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SHORT COMMUNICATIONS

A DYNAMIC NET PRESENT VALUE RULE IN A FINANCIAL ADJUSTMENT COST MODEL*

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SUMMARY

A dynamic model of a self-financing firm is presented in which a convex adjustment cost function is incorporated. The horizon date is assumed to be infinite. It turns out that the concept of 'net present value of marginal investment' is a useful tool to develop the firm's optimal investment policy. When this net present value is positive, negative or zero, it is optimal for the firm to fix the investment rate at its maximum level, minimum level or equilibrium level respectively.

KEY WORDS Adjustment costs Net present value Constrained optimal control

1. INTRODUCTION

The problem of a firm facing convex adjustment costs has received a lot of attention in the literature.[1–4] One of the most important contributions in this respect is by Gould,[5] who studies a competitive firm that maximizes the present value of all future net cash flows. In this paper we extend Gould's model by allowing the firm to have price-making power in the output market[6] and by incorporating a financial structure which implies, roughly stated, that the firm must earn the money first before it can invest.

We determine the firm's optimal investment policy by using a new concept termed 'the net present value of marginal investment.'[7] From this value it can be derived whether the firm is in equilibrium, and if it is not, how to reach this equilibrium as soon as possible. In standard books of corporate finance[8,9] the net present value criterion is used as a method to evaluate an investment proposal and to compare alternative investment proposals. The net present value of such an investment is defined as the sum of the net cash receipts minus the initial investment outlay (see Reference 9, pp. 33–34). In this paper the net present value approach is extended to a dynamic context. We show that on the equilibrium path the net present value of marginal investment is equal to zero. If the net present value is not zero at the start, the firm needs an adjustment phase in which it invests at its maximum if the net present value is positive and it does not invest at all if the net present value of marginal investment is negative.

* The proofs of the results presented in this paper can be obtained from the author upon request.

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2. MODEL FORMULATION

In this section we present our dynamic model of the firm. Assume that the firm behaves so as to maximize the shareholders' value of the firm. This value consists of the discounted dividend stream over the planning period. The horizon date is assumed to be infinite. Hence

$$\text{maximize} \int_0^\infty D(t)\exp(-it)\,\mathrm{d}t \tag{1}$$

where $D = D(t)$ is the rate of dividend pay-out at time t and i is the shareholders' time preference rate ($i > 0$ and constant).

If depreciation is proportional to the stock of capital goods, we can describe the impact of investment on the amount of capital goods by the commonly used formulation of net investment

$$\dot{K}(t) = I(t) - aK(t), \qquad K(0) = K_0 > 0 \tag{2}$$

where $I = I(t)$ is the rate of gross investment and a is the depreciation rate ($a > 0$ and constant). We assume the value of a capital good to be equal to unity and that borrowing is not allowed. In this way the balance sheet becomes

$$K(t) = X(t) \tag{3}$$

where $X = X(t)$ is the stock of equity at time t. To construct the state equation of the stock of equity, we introduce the following assumptions.

(i) The rate of earnings is a strictly concave differentiable function of capital stock. If we assume a fixed labour-to-capital rate and constant returns to scale, this implies that the firm has price-making power in the output market.

(ii) The rate of adjustment costs is a strictly convex differentiable function of gross investment.

Then the state equation for X is given by

$$\dot{X}(t) = S(K(t)) - aK(t) - A(I(t)) - D(t), \qquad X(0) = X_0 > 0 \tag{4}$$

where $S = S(K)$ is the rate of earnings ($S(K) > 0$, $S'(K) > a$, $S''(K) < 0$) and $A = A(I)$ is the rate of adjustment costs ($A(I) \geqslant 0$, $A'(I) > 0$, $A''(I) > 0$, $A(0) = 0$). Dividend is restricted by a rational lower bound and investment is assumed to be irreversible:

$$D(t) \geqslant 0 \tag{5}$$

$$I(t) \geqslant 0 \tag{6}$$

Using (2)–(4) we get

$$D(t) = S(K(t)) - I(t) - A(I(t)) \tag{7}$$

By using (7) we can state the problem as follows:

$$\underset{I}{\text{maximize}} \int_0^\infty (S(K) - I - A(I))\exp(-it)\,\mathrm{d}t \tag{8}$$

subject to

$$\dot{K} = I - aK, \qquad K(0) = K_0 > 0 \tag{9}$$

$$S(K) - I - A(I) \geqslant 0 \tag{10}$$

$$I \geqslant 0 \tag{11}$$

Table I. Features of feasible paths

Path	λ_1	λ_2	I	D	Policy
1	+	0	Max	0	Maximum growth
2	0	0	>0	>0	Equilibrium policy
3	0	+	0	Max	Contraction

