_s^{-1}(A + \lambda X_s).$$

So X_s is the unique solution for which $A + \lambda X_s$ is invertible for $\lambda \in D$. Equivalently, $(A + \bar{\lambda}^{-1}X_s)^*$ is invertible for $|\lambda| > 1$. But

$$\left(A + \bar{\lambda}^{-1}X_s\right)^* = A^* + \lambda^{-1}X_s = \lambda^{-1}(X_s + \lambda A^*).$$

So X_s is the only solution such that $X_s + \lambda A^*$ is invertible for $|\lambda| > 1$. ■

We can generalize the last statement of Theorem 3.3 to the case of Equation (2.1).

THEOREM 3.4. *Suppose $Q > 0$, and assume the equation $X + A^*X^{-1}A = Q$ has a positive solution. Then this equation has a largest and a smallest solution X_L and X_s, respectively. Moreover, X_L is the unique solution for which $X + \lambda A$ is invertible for $|\lambda| < 1$, while X_s is the unique solution for which $X + \lambda A^*$ is invertible for $|\lambda| > 1$.*

Proof. First we show the existence of a smallest solution. The reduction process outlined in Theorem 3.2 and Proposition 3.1 preserves the ordering of the solution. Thus we may apply Theorem 3.3 to see that there exists a smallest solution.

To prove the second part of the theorem we only need to show that X_s is the unique solution for which $X + \lambda A^*$ is invertible for $|\lambda| > 1$. It is not hard to see that this property is also preserved under the reduction process of Theorem 3.2 and Proposition 3.1. Thus, again, this follows from Theorem 3.3. ■

As a corollary we have the following theorem which tells us exactly when there is a unique solution.

THEOREM 3.5. *Suppose $Q > 0$. Then the equation* (2.1) *has exactly one solution if and only if the following three conditions hold*:

(i) ψ *is regular*;
(ii) $\psi(\lambda) \geqslant 0$ *for* $|\lambda| = 1$;
(iii) *any zeros of* $\det \psi(\lambda)$ *are on the unit circle.*

Proof. Suppose Equation (2.1) has exactly one solution. Then (i) and (ii) must hold by Theorem 2.1. Moreover,

$$\psi(\lambda) = (X + \lambda^{-1}A^*)X^{-1}(X + \lambda A).$$

Therefore, $\det \psi(\lambda) = \det X^{-1} \det(X + \lambda A)\det(X + \lambda^{-1}A^*)$. As X is the unique solution, we have $X = X_s = X_L$. By Theorem 3.4, $\det(X + \lambda A)$ and $\det(X + \lambda^{-1}A^*)$ are both invertible for $|\lambda| < 1$. Thus $\psi(\lambda)$ is invertible for all λ inside the unit circle (with the exception of zero). As ψ is self-adjoint, it follows that $\psi(\lambda)$ must be invertible for all nonzero λ not on the unit circle. Thus (iii) holds.

Conversely, assume (i), (ii), (iii) hold. Then there is at least one solution by Theorem 2.1. Moreover, by (iii) we have for any solution X of (2.1) that $X + \lambda A$ and $X + \lambda^{-1}A^*$ must be invertible for $|\lambda| < 1$. Thus by Theorem 3.4, $X = X_L = X_s$. ■

4. TWO RECURRENCE EQUATIONS

In the previous section we saw that whenever our matrix equation (2.1) has a solution, it automatically has a largest and a smallest solution, denoted by X_L and X_s, respectively. Moreover, we presented an algorithm to calculate these solutions X_L and X_s. In this section we show that these solutions can also be obtained via a recurrence equation. The advantage of these recurrence equations are that they are directly related to the original equation (2.1) and very simple to implement. Whether both solutions X_L and X_s are obtained from these equations in a numerically reliable way remains at this point an open question, and therefore a problem for future research.

We shall see that the algorithm to calculate the largest (real) solution X_L is the easiest one. To calculate X_s, we will in fact implement the dual algorithm for calculating X_L. However, since the dual algorithm only works if matrix A is invertible, in general we first have to apply some transformations, already mentioned in the previous section, to Equation (2.1).

The algorithm to calculate X_L is as follows.

