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A dynamic model of the firm with uncertain earnings and adjustment costs

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Abstract: In this paper a stochastic dynamic model of the firm developed by Bensoussan and Lesourne is extended to allow for adjustment costs. The optimal solution is derived for different scenarios dependent on the shapes of the expected earnings function and the adjustment cost function, and on the different parameters of the model. It turns out that, besides pure investment, dividend and saving policies, mixed policies can also be optimal for the firm. The latter do not occur in the solution of the Bensoussan and Lesourne model, and, therefore, the solutions derived in this paper come closer to reality.

Keywords: Investment; Finance; Dynamic programming

1. Introduction and model formulation

This paper deals with the influence of uncertain earnings and adjustment costs on the optimal investment, dividend and saving policies of a firm. The basis of the research carried out in this paper lies in two different areas. On the one hand we have the deterministic adjustment cost literature where the influence of adjustment costs on dynamic investment behavior is studied (see, e.g. Gould, 1968; Kort, 1988a, 1990a). Adjustment costs arise due to investment expenditures of the firm. The assumption is that capital inputs are adjustable (quasi-fixed) at a positive cost. This cost can be caused by processes within the firm (internal adjustment costs, e.g. a decrease of productivity due to reorganization of the production line upon the installation of new machinery) or be due to increasing prices of new capital goods because of monopsony in markets for such goods (external adjustment costs).

On the other hand, we refer to Bensoussan and Lesourne (1980) for the investment/dividend/saving decision studied within a stochastic dynamic model for a self-financing firm (see also Kort, 1988b, 1989). In Bensoussan and Lesourne (1981) this model was further extended to allow for borrowing, but, unfortunately, they had to rely on numerical results in stead of analytical derivations to characterize the solution. Another related model can be found in Kort (1990b) where dividend is an argument of a concave utility function. Tapiero used the Bensoussan–Lesourne framework to establish optimal policies of the insurance firm (see Tapiero, 1984, 1985).

In this paper we extend the model of Bensoussan and Lesourne (1980) by incorporating adjustment costs. The stochastic dynamic optimization model which results can be defined as follows:

$$\max_{I(t),D(t)} \quad E_0\left(\int_0^T D(t)\exp(-it)\,\mathrm{d}t\right) \tag{1}$$

$$\mathrm{d}K(t) = I(t)\mathrm{d}t, \quad K(0) = K_0, \tag{2}$$

$$\mathrm{d}M(t) = (S(K(t)) - I(t) - A(I(t)) - D(t)) \times \mathrm{d}t + \sigma S(K(t))\,\mathrm{d}B(t), \quad M(0) = M_0, \tag{3}$$

$$D(t) \geq 0, \tag{4}$$

$$I(t) \geq 0, \tag{5}$$

$$S(K(t)) - I(t) - A(I(t)) - D(t) \geq 0, \tag{6}$$

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1. Introduction and model description

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in which:

t = Time.

B = $B(t)$ = A standard Wiener process with independent increments $\mathrm{d}B(t)$, which are normally distributed with mean zero and variance $\mathrm{d}t$.

D = $D(t)$ = Dividend rate at time t.

I = $I(t)$ = Investment rate at time t.

K = $K(t)$ = Stock of capital goods at time t.

M = $M(t)$ = Cash balance at time t.

$A(I(t))$ = Rate of adjustment costs, $A(0)=0$, $A'(I)>0$, $A''(I)>0$.

$S(K(t))$ = Usual deterministic earnings function, $S(0)=0$, $S'(K)>0$, $S''(K)<0$, $S'(0)>i(1+A'(0))$.

i = Shareholders' time preference rate ($i>0$ and constant).

T = Horizon date, $T=\inf\{t \mid M(t)\le 0\}$.

σ = A constant.

The expressions (1), (2), (4) and (5) are the same as in the original model of Bensoussan and Lesourne (1980). The firm behaves as if it maximizes the shareholders' value of the firm which can be expressed as the mathematical expectation of the discounted dividend stream over the planning period. As soon as the cash balance becomes negative the firm is bankrupt. Here, it is important to remark that capital goods cannot be sold to increase cash, because investments are assumed to be irreversible (see Pindyck, 1988): the firm cannot disinvest, so the expenditures are sunk costs. Irreversibility usually arises because capital is industry- or firm-specific, that is, it cannot be used in a different industry or by a different firm. Capital stock is of the non-depreciating type and can be increased by investment.

Earnings in $\mathrm{d}t$ are

$$R(K(t))\,\mathrm{d}t = S(K(t))\,\mathrm{d}t + \sigma S(K(t))\,\mathrm{d}B(t). \tag{7}$$

Hence earnings consist of a deterministic ($S(K)\,\mathrm{d}t$) and a stochastic part ($\sigma S(K)\,\mathrm{d}B$). These earnings can be used for investment, which also generates adjustment costs, for dividend payments and for increasing the cash balance. The value per unit of capital goods is fixed at one unit of money. Taking these into account we obtained (3).

As in Bensoussan and Lesourne (1980) it is also assumed here that the firm cannot spend more on investment and dividend than the expected earnings. This is achieved by (6) and the difference with the comparable constraint in Bensoussan and Lesourne (1980) is the occurrence of adjustment costs.

2. The optimal policies

To solve the model we use dynamic programming. To do so we need a value function which is defined by

$$V(M(t), K(t)) = \max_{\substack{I,D\ge 0\\ I+A(I)+D\le S(K)}} E_t\left(\int_t^T D(s)\exp(-i(s-t))\,\mathrm{d}s\right), \tag{8}$$

where V is the expected discounted dividend stream during a time interval that begins at an arbitrary instant $t\in[0,T]$ and ends at the horizon date T. V can be interpreted as the value of the firm at time t. Because the horizon date T depends completely on the value of M, we can conclude that V depends only on M and K, and not explicitly on t.