The following assumption is required to ensure that capital stock increases when investment is at its upper bound (see (10)):

$$S(K) - aK - A(aK) > 0 \tag{12}$$

By using standard control theory[10] we define the (current value) Lagrangian

$$L = (S(K) - I - A(I))(1 + \lambda_1) + \psi(I - aK) + \lambda_2 I \tag{13}$$

The necessary conditions are

$$\psi = (1 + A'(I))(1 + \lambda_1) - \lambda_2 \tag{14}$$

$$\dot{\psi} = i\psi - \partial L/\partial K = (i + a)\psi - S'(K)(1 + \lambda_1) \tag{15}$$

$$\lambda_1 \geqslant 0, \qquad \lambda_1(S(K) - I - A(I)) = 0 \tag{16}$$

$$\lambda_2 \geqslant 0, \qquad \lambda_2 I = 0 \tag{17}$$

Because of (9) and (11), K is positive. Therefore the constraint qualification (see Reference 10, p. 161) is satisfied, which means that conditions (14)–(17) are applicable.

In the direct adjoining approach that we have chosen, the adjoint function is continuous everywhere since the Hamiltonian is strictly concave in I. If, furthermore, the transversality condition

$$\lim_{t \to \infty} \exp(-it)\psi(t)(\tilde{K}(t) - K(t)) \geqslant 0 \tag{18}$$

holds for every feasible solution $\tilde{K}$, then (14)–(17) are also sufficient for optimality[10] since the maximized Hamiltonian is strictly concave in K.

To facilitate the analysis later on, we distinguish between different paths. Since the Lagrange multipliers λ_i $(i = 1, 2)$ can be positive or zero, each path is characterized by a combination of positive λs. Using the fact that K is positive we can easily derive that λ_1 and λ_2 cannot be positive at the same time. The remaining paths are feasible and are presented in Table I.

3. THE OPTIMAL SOLUTION

We first study the case where $\lambda_1 = \lambda_2 = 0$ (path 2). The steady state follows from (2), (14) and (15) and can be expressed as

$$\psi^* = 1 + A'(aK^*) \tag{19}$$

$$S'(K^*) = (i + a)(1 + A'(aK^*)) \tag{20}$$

The determinant of the Jacobian of the system (2), (14), (15) is negative so that the dynamics corresponds to a saddle point.[10]

From (14), (15), (19) and (20) we can show that on path 2

$$\int_t^{\infty} S'(K(s))\exp[-(i+a)(s-t)]\ ds \quad - \quad (1 + A'(I)) = 0 \quad \text{(path 2)} \tag{21}$$

The first term is equal to the marginal earnings of investment, which consist of the discounted value of the additional earnings due to the new equipment. Capital decays and therefore at each time $s > t$ it contributes only a fraction of what a whole unit of capital would add (Reference 11, p. 129). The second term represents the initial outlay, including adjustment costs, that is required to increase the stock of capital goods by one dollar at time 't'.

Hence equation (21) is equal to the benefit of an investment of one dollar and therefore can be interpreted as the net present value of marginal investment. From equation (21) we can show that the net present value of marginal investment is equal to zero on path 2. Therefore marginal earnings equal marginal expenses and the firm is in equilibrium.