ALGORITHM 4.1. Consider the recurrence equation

$$\begin{aligned} X_0 &= I \\ X_{n+1} &= I - A^*X_n^{-1}A \end{aligned} \tag{4.1}$$

If Equation (2.1) has a solution $X > 0$, then $X_n \to X_L$.

Proof. We show that X_n is a monotonically decreasing sequence that is bounded from below, and thus converges. To that end we first show by induction that $X_k \geqslant X\ \forall k \in \mathbb{N}$. Note that as a consequence then $X_k > 0$ for any $k \in \mathbb{N}$, and since X is an arbitrarily chosen solution of (2.1), we have that $X_k \geqslant X_L\ \ \forall k \in \mathbb{N}$.

For $k = 0$, the statement is trivially satisfied. So assume that the statement holds for $k = n$. Then, $X_{k+1} - X = A^*(X^{-1} - X_k^{-1})A \geqslant 0$, since $X_k \geqslant X > 0$, which completes the first part of our argument.

Next we show that X_k is a monotonically decreasing sequence. The proof is quite similar to the previous argument. First, consider $X_0 - X_1$. From the definition of X_n we have that $X_0 - X_1 = I - (I - A^*X_0^{-1}A) = A^*A \geqslant 0$. So the statement holds for $k = 0$. Next, assume that $X_k - X_{k+1} \geqslant 0$ for $k = n$. Then, using the induction argument and the fact that $X_k > 0$ for any k, we have $X_{n+1} - X_{n+2} = A^*(X_{n+1}^{-1} - X_n^{-1})A \geqslant 0$. So the induction argument is complete. Combination of both results yields that $X_n \to X_L$. ■

To calculate X_s, the following algorithm can be used.

ALGORITHM 4.2. If Equation (2.1) has a solution $X > 0$, then this algorithm gives us the smallest (real) solution X_s of the equation.

1. (i) If A is invertible then go to part 2.
   (ii) Else apply a unitary transformation T such that

$$A = T^* \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} T.$$

   (iii) If $A_{11} = 0$, then

$$X_S := T^* \begin{pmatrix} I - A_{21}^* A_{21} & 0 \\ 0 & I \end{pmatrix} T$$

   and the algorithm stops.
   (iv) Else

$$X_S := T^* \begin{pmatrix} Y_S & 0 \\ 0 & I \end{pmatrix} T,$$

   with $Y_S > 0$ the smallest solution of Equation (2.1), where A is replaced by $(I - A_{21}^* A_{21})^{-1/2} A_{11} (I - A_{21}^* A_{21})^{-1/2}$. Now return to (i).
2. Consider the recurrence equation

$$X_0 := AA^*,$$

$$X_{n+1} := A(I - X_n)^{-1} A^*.$$

   Then $X_n \to X_s$.

Proof. Part 1 of the algorithm follows from the reduction process of Section 3. So what is left to be proved is that part 2 works under the assumption that A is invertible. Using Theorem 3.3 and Algorithm 4.1, this is however straightforward to prove, and therefore the proof is omitted. ■

For Algorithm 4.1 compare also [3, 2].

5. ANOTHER NECESSARY AND SUFFICIENT CONDITION

In this section we shall assume A is invertible and $Q = I$. Recall that the set $\{\langle Ax, x\rangle \mid \|x\| = 1\}$ is called the numerical range of A; we shall denote this set by $W(A)$. Furthermore, let us denote by $\omega(A)$ the numerical radius of A, i.e.,

$$\omega(A) = \max\{|z| \,|\, z \in W(A)\}.$$

With this notation the following theorem holds.

THEOREM 5.1. *Suppose A is invertible. Then there is a positive definite solution X of the equation $X + A^*X^{-1}A = I$ if and only if $\omega(A) \leqslant \frac{1}{2}$.*

Proof. Suppose $X > 0$ and solves the equation $X + A^*X^{-1}A = I$. From Theorem 2.1 we know that the rational matrix function $\psi(\lambda) = I + \lambda A + \lambda^{-1}A^*$ is positive semidefinite for $|\lambda| = 1$. Now take x with $\|x\| = 1$. Then, for $|\lambda| = 1$,

$$0 \leqslant \langle \psi(\lambda)x, x\rangle = \langle x, x\rangle + \lambda\langle Ax, x\rangle + \bar{\lambda}\,\overline{\langle Ax, x\rangle}.$$

Hence for $z = \langle Ax, x\rangle \in W(A)$ and $|\lambda| = 1$ we have $0 \leqslant 1 + \lambda z + \bar{\lambda}\bar{z}$. But this is easily seen to be equivalent to $|z| \leqslant \frac{1}{2}$.