Throughout the rest of the paper we assume that the partial derivatives V_M, V_K, V_{MM}, V_{KK} and V_{MK} exist. The Hamilton–Jacobi–Bellman equation is (see also Bensoussan and Lesourne, 1980, pp. 244–245):

$$iV = \max_{\substack{I,D\ge 0\\ I+A(I)+D\le S(K)}} \{D + V_M(S(K) - I - A(I) - D) + V_K I\} + \tfrac{1}{2}\sigma^2 S^2(K) V_{MM} \tag{9}$$

while at the boundary $M=0$,

$$V(0, K) = 0. \tag{10}$$

Application of Bellman's principle for the controls optimization yields

$$\max_{I,D}\{D + V_M(S(K) - I - A(I) - D) + V_K I\} \tag{11}$$

$$\text{s.t.}\quad I\ge 0,\ D\ge 0,\ I + A(I) + D \le S(K), \tag{12}$$

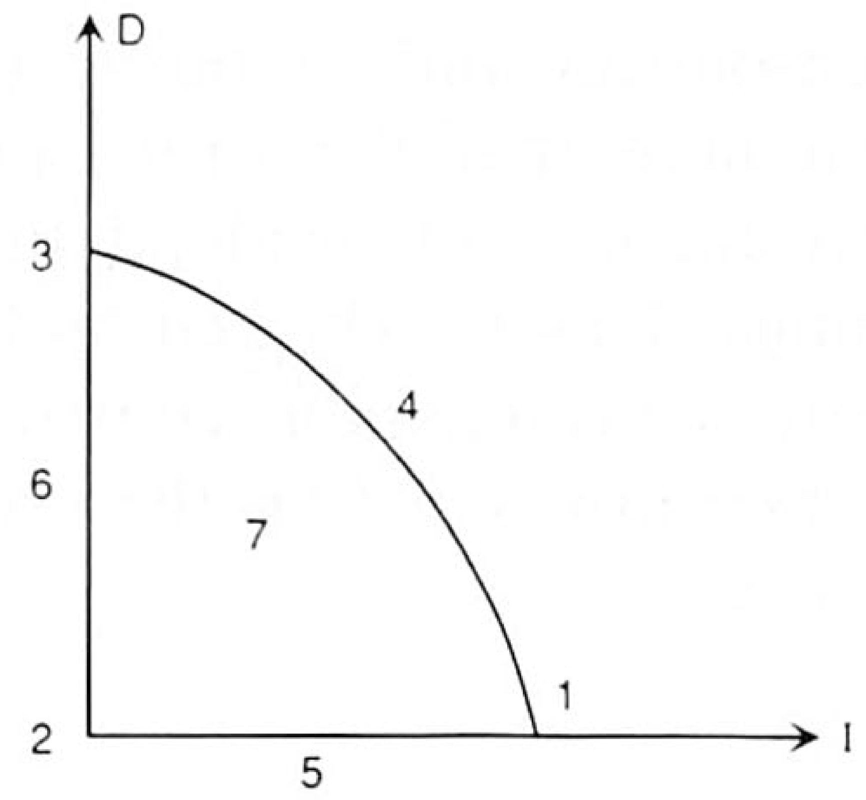

Figure 1. The set of feasible pairs (I, D)

while in Figure 1 we illustrate the set of feasible pairs (I, D). The optimal solution can only lie on the corners, on the edges or in the interior, i.e. there are seven possible cases to investigate. In Appendix A the Kuhn–Tucker conditions are used to transform these cases into the optimal policies of the firm. It turns out that cases 6 and 7 do not lead to well defined policies. The conditions under which these cases occur imply that it is also optimal to carry out two of the other policies. This derivation of the optimal policies has strong similarities with step 1 of the so-called two-step procedure, which can be used to solve a certain class of deterministic optimal control models (see e.g. Hartl, 1988).

In Appendix A we show that five candidate policies have to be considered for optimality. It turns out that the optimal policy depends on the relationship between:

V_M : Marginal value of the firm due to one extra dollar kept in cash.

$V_K/(1+A'(I))$: Marginal value of the firm per dollar invested discounted by a factor which includes the adjustment costs (to see this divide both nominator and denominator by I).

1 : Profitability of an additional dollar used to increase dividend.

Before presenting the optimal policies, we first define the total cost function $C(I)$:

$$C(I) = I + A(I). \tag{13}$$

From (4), (6) and (13) it can be seen that the firm invests maximally if

$$I = C^{-1}(S(K)). \tag{14}$$

The five optimal policies can then be summarized by the following:

Investment policy:

$$\mathrm{d}K = C^{-1}(S(K))\ \mathrm{d}t, \qquad dM = \sigma S(K)\ \mathrm{d}B,$$

$$D = 0,$$

optimal if

$$V_K/\{1 + A'(C^{-1}(S(K)))\} \geq \max(1, V_M). \tag{15}$$

(15) implies that for this policy it is marginally better:

- to invest maximally than to pay out dividend;
- to invest maximally than to increase cash.

Cash policy:

$$\mathrm{d}K = 0, \qquad \mathrm{d}M = S(K)\ \mathrm{d}t + \sigma S(K)\ \mathrm{d}B,$$

$$D = 0,$$

optimal if

$$V_M \geq \max(1, V_K/\{1 + A'(0)\}). \tag{16}$$

Thus for this policy it is marginally better:

- to increase cash than to pay out dividend;
- to increase cash than to invest.

Dividend policy:

$$\mathrm{d}K = 0, \qquad \mathrm{d}M = \sigma S(K)\ \mathrm{d}B, \qquad D = S(K),$$

optimal if

$$1 \geq \max(V_M, V_K/\{1 + A'(0)\}). \tag{17}$$

Due to (17) we can conclude that for this policy it is marginally better:

- to pay out dividend than to increase cash;
- to pay out dividend than to invest.