Next, suppose that at the initial time capital stock is so low (this implies a high level of marginal earnings owing to the strict concavity of the earnings function) that equation (21) dictates an investment level to the firm which exceeds its upper bound described by (10). Then the optimal policy is to approach this level as rapidly as possible, which implies that investment is situated on this upper bound (path 1). From the optimality conditions we can obtain the following relation which holds on path 1:

$$(1 + A'(I))\lambda_1 = \int_t^{\infty} S'(K(s))\exp[-(i+a)(s-t)]\ ds$$
$$+ \int_t^{\infty} S'(K(s))\exp[-a(s-t)]\lambda_1(s)\exp[-i(s-t)]\ ds \quad - \quad (1 + A'(I)) \quad \text{(path 1)} \tag{22}$$

Recall that λ_1 is the Lagrange multiplier of the upper bound of investment plus adjustment costs (see (16)). Therefore λ_1 is equal to the extra value of the Hamiltonian gained if the upper bound of investment plus adjustment costs ($S(K)$) is increased by one. In this way the left-hand side of (22) represents the gain due to an increase of this upper bound with $1 + A'(I)$. Notice that an extra expenditure on investment plus adjustments costs of $1 + A'(I)$ implies a one dollar increase of the stock of capital goods.

The first and third terms on the right-hand side of (22) can also be found in equation (21). The second term represents the indirect marginal earnings of investment. An extra dollar of investment at the instant 't' implies an increase in the stock of capital goods of $\exp[-a(s-t)]$ at time $s > t$, generating an extra return of $S'(K(s))\exp[-a(s-t)]$. The upper bound of investment plus adjustment costs will be increased by this value. In this way the Hamiltonian discounted to 't' is increased by $S'(K(s))\exp[-a(s-t)]\lambda_1(s)\exp[-i(s-t)]$ because $\lambda_1(s)\exp[-i(s-t)]$ is the shadow price of this upper bound discounted to 't'.

We conclude that the right-hand side of (22) is equal to the net present value of marginal investment on path 1. Since λ_1 is greater than zero on path 1 (see Table I), we see that this net present value is greater than zero. Thus marginal earnings are greater than marginal expenses of investment and therefore it is optimal for the firm to invest at its maximum rate.

Because the firm grows at its maximum on path 1, $A'(I)$ increases (because I increases) and $S'(K)$ decreases (because K increases). Therefore the net present value will be equal to zero at some instant. As soon as this happens, path 1 will pass into path 2 and (22) turns into (21). Then the firm is in equilibrium. This solution is depicted in Figure 1.

Finally, we turn to the case where capital stock is so high that the net present value of marginal investment is negative at the start of the planning period. Notice that, after assuming that S' is sufficiently high compared to $A'(0)$, this case cannot occur when S is a linear