Conversely, assume $\omega(A) \leqslant \frac{1}{2}$. Then for $\|x\| = 1$ and $|\lambda| = 1$,

$$\langle \psi(\lambda)x, x\rangle = 1 + 2\,\mathrm{Re}\,\lambda\langle Ax, x\rangle.$$

Now $|\lambda\langle Ax, x\rangle| \leqslant \frac{1}{2}$ so $\psi(\lambda) \geqslant 0$ for $|\lambda| = 1$. But as A is invertible, $\psi(\lambda)$ is regular. Indeed,

$$\psi(\lambda) = \lambda^{-1}A^*(\lambda A^{*-1} + \lambda^2 A^{*-1}A + I).$$

Now $\lambda^{-1}A^*$ is invertible for $\lambda \neq 0$, while $\lambda^{-1}A^* + \lambda^2 A^{*-1}A + I$ is a regular matrix polynomial. Thus $\psi(\lambda)$ is regular and positive semidefinite for $|\lambda| = 1$. By Theorem 2.1 the equation $X + A^*X^{-1}A = I$ has a positive solution. ■

This theorem was essentially obtained by different methods in [3]. In one direction the result can also be derived straightforwardly from Lemma 1 in [4]. Using again different methods, the theorem was derived for the special case of normal matrices A in [5].

Next we consider a similar condition for the more general equation $X + A^*X^{-1}A = Q$, where we assume $Q > 0$ and A nonsingular. The Q-numerical radius of A is defined as

$$\omega_Q(A) = \omega(Q^{-1/2}AQ^{-1/2}).$$

THEOREM 5.2. *Suppose A is nonsingular. Then the equation $X + A^*X^{-1}A = Q$ has a positive definite solution X if and only if $\omega_Q(A) \leqslant \frac{1}{2}$.*

As the proof follows essentially the same lines as the proof of Theorem 5.1, it is omitted. ■

6. DESCRIPTION OF THE SET OF SOLUTIONS IN TERMS OF INVARIANT SUBSPACES IN CASE A IS INVERTIBLE

The equation we study here is the one obtained after application of the reduction process of Section 3. In other words, we consider the equation

$$X + A^*X^{-1}A = I, \tag{6.1}$$

where A is nonsingular. Introduce the matrices

$$H = \begin{bmatrix} 0 & -A^{-1} \\ A^* & -A^{-1} \end{bmatrix}, \qquad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \tag{6.2}$$

Note that H is J-unitary, i.e., $H^*JH = J$. The next theorem gives necessary and sufficient conditions for solvability of Equation (6.1) in terms of H and its invariant subspaces.

THEOREM 6.1. *The following are equivalent:*

(i) *there is a positive solution X of* (6.1);

(ii) $\psi(\lambda) = I + \lambda A + \lambda^{-1}A^*$ *is regular and positive semidefinite for* $|\lambda| = 1$;

(iii) *there is a number η, $|\eta| = 1$, such that $\psi(\eta) > 0$ and the partial multiplicities of H corresponding to its eigenvalues on the unit circle (if any) are all even;*

(iv) *there is a number η, $|\eta| = 1$, such that $\psi(\eta) > 0$ and there exists an H-invariant subspace M such that $JM = M^{\perp}$.*

Proof. The equivalence of (i) and (ii) is already observed in Theorem 2.1. First we show that (ii) implies (iii). The existence of η, $|\eta| = 1$, such that

$\psi(\eta) > 0$ is immediate from (ii). To show the second part of (iii), first note that

$$\begin{bmatrix} I & 0 \\ \frac{1}{\lambda}A^* & I \end{bmatrix} (H - \lambda I) \begin{bmatrix} -\frac{1}{\lambda}I & \frac{1}{\lambda}I \\ 0 & -A \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \psi(\lambda) \end{bmatrix}. \tag{6.3}$$