Investment / dividend policy:

$$\mathrm{d}K = I\ \mathrm{d}t, \qquad \mathrm{d}M = \sigma S(K)\ \mathrm{d}B,$$

$$D = S(K) - I - A(I),$$

optimal if

$$V_K/\{1 + A'(I)\} = 1 \geq V_M. \tag{18}$$

Due to (18) and the strict convexity of $A(I)$ it is marginally better:

- to use part of the expected earnings to invest and the rest for paying out dividend, than to increase cash;

– to use part of the expected earnings to invest and the rest for paying out dividend, than to use all expected earnings for investment;

– to use part of the expected earnings to invest and the rest for paying out dividend, than to use all expected earnings for paying out dividend.

Investment / cash policy:

$$\mathrm{d}K = I\ \mathrm{d}t,$$

$$\mathrm{d}M = (S(K) - I - A(I))\ \mathrm{d}t + \sigma S(K)\ \mathrm{d}B,$$

$$D = 0,$$

optimal if

$$V_K/(1 + A'(I)) = V_M \geq 1. \tag{19}$$

Because of (19) and the strict convexity of $A(I)$ we note that it is marginally better:

– to use part of the expected earnings to invest and the rest to increase cash, than to pay out dividend;

– to use part of the expected earnings to invest and the rest to increase cash, than to use all expected earnings for investment;

– to use part of the expected earnings to invest and the rest to increase cash, than to use all expected earnings for increasing cash.

3. The optimal solutions for different scenarios

In the previous section we have established the potential five optimal policies. As already stated in Section 2, the value of the firm V only depends on M and K and not on t. Hence, we can divide the M–K plane in five regions, each of them belonging to one of the five candidates for an optimal policy, which are collections of those values of M and K for which the corresponding policy is optimal. In this way we get the following regions: investment region, cash region, dividend region, investment/dividend region, investment/cash region. In what follows these regions will be denoted by I-region, M-region, D-region, I/D-region and I/M-region, respectively.

Due to the conditions (15)–(19) and the assumption that the partial derivatives V_{MK}, V_{MM} and V_{KK} exist, we can establish the following general features for the positions of the regions in the M–K plane:

(F1): The boundary between the M-region and the I-region does not exist for K positive. This is because in the M-region it holds that

$$V_M \geq V_K/\{1 + A'(0)\}$$

and in the I-region we have

$$V_M \leq V_K/\{1 + A'(C^{-1}(S(K)))\}.$$

Due to the strict convexity of $A(I)$ we can conclude that for K positive we get

$$A'(C^{-1}(S(K))) > A'(0).$$

Hence, in the M–K plane the M-region and the I-region can only meet for K equal to zero. Therefore for K positive there will be always an I/M-region which is situated between the M-region and the I-region.

(F2): Due to the same reasoning as in (F1) we can argue that for K positive the I/D-region must always be situated between the I-region and the D-region.

(F3): For K positive the boundaries between the M-region and the I/M-region –

$$V_M = V_I/\{1 + A'(0)\},$$

which holds on this boundary because both (16) and (19) must be satisfied – and between the I-region and the I/D-region,

$$V_K/\{1 + A'(C^{-1}(S(K)))\} = 1,$$

do not intersect because it can never be optimal to pay out dividend in a region that hits this intersection point (this because the two equalities imply that $V_M > 1$).

(F4): Following the same reasoning as in (F3) for K positive, we can argue that the boundaries between the I/M-region and the I-region,

$$V_M = V_K/\{1 + A'(C^{-1}(S(K)))\},$$

and between the D-region and the I/D-region,

$$V_K/\{1 + A'(0)\} = 1,$$

do not intersect.

Notice that (F1) implies that for K positive the boundary between the M-region and the I/M-region and the boundary between the I/M-region and the I-region do not intersect. The implication of (F2) is that the boundaries between the I-region and the I/D-region and between the I/D-region and the D-region do not intersect.

Except that the contents and the proof of the Propositions 3 and 4 are slightly adjusted for the presence of adjustment costs, the following propositions and their proofs also hold for the original Bensoussan–Lesourne model without adjustment costs. Therefore, we only present the propositions and for their proofs refer to Kort (1989)

Proposition 1. If $1/i - \sigma/\sqrt{2i} > 0$, *only the M-region includes the K-axis.*

Proposition 2. *The boundary between the M-region and the D-region is given by* $M = \rho S(K)$, *in which* ρ *is a constant which satisfies*

$$\exp((r_1 - r_2)\rho) = \frac{1 - r_2(1/i - \sigma/\sqrt{2i})}{1 - r_1(1/i - \sigma/\sqrt{2i})}, \tag{20}$$

where

$$r_1 = \left[-1 + \sqrt{1 + 2\sigma^2 i}\right]/\sigma^2, \tag{21}$$

$$r_2 = \left[-1 - \sqrt{1 + 2\sigma^2 i}\right]/\sigma^2. \tag{22}$$

Proposition 3. *The boundary between the I/D-region and the D-region increases in the M–K-plane and lies below a horizontal asymptote which is situated on the level* K^*, *determined by*

$$S'(K^*) = i(1 + A'(0)).$$

The D-region lies at the left-hand side of this boundary.