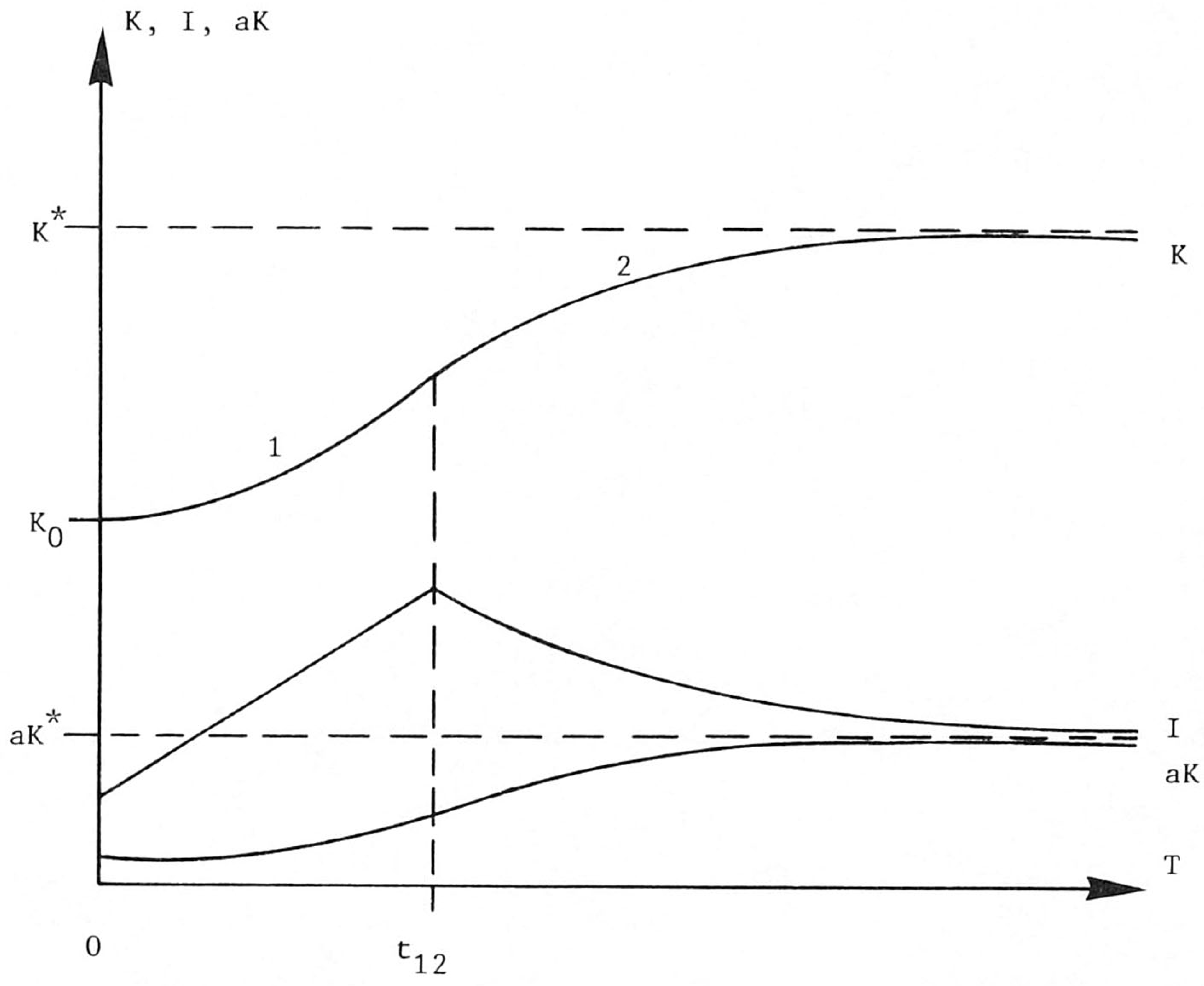

Figure 1. Optimal solution in the case of a positive net present value at the initial time