So $\psi(\lambda)$ and $H - \lambda I$ are equivalent (in the sense of analytic matrix functions) on $\mathbb{C} \setminus \{0\}$. Hence the partial multiplicities of H and $\psi(\lambda)$ at their eigenvalues on the unit circle coincide (see, e.g., [6]). Next, for $z \in \mathbb{R}$ define

$$\begin{aligned} \varphi(z) &= (z - i)\psi\left(\frac{z + i}{z - i}\right)(z + i) \\ &= (z^2 + 1)I + (z^2 - 1)(A + A^*) + 2iz(A - A^*). \end{aligned}$$

Then $\varphi(z) \geqslant 0$ for $z \in \mathbb{R}$, and the partial multiplicities of φ at z coincide with those of ψ at $\lambda = (z + i)/(z - i)$. But the partial multiplicities of φ at real zeros are all even [8, Chapter 12]. Hence, those of ψ at its zeros on the unit circle are all even, and so (iii) is proved.

Now we prove (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Since H is J-unitary and the partial multiplicities of H at its eigenvalues on the unit circle are all even, there is a H-invariant subspace M such that $JM = M^{\perp}$ (see [12, 13]). Let

$$M = \operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}.$$

Then we shall show that X_1 is invertible. For this, note that for any η on the unit circle for which $H - \eta$ and $H + \eta$ are invertible,

$$(H + \eta)(H - \eta)^{-1}M = M.$$

Using (6.3), we have

$$(H + \eta)(H - \eta)^{-1} = \begin{bmatrix} * & 2\psi(\eta)^{-1} \\ * & * \end{bmatrix}.$$

Now, assume $X_1 x = 0$. Then

$$\begin{bmatrix} 0 \\ X_2 x \end{bmatrix} \in M.$$

Hence also

$$(H+\eta)(H-\eta)^{-1}\begin{bmatrix}0\\ X_2x\end{bmatrix}=\begin{bmatrix}2\psi(\eta)^{-1}X_2x\\ *\end{bmatrix}\in M.$$

Since $JM = M^{\perp}$, we have

$$0=\left\langle J\begin{bmatrix}0\\ X_2x\end{bmatrix},(H+\eta)(H-\eta)^{-1}\begin{bmatrix}0\\ X_2x\end{bmatrix}\right\rangle=\left\langle X_2x,2\psi(\eta)^{-1}X_2x\right\rangle.$$

Now, by assumption $\psi(\eta)$ is positive definite. So $X_2x = 0$. But $\dim M = n$, and so $\operatorname{Ker} X_1 \cap \operatorname{Ker} X_2 = (0)$. Hence $x = 0$. Now put $X = X_2X_1^{-1}$. Then

$$M=\operatorname{Im}\begin{bmatrix}I\\ X\end{bmatrix}.$$

From $JM = M^{\perp}$ we obtain

$$0=[I\quad X^*]\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}\begin{bmatrix}I\\ X\end{bmatrix}=X-X^*.$$

So X is Hermitian. Since $HM = M$, we have

$$M=\operatorname{Im}H\begin{bmatrix}I\\ X\end{bmatrix}=\operatorname{Im}\begin{bmatrix}-A^{-1}X\\ A^*-A^{-1}X\end{bmatrix}$$

Applying the result above, we see that $-A^{-1}X$ is invertible and $X = (A^* - A^{-1}X)(-A^{-1}X)^{-1}$. Consequently, $-XA^{-1}X = A^* - A^{-1}X$, which yields $A^*X^{-1}A - I = -X$, i.e., $X + A^*X^{-1}A = I$.

Next, put $P(\lambda) = I + \lambda X^{-1}A$. Then

$$\begin{aligned}P\left(\frac{1}{\lambda}\right)^*XP(\lambda)&=\left(I+\frac{1}{\lambda}A^*X^{-1}\right)X(I+\lambda X^{-1}A)\\&=X+A^*X^{-1}A+\frac{1}{\lambda}A^*+\lambda A=I+\lambda A+\lambda^{-1}A^*\\&=\psi(\lambda).\end{aligned}$$

In particular, as $\psi(\eta) > 0$, we obtain that $I + \eta X^{-1}A$ is invertible and $X > 0$. ■

From the last paragraph of the proof also the following corollary is obtained.