At the intersection point $(\overline{M}, \overline{K})$ *of the boundary between the I/D-region and the D-region and the boundary between the M-region and the D-region* ($M + \rho S(K)$, see Proposition 2) *it must hold that*

$$S'(\overline{K})\left(\frac{1}{i} - \frac{\sigma}{\sqrt{2i}} - \rho\right) = 1 + A'(0).$$

Proposition 4. *The boundary between the M-region and the M/I-region starts at the origin and ends at the intersection point* $(\overline{M}, \overline{K})$ *of the boundaries between the M-region and the D-region and between the I/D-region and the D-region.*

In addition we developed the following propositions, from which the proofs are presented in Appendix B:

Proposition 5. *The D-region and the I/M-region meet at one and only one point, namely at* $(\overline{M}, \overline{K})$, *which is the intersection point of the boundary between the I/D- and the D-region and the boundary between the M- and the D-region* (see Proposition 3). *The same holds for the M-region and the I/D-region.*

Proposition 6. *If the I-region exists for* $M \to \infty$, *then the boundary between this region and the I/D-region is situated at a level* $K = \hat{K}$ *for* $M \to \infty$, *where* $\hat{K}$ *is given by*

$$\frac{S'(\hat{K})}{i\{1 + A'(S(\hat{K}))\}}\left[1 + A'(C^{-1}(S(\hat{K}))) + C^{-1}(S(\hat{K}))A''(C^{-1}(S(\hat{K})))\right] = 1 + A'(C^{-1}(S(\hat{K}))) \tag{23}$$

From (23) we derive the following sufficient condition for $\hat{K}$ positive:

$$S'(0) \geq i\{1 + A'(S(\hat{K}))\}. \tag{24}$$

Notice that (24) is not a sufficient condition for an investment policy being optimal when M is sufficiently large and K is below $\hat{K}$. However due to economic reasons it is clear that this is the preferable policy, because the expected marginal earnings are high while there is no immediate risk for bankruptcy. Therefore, in the sequel we take the view that the firm invests maximally when K is small and M sufficiently high.

From Propositions 3, 4 and 5 it is clear that the intersection point $(\overline{M}, \overline{K})$ of the boundary between the M-region and the D-region and the boundary between the I/D-region and the D-region plays a crucial role in the optimal solution. We first pay attention to the solution for the scenarios where this intersection point exists. From Proposition 3 we obtain that existence is assured if the following inequality holds:

$$S'(0)\left(\frac{1}{i} - \frac{\sigma}{\sqrt{2i}} - \rho\right) > 1 + A'(0). \tag{25}$$

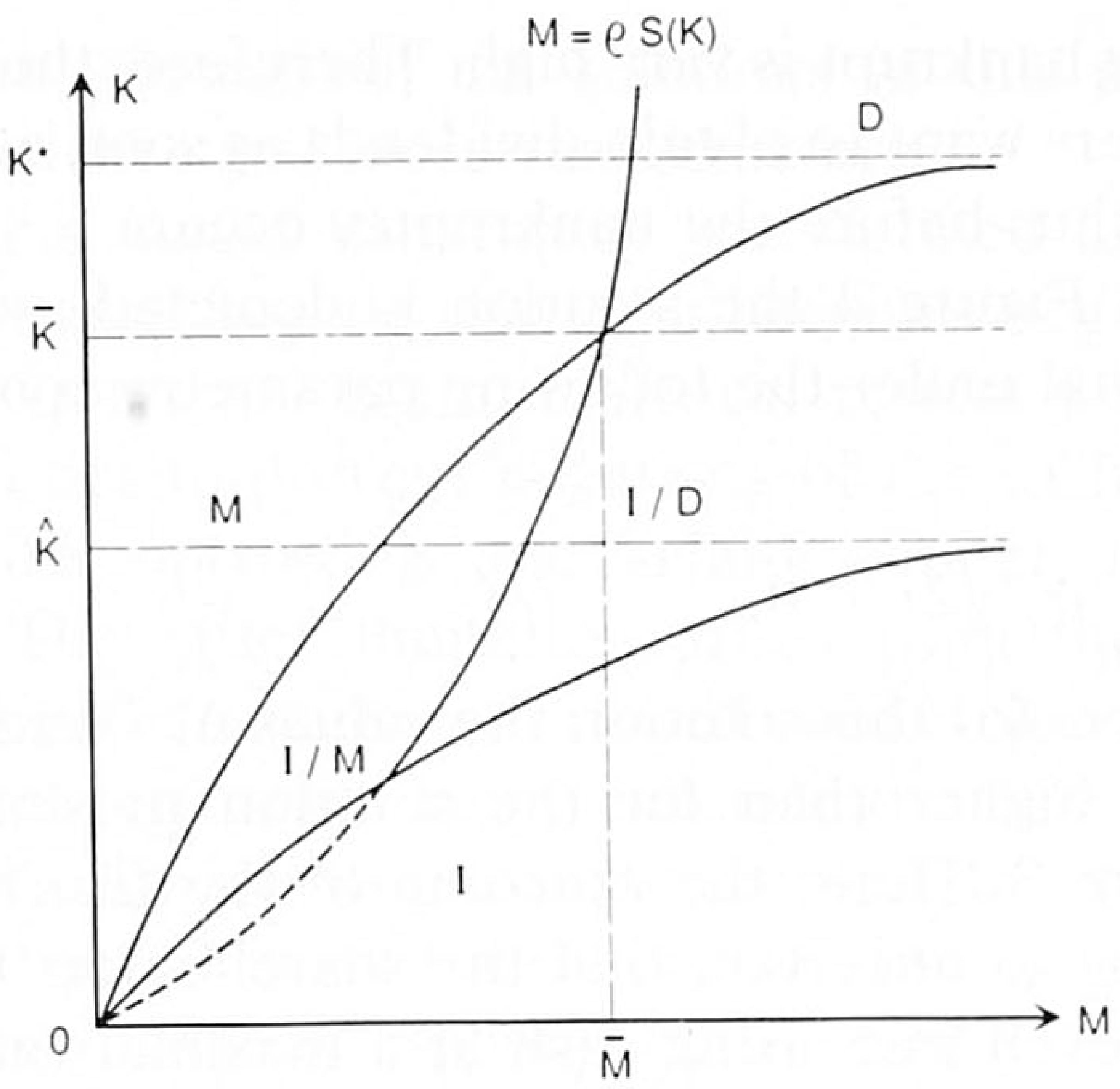

Figure 2. The optimal solution for the scenario where it holds that $S'(0)(1/i-\sigma/\sqrt{2i}-\rho)>1+A'(0)$