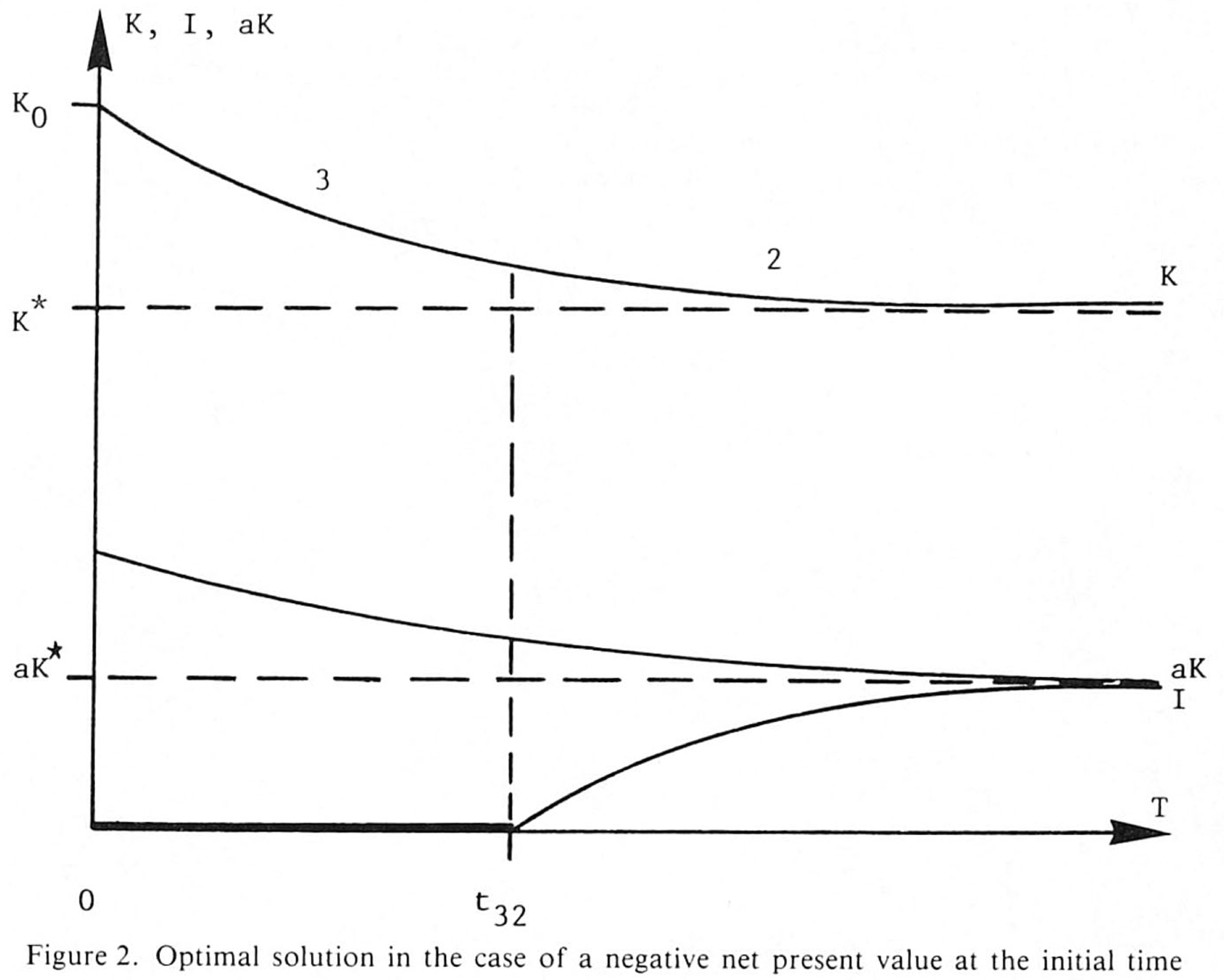

Figure 2. Optimal solution in the case of a negative net present value at the initial time

function of K and thus when the firm does not have price-making power in the output market. Now, marginal expenses exceed marginal earnings of investment and it is optimal to invest nothing at all (path 3). To confirm this we can show from (14) and (15) that the net present value relation along this path is

$$-\lambda_2 = \int_t^{\infty} S'(K(s))\exp[-(i+a)(s-t)]\ \mathrm{d}s \quad - \quad (1 + A'(I)) \quad \text{(path 3)} \tag{23}$$

On path 3 the stock of capital goods decreases and $S'(K)$ increases. Therefore the net present value of marginal investment will increase and become equal to zero after some time. Then path 3 passes into path 2 and (23) turns into (21). The solution with negative net present value at the initial time is present in Figure 2.

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REFERENCES

1. Lucas, R. E. Jr., 'Optimal investment policy and the flexible accelerator', *Int. Econ. Rev.*, **8**, 78–85 (1967).
2. Söderström, H. T., 'Production and investment under cost of adjustment, a survey', *Z. Nationalökon.*, **36**, 369–388 (1976).
3. Lucas, R. E. Jr. and T. Sargeant (eds), *Rational Expectations and Econometric Practice*, University of Minnesota Press, Minneapolis, MN, 1981.
4. Sargent, T., *Dynamic Macroeconomic Theory*, Harvard University Press, Cambridge, MA, 1987.
5. Gould, J. P., 'Adjustment costs in the theory of investment of the firm', *Rev. Econ. Stud.*, **35**, 47–56 (1968).
6. Sargent, T., *Macroeconomic Theory*, 2nd edn, Academic Press, Orlando, FL, 1987.
7. Kort, P. M., *Optimal Dynamic Investment Policies of a Value Maximizing Firm*, Springer, Berlin, 1989.
8. Brealy, R. and S. Myers, *Principals of Corporate Finance*, McGraw-Hill, New York, 1981.
9. Levy, H. and M. Sarnat, *Capital Investment and Financial Decisions*, Prentice-Hall, Englewood Cliffs, NJ, 1986.
10. Feichtinger, G. and R. G. Hartl, *Optimale Kontrolle Oekonomischer Prozesse: Anwendungen des Maximumprinzips in den Wirtschaftswissenschaften*, De Gruyter, Berlin, 1986.
11. Kamien, M. I. and N. L. Schwartz, *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, North-Holland, New York, 1981.