COROLLARY 6.2. *If* (6.1) *has a positive definite solution then all its hermitian solutions are positive definite.* ■

The next theorem provides a description of the set of solutions in terms of invariant subspace M of H for which $JM = M^{\perp}$. Such subspaces are called Lagrangian subspaces.

THEOREM 6.3. *Suppose* (i)–(iv) *of Theorem* 6.1 *hold. Then for any solution* $X > 0$ *of* (6.1) *the subspace* $M = \operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$ *is a Lagrangian H-invariant subspace. Conversely, any Lagrangian H-invariant subspace M is of the form* $M = \operatorname{Im}\begin{bmatrix} I \\ X \end{bmatrix}$ *for some X satisfying* (6.1).

Proof. Let $X > 0$ be a solution. It is a straightforward computation that $JM = M^{\perp}$. Furthermore, by (6.1)

$$H\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} -A^{-1}X \\ A^* - A^{-1}X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}(-A^{-1}X).$$

The converse was proved in the proof of Theorem 6.1. ■

The pair (H, iJ) has extra properties connected with the sign characteristic which are extremely important for determining the fine structure of the set of solutions of (6.1). Recall from [9] that the sign characteristic of the pair (H, iJ) may be defined to be the sign characteristic of the pair $(i(H + \eta)(H - \eta)^{-1}, iJ)$, the latter being defined from a canonical form for matrices self-adjoint in an indefinite scalar product (see [9]). Let $\varphi(\lambda) = (\lambda + \eta)(\lambda - \eta)^{-1}$; then $(H + \eta)(H - \eta)^{-1} = \varphi(H)$.

In the following theorem we shall denote by $X_+(H)$ the spectral invariant subspace of H corresponding to its eigenvalues outside the closed unit disc.

THEOREM 6.4. *Statements* (i)–(iv) *in Theorem* 6.1 *are also equivalent to*

(v) *there is a number* η, $|\eta| = 1$ *with* $\psi(\eta) > 0$; *H has only even partial multiplicities corresponding to its eigenvalues on the unit circle; and the signs in the sign characteristic of* (H, iJ) *are all* 1.

Proof. Clearly (v) implies (iii). So assume there is a solution X of (6.1). Then (iii) holds, and to prove (v) it remains to show that the statement on the sign characteristic is correct. To see this, compute

$$\varphi(H) = (H+\eta)(H-\eta)^{-1}$$
$$= \begin{bmatrix} -I + 2\psi(\eta)^{-1}\dfrac{1}{\eta}A^* & 2\psi(\eta)^{-1} \\ -2A\psi(\eta)^{-1}A^* & I - 2\eta A\psi(\eta)^{-1} \end{bmatrix}.$$

By Theorem 6.3

$$H\,\mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix} \subset \mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix}.$$

But then

$$\varphi(H)\,\mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix} \subset \mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix},$$

which implies that X solves the algebraic Riccati equation

$$2X\psi(\eta)^{-1}X + X\left(-I + 2\psi(\eta)^{-1}\frac{1}{\eta}A^*\right) + \left[-I + 2\eta A\psi(\eta)^{-1}\right]X$$
$$+ 2A\psi(\eta)^{-1}A^* = 0.$$

From the positivity of $\psi(\eta)$ it follows that we may apply [9, Corollary II.4.7]. According to this corollary the signs in the sign characteristic of $(i\varphi(H), iJ)$ are all 1's. ■

As a consequence of this theorem and the one preceding it we can now describe the structure of the set of solutions of (6.1) in terms of the set of invariant subspaces of a matrix.

THEOREM 6.5. *Suppose* (i)–(v) *hold. Then for every H-invariant subspace N contained in $X_+(H)$ there is a unique solution X of* (6.1) *such that*

$$\mathrm{Im}\begin{bmatrix} I \\ X \end{bmatrix} \cap X_+(H) = N.$$

Proof. From [12, Sections 7 and 2] it follows that (v) implies that given N as in the theorem, there is a unique H-invariant Lagrangian subspace M

such that $M \cap X_+(H) = N$. But Theorem 6.3 gives a one-one correspondence between such subspaces M and solutions X of (6.1). ■

As a corollary we present the following.