The solution for this scenario is depicted in Figure 2. Notice that (25) implies that $1/i-\sigma/\sqrt{2i}>0$, so according to Proposition 1 the M-region includes the K-axis. The reader can check for him/herself that the features (F1)–(F4) and the remaining propositions are also satisfied. Concerning this solution it has to be remarked that we cannot prove anything about the shape of the boundary between the I/M-region and the I/D-region, except that it ends at $(\bar{M}, \bar{K})$ (Cf. Proposition 5). In Figure 2 we assume that it lies on the curve $M=\rho S(K)$ which seems reasonable because, like on the boundary between the M-region and the D-region, on this boundary it must also hold that $V_M=1$ (cf. (18) and (19)).

The solution depicted in Figure 2 has some similarities with the corresponding solution of the original Bensoussan–Lesourne model without adjustment costs (see Kort, 1988b, Figure 4.1, Panel D), namely that the firm saves money if the amount of equipment is high enough while the cash situation is poor, that the firm invests if the amount of equipment is low while there is plenty of cash to limit the risk of bankruptcy and that the firm pays out dividends if M and K are such that the expected marginal earnings are too small to justify additional growth and the amount of cash available high enough to guarantee a sufficiently safe situation. The difference is that the present solution contains two more regions in which it is optimal to carry out a mixed investment/saving policy (I/M-region) and a mixed investment/dividend policy (I/D-region), respectively. Concerning the I/M-region, on its boundary with the M-region it holds that $I=0$, and on the boundary with the I-region the firm invests at its maximum implying that $I+A(I)=S(K)$. In between investment is such that $V_M=V_K/(1+A'(I))$, so the increase of the value of the firm due to one extra dollar in cash is equal to the increase of the value of the firm due to one extra dollar of capital goods, corrected for the fact that additional adjustment costs must be paid in order to increase the stock of capital goods.

Concerning the I/D-region, on its boundary with the I-region it holds that

$$I+A(I)=S(K) \quad \text{and} \quad D=0$$

and on the boundary with the D-region

$$I=0 \quad \text{and} \quad D=S(K).$$

In the rest of the region I and D are such that the marginal profitability of an additional dollar used to increase dividend ($=1$) is equal to $V_K/(1+A'(I))$.

In the evolution of the firm over time M and K are continuous (cf. (2)–(6)). Therefore, the existence of the two intermediate regions where mixed policies are optimal, imply that the investment level develops gradually over time. Such an investment behavior is also concluded from the deterministic adjustment cost literature (cf. Kort, 1990a), in which it is shown that the investment rate is a continuous function of time.

In Figure 2 it can also be seen that the firm never invests when K is greater than K^*. The reason is that, due to the concavity of $S(K)$, the expected marginal earnings ($S'(K)$) then fall below the minimal return the shareholders demand ($=i(1+A'(0))$). This feature also plays an important role in deterministic adjustment most models.

Next, we study the solution under the following scenario:

$$S'(0)\left(\frac{1}{i}-\frac{\sigma}{\sqrt{2i}}-\rho\right)<1+A'(0) \quad \text{and}$$

$$\frac{1}{i}-\frac{\sigma}{\sqrt{2i}}>0. \tag{26}$$

From Proposition 1 we obtain that the M-region is still the only region that includes the K-axis, but due to Proposition 3 we can conclude that now the intersection point $(\bar{K}, \bar{M})$ does not exist.