COROLLARY 6.6. *Suppose* (6.1) *has a solution* $X > 0$. *Then there is a finite number of solutions if and only if* $\dim \operatorname{Ker}(H - \lambda) = 1$ *for every eigenvalue* λ *of* H *not on the unit circle. Otherwise there is a continuum of solutions.*

Proof. If $\dim \operatorname{Ker}(H - \lambda) = 1$ for all λ, $|\lambda| \neq 1$, which are eigenvalues of H, then clearly the number of H-invariant subspaces $N \subset X_+(H)$ is finite. So there is a finite number of solutions. Conversely, if there is a finite number of solutions, there can be only finitely many H-invariant subspaces $N \subset X_+(H)$. This implies $\dim \operatorname{Ker}(H - \lambda) = 1$ whenever $|\lambda| > 1$ and λ is an eigenvalue. However, as H is J-unitary $\dim \operatorname{Ker}(H - \lambda) = \dim \operatorname{Ker}(H - \bar{\lambda}^{-1})$. So $\dim \operatorname{Ker}(H - \lambda) = 1$ also when $|\lambda| < 1$ and λ is an eigenvalue.

In case $\dim \operatorname{Ker}(H - \lambda) > 1$ for some eigenvalue λ not on the unit circle, there is a continuum of H-invariant subspaces in $X_+(H)$. (See [10, Proposition 2.5.4].) ■

Actually, in case there is a finite number of solutions, one can be more precise. Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of H outside the closed unit disc, assume $\dim \operatorname{Ker}(H - \lambda_i) = 1$, and let $n_1, \ldots, n_k$ be the algebraic multiplicities of $\lambda_1, \ldots, \lambda_k$. Then the number of solutions is exactly $\prod_{j=1}^{k} (n_j + 1)$. Indeed, in general every invariant subspace N of H such that $N \subset X_+(H)$ can be decomposed (uniquely) as $N = N_1 \dotplus \cdots \dotplus N_k$, where N_i is H-invariant and $\sigma(H|_{N_i}) \subset \{\lambda_i\}$. As $\dim \operatorname{Ker}(H - \lambda_i) = 1$, we have $n_i + 1$ possible choices for N_i, namely $N_i = \operatorname{Ker}(H - \lambda_i)^p$, $p = 0, 1, \ldots, n_i$. Making all possible combinations, we arrive at the total of $\prod_{j=1}^{k} (n_j + 1)$ possibilities for N.

Next, we analyze the number of solutions in the particular case when A is a normal matrix. Recall that for a normal matrix the numerical radius $\omega(A)$ equals the spectral radius $r(A)$.

THEOREM 6.7. *Let* A *be normal, and assume* $r(A) \leqslant \frac{1}{2}$. *Let* $S_A = \{\lambda \in \sigma(A) \,|\, |\lambda| = \frac{1}{2}\}$ *and let* $p = \#S_A$. *Then* (6.1) *has*

(a) *exactly one solution if and only if* $p = n$,

(b) 2^{n-p} *solutions if and only if* $\dim \operatorname{Ker}(A - \lambda) = 1$ *for all* $\lambda \in \sigma(A) \setminus S_A$,

(c) *a continuum of solutions in all other cases.*

Proof. Making a unitary transformation, we may assume A to be diagonal, $A = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. It is a straightforward calculation to see that the eigenvalues of

$$H = \begin{bmatrix} 0 & -A^{-1} \\ A^* & -A^{-1} \end{bmatrix}$$

are given by

$$\mu_{i\pm} = \left(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4|\lambda_i|^2} \right) \lambda_i^{-1}$$

and $\dim \operatorname{Ker}(H - \mu_{i\pm}) = \dim \operatorname{Ker}(A - \lambda_i)$. Clearly $|\mu_{i\pm}| = 1$ if and only if $|\lambda_i| = \frac{1}{2}$. Thus, all eigenvalues of H are on the unit circle if and only if $p = n$. So (a) holds. Also (c) is easily seen. To prove (b), assume $\dim \operatorname{Ker}(A - \lambda_i) = 1$ for all $\lambda_i \in \sigma(A) \setminus S_A$. The algebraic multiplicity of $\mu_{i\pm}$ as eigenvalue of H is one as well, and exactly one of the numbers μ_{i+} and μ_{i-} lies outside the unit circle. (To be precise, μ_{i-} is outside the unit circle.) So H has $n - p$ eigenvalues outside the unit circle, all with geometric multiplicity one, and algebraic multiplicity also one. Therefore the number of solutions of (6.1) is 2^{n-p}. ■

7. CONNECTIONS WITH ALGEBRAIC RICCATI EQUATIONS

Many of the results in the previous section are very reminiscent of theorems on the discrete algebraic Riccati equation (compare [14] for instance). That this is no coincidence is seen from the following statement.