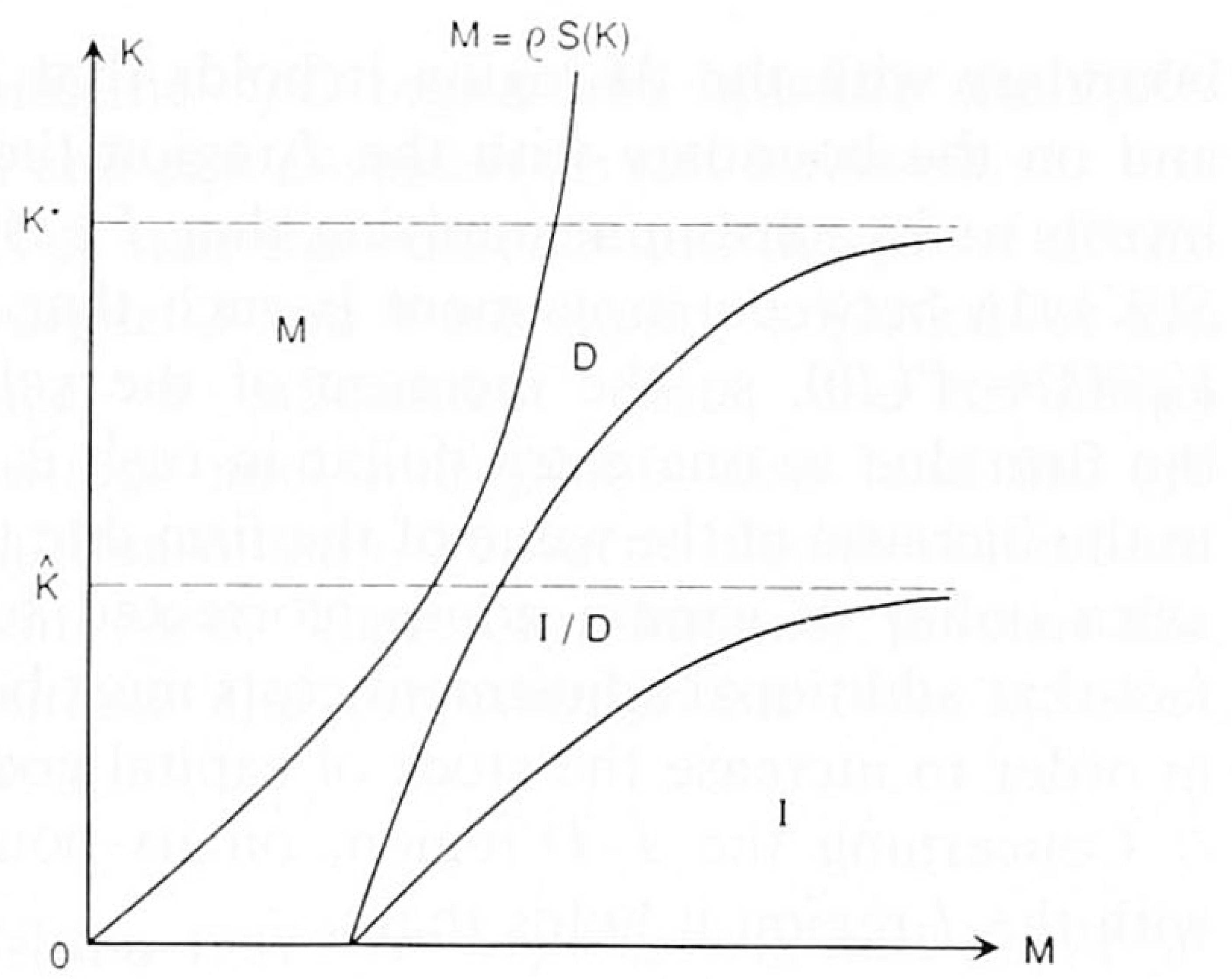

Figure 3. The optimal solution for the scenario where it holds that $S'(0)(1/i - \sigma/\sqrt{2i} - \rho) < 1 + A'(0)$ and $1/i - \sigma/\sqrt{2i} > 0$

Hence, the general features (F1)–(F4) and Propositions (1)–(6) now lead to the solution depicted in Figure 3.

The difference between this solution and the solution of Figure 2 is that here it is optimal to pay out dividend for some low levels of capital stock and cash balance. The reason could be that in this solution the firm has to deal with larger values of time preference rate i and risk parameter σ (cf. (25), (26) and the fact that tedious calculations show that both the signs of the derivatives from ρ to i and σ are not clearly positive or negative; see also Kort, 1988b, Appendix 2). A high time preference rate implies that the shareholders of the firm can obtain a high return through investing there money outside the firm and therefore they like to receive lots of dividends. A high σ means that the outcome of the firm's earnings is very uncertain (cf. (7)). Hence, for low levels of M the chance of going bankrupt is very high. Therefore, the shareholders want to obtain dividends as soon as possible, thus before the bankruptcy occurs.

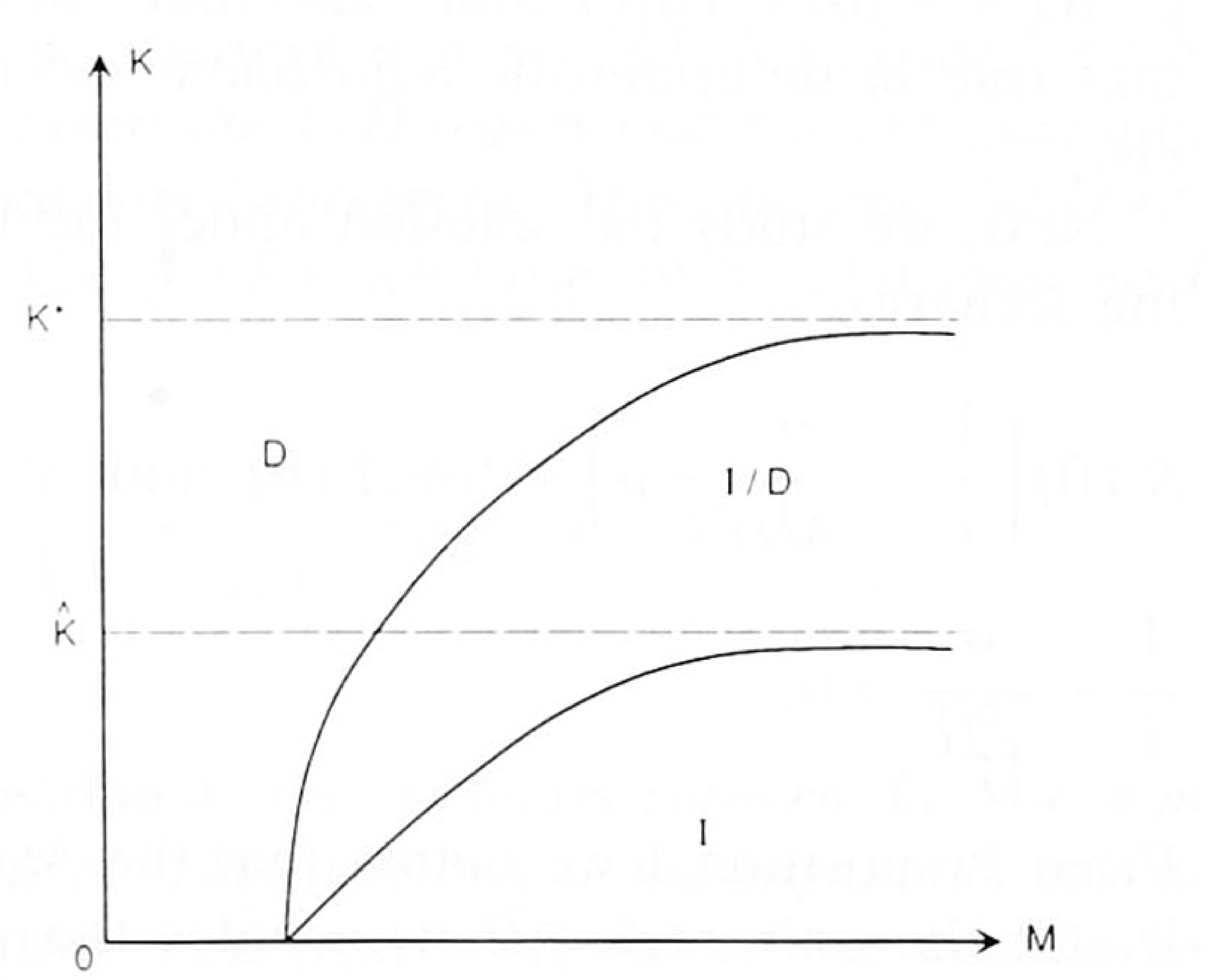

Figure 4. The optimal solution for the scenario where it holds that $1/i - \sigma/\sqrt{2i} < 0$

In Figure 4 the solution is depicted, which is optimal under the following parameter configuration:

$$1/i - \sigma/\sqrt{2i} < 0. \tag{27}$$

Hence, for this solution the values of i and σ are even higher than for the solution presented in Figure 3. Here, the outcome of the firm's earnings is so uncertain that the shareholders believe that even increasing cash at a maximal rate cannot prevent bankruptcy when M is small. Instead, the shareholders want to receive dividends that they can use for investment outside the firm in order to generate a return which equals the high time preference rate i. The fact that only the D-region includes the K-axis follows from the proof of Proposition 1 (see Kort, 1989, p. 161).

4. Conclusions

In this paper the stochastic model of Bensoussan and Lesourne (1980) was extended by incorporating adjustment costs. The influence of both adjustment costs and uncertainty on the behavior of the firm was already modelled by Pindyck (1982), but our approach differs in that we also consider the possibility of bankruptcy. The problem was modelled as a dynamic model of the firm. Models belonging to this category attempt to describe a firm, in broad terms and over its entire lifetime, with respect to basic characteristics such as its objective, production technology and financial structure. Taking primarily a normative point of view, the theory attempts to derive optimal time paths for key decision variables of the firm such as investment in productive capital, employment and dividend policy. One of the main purposes of designing and analyzing dynamic models of the firm is to develop a theoretical background for managerial policies. An extensive survey of the area of 'The dynamics of the firm' is provided by Jørgensen (1991).

Besides pure investment, dividend and saving policies, the results obtained in this paper show that, contrary to Bensoussan and Lesourne (1980), a mixed investment/dividend policy and a mixed investment/saving policy can also be optimal for the firm. Therefore, the solutions in this paper have a richer and more real structure.

In this model bankruptcy was defined as the first moment in time that cash becomes negative. In our quest to obtain analytical results we left possibilities to prevent negativity of the cash balance like borrowing and selling capital goods aside. The latter limitation arises from the assumption that markets for used capital goods do not exist, i.e. investment expenditures are irreversible. An immediate extension of the paper would be to assess the effects of incorporating the possibility of selling used capital goods, thereby provide for capital reversibility. This extension will be carried out in forthcoming research by using this study as a departure point.

Appendix A. The optimal policies

In order to solve problem (11)–(12) we set up the Lagrangian

$$\begin{aligned} L = {} & D(1-V_M) + I(V_K - V_M) \\ & - A(I)V_M + S(I)V_M + \lambda_1 I + \lambda_2 D \\ & + \lambda_3(S(K) - I - A(I) - D) \end{aligned} \quad (A.1)$$

The Kuhn–Tucker optimality conditions are

$$L_D = 1 - V_M + \lambda_2 - \lambda_3 = 0, \quad (A.2)$$

$$\begin{aligned} L_I = {} & V_K - V_M(1 + A'(I)) \\ & + \lambda_1 - \lambda_3(1 + A'(I)) = 0, \end{aligned} \quad (A.3)$$

$$\lambda_1 I = 0, \quad \lambda_1 \geq 0, \quad (A.4)$$

$$\lambda_2 D = 0, \quad \lambda_2 \geq 0, \quad (A.5)$$

$$\lambda_3(S(K) - I - A(I) - D) = 0, \quad \lambda_3 \geq 0. \quad (A.6)$$

We now derive the conditions for which cases 1–7 of Figure 1 can occur.

Case 1: $I > 0, \; D = 0, \; I + A(I) = S(K) \Rightarrow$ investment policy. In this case we have $\lambda_1 = 0$ from (A.4) and $I = C^{-1}(S(K))$ from (13). Now, we obtain from (A.3)

$$V_K / \{1 + A'(C^{-1}(S(K)))\} \geq V_M. \quad (A.7)$$

From (A.2) and (A.3) we can derive

$$\lambda_2 = -1 + V_K/\{1 + A'(C^{-1}(S(K)))\} \geq 0. \quad (A.8)$$

(A.7)–(A.8) lead to (15).

Case 2: $I = D = 0, \; S(K) - I - A(I) - D > 0 \Rightarrow$ cash policy. Here $\lambda_3 = 0$ because of (A.6). Hence, (A.2)–(A.3) lead to (16).

Case 3: $I = 0, \; D > 0, \; D = S(K) \Rightarrow$ dividend policy. Here $\lambda_2 = 0$ (cf. (A.5)). Hence, (A.2) leads to

$$V_M \leq 1. \quad (A.9)$$

From (A.2)–(A.3) we can conclude

$$\lambda_1 = 1 + A'(0) - V_K \geq 0. \quad (A.10)$$

(A.9)–(A.10) lead to (17).

Case 4: $I > 0, \; D > 0, \; I + A(I) + D = S(K) \Rightarrow$ investment/dividend policy. Here $\lambda_1 = \lambda_2 = 0$ because of (A.4)–(A.5). Hence (A.2)–(A.3) lead to (18).