PROPOSITION 7.1. *Let A be invertible. Then X is a solution of $X + A^*X^{-1}A = I$ if and only if X is a solution of the discrete algebraic Riccati equation*

$$X = AXA^* + AA^* - AX(-I + X)^{-1} XA^* \tag{7.1}$$

Proof. Rewrite $X + A^*X^{-1}A = I$ as

$$X = A(I - X)^{-1} A^*$$

(use Theorem 3.3). The result follows from

$$(I - X)^{-1} = I + X + X(I - X)^{-1}X. \qquad \blacksquare$$

Note also that in the course of proving Theorem 6.4 we have found that the solutions of (6.1) coincide with the solutions of the continuous algebraic Riccati equation (6.4).

Using Proposition 7.1, some of the results of Section 6 might have been derived directly from [14, Section 1]. We have chosen to give full proofs here, independent of this observation. Algorithm 4.1 may be compared with recursive algorithms to compute the largest solution of a discrete algebraic Riccati equation; see e.g. [11].

8. THE REAL CASE

The case where A and Q are real and we are looking for real symmetric positive definite solutions X is also of interest.

THEOREM 8.1. *The solutions X_L and X_S of $X + A^T X^{-1} A = Q$ are real.*

Proof. Note that the reduction procedure of Proposition 3.1 and Theorem 3.2 preserves real solutions. So we may assume A is invertible and $Q = I$. Consider Algorithm 4.1. The matrices X_n in this algorithm are all real. Hence X_L is real. Also the matrices X_n of Algorithm 4.2, step 2 are all real. Thus X_S is real. ■

Note that also the matrix H in (6.2) is a real matrix. Moreover, real solutions X of $X + A^T X^{-1} A = I$, with A invertible, correspond to real H-invariant Lagrangian subspaces. So all results of the previous sections hold for real solutions as well, with the exception of the result on the precise number of solutions stated after Corollary 6.6. We now give the version of that result for the real case.

PROPOSITION 8.2. *Let A be invertible, and let H be given by* (6.2). *Let $\lambda_1, \dots, \lambda_k$ be the real eigenvalues of H outside the closed unit disc, and $\lambda_{k+1}, \overline{\lambda}_{k+1}, \dots, \lambda_{k+q}, \overline{\lambda}_{k+q}$ the nonreal eigenvalues of H outside the closed unit disc. Assume $\dim \operatorname{Ker}(H - \lambda_i) = 1$ for $i = 1, \dots, k+q$. Denote by n_i the algebraic multiplicity of λ_i. Then there are exactly $\prod_{j=1}^{k+q}(n_j + 1)$ real symmetric positive definite solutions of the equation $X + A^T X^{-1} A = I$.*

Proof. The number of real solutions is equal to the number of real H-invariant subspace N such that $N \subset X_+(H)$. Such a subspace can be decomposed uniquely as $N = N_1 \dotplus \cdots \dotplus N_k \dotplus N_{k+1} \dotplus \cdots \dotplus N_{k+q}$, where $N_i \subset \operatorname{Ker}(H - \lambda_i)^{n_i}$, $i = 1, \ldots, k$, and $N_i \subset \operatorname{Ker}(H - \lambda_i)^{n_i} \dotplus \operatorname{Ker}(H - \overline{\lambda}_i)^{n_i}$, $i = k+1, \ldots, q$. In case λ_i is real, there are exactly $n_i + 1$ real H-invariant subspaces $N_i \subset \operatorname{Ker}(H - \lambda_i)^{n_i}$. In case λ_i is not real, there are $n_i + 1$ real H-invariant subspaces $N_i \subset \operatorname{Ker}(H - \lambda_i)^{n_i} \dotplus \operatorname{Ker}(H - \overline{\lambda}_i)^{n_i}$. This proves the theorem. ■

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