Case 5: $I > 0, \; D = 0, \; I + A(I) < S(K) \Rightarrow$ investment/cash policy. Here $\lambda_1 = \lambda_3 = 0$ because of (A.4) and (A.6). Now (A.2)–(A.3) lead to (19).

Case 6: $I = 0, \; D > 0, \; S(K) - D > 0$. Here $\lambda_2 = \lambda_3 = 0$ due to (A.5)–(A.6). Now (A.2)–(A.3) lead to

$$1 = V_M \geq V_K/\{1 + A'(0)\}. \quad (A.11)$$

Here the marginal value of the dividend payout equals the marginal value of saving money. Therefore the firm can adopt either a pure cash policy or a pure dividend policy. This is optimal because (A.11) does not contradict (16) or (17).

Case 7: $I > 0, \; D > 0, \; S(K) - I - A(I) - D > 0$. Here $\lambda_1 = \lambda_2 = \lambda_3 = 0$ due to (A.4)–(A.7). Now (A.2)–(A.3) result in

$$1 = V_M = V_K/\{1 + A'(I)\}. \quad (A.12)$$

Due to the same reasoning as in Case 6 it can be concluded that under (A.12) both an investment/dividend policy as well as an investment/cash policy are optimal.

Appendix B. Proofs of Propositions 5 and 6

Proof of Proposition 5. In the *D*-region it holds that $D = S(K)$ and $I = 0$. If we substitute these values into (9) we obtain

$$iV = S(K) + \tfrac{1}{2}\sigma^2 S^2(K) V_{MM}. \quad (B.1)$$

Solving this differential equation implies

$$V = S(K)/i + c_1(K)\exp\left[M\sqrt{2i}/(\sigma S(K))\right] + c_2(K)\exp\left[-M\sqrt{2i}/(\sigma S(K))\right]. \quad \text{(B.2)}$$

From the previous propositions we know that the D-region exists for finite K and infinite M. Due to (8) we derive that V must always have a finite value, so from (B.2) we can conclude that $c_1(K) = 0$. From (B.2) we also obtain

$$V_M = -c_2(K)\sqrt{2i}\exp\left[-M\sqrt{2i}/(\sigma S(K))\right] /(\sigma S(K)). \quad \text{(B.3)}$$

Due to Proposition 2 we know that the boundary between the M-region and the D-region ($V_M = 1$, cf. (16), (17)) is given by $M = \rho S(K)$. After substitution of $M = \rho S(K)$ into (B.3) and equating V_M to 1 we obtain the following expression for $c_2(K)$:

$$c_2(K) = -\sigma S(K)\exp\left[\rho\sqrt{2i}/\sigma\right]/\sqrt{2i}. \quad \text{(B.4)}$$

Knowing $c_1(K)$ and $c_2(K)$ we now get from (B.2)

$$V_K = S'(K)\left\{\frac{1}{i} - \left(\frac{\sigma}{\sqrt{2i}} + \frac{M}{S(K)}\right) \times \exp\left[\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{2i}}{\sigma}\right]\right\}, \quad \text{(B.5)}$$

$$V_M = \exp\left[\left(\rho - \frac{M}{S(K)}\right)\frac{\sqrt{2i}}{\sigma}\right]. \quad \text{(B.6)}$$

On the boundary between the D-region and the I/M-region it must hold that

$$1 = V_M = V_K/\{1 + A'(0)\}. \quad \text{(B.7)}$$

Equating V_M to 1 gives that for this boundary it must hold that

$$M = \rho S(K). \quad \text{(B.8)}$$

After substitution of (B.8) into (B.5) and equating V_K to $1 + A'(0)$ we get

$$S'(K)\{1/i - \sigma/\sqrt{2i} - \rho\} = 1 + A'(0). \quad \text{(B.9)}$$

Comparing these results with Proposition 3 we conclude that (B.8) and (B.9) are exactly the conditions that fix $(\bar{M}, \bar{K})$.

It is left to the reader to check that manipulating the information for the M-region in the same way as done above for the D-region leads to the conclusion that $(\bar{M}, \bar{K})$ is the only point in common for the M-region and the I/D-region, too. □

Proof of Proposition 6. On the boundary between the I-region and the I/D-region it holds that $I = C^{-1}(S(K))$, $D = 0$ and $V_K = 1 + A'(C^{-1}(S(K)))$. Combining this with (9) leads to

$$iV = \{1 + A'(C^{-1}(S(K)))\}C^{-1}(S(K)) + \tfrac{1}{2}\sigma^2 S^2(K)V_{MM}. \quad \text{(B.10)}$$

This differential equation can be solved:

$$V = \{1 + A'(C^{-1}(S(K)))\}C^{-1}(S(K))/i + c_3(K)\exp\left[\frac{M\sqrt{2i}}{\sigma S(K)}\right] + c_4(K)\exp\left[-\frac{M\sqrt{2i}}{\sigma S(K)}\right]. \quad \text{(B.11)}$$

From (F2) in Section 3 we obtain that if it exists, this boundary is situated below the D–I/D-boundary, so it then exists for infinite M and finite K. Therefore c_3 must be equal to zero, because V must have a finite value. After differentiating with respect to K and equating this to $1 + A'(C^{-1}(S(K)))$ we obtain the following expression for the I–I/D-boundary:

$$\frac{S'(K)}{iC'(S(K))}\{1 + A'(C^{-1}(S(K))) + C^{-1}(S(K))A''(C^{-1}(S(K)))\} + \left\{c_4'(K) + \frac{M\sqrt{2i}\,S'(K)c_4(K)}{\sigma S^2(K)}\right\} \times \exp\left[-\frac{M\sqrt{2i}}{\sigma S(K)}\right] = 1 + A'(C^{-1}(S(K))). \quad \text{(B.12)}$$

Because c_4 and c_4' are finite (this follows from the assumption that V_K exists), taking M to infinity in (B.12) leads to expression (23). □

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