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Magnus, J.R.; Pesaran, B.

Published in:
Computational Statistics

Publication date:
1993

Computational Statistics (1993) 8:39-45

Computational Statistics

The Evaluation of Cumulants and Moments of Quadratic Forms in Normal Variables (CUM): Technical Description

J. R. Magnus

London School of Economics, Houghton Street, London WC2A 2AE, England and CentER for Economic Research, P.O. Box 90153, 5000 LE Tilburg, The Netherlands

B. Pesaran

Economics Division, Bank of England, Threadneedle Street, London EC2R 8AH, England

SUMMARY

This paper considers quadratic forms, $Q = x'Ax$, where x is normally distributed with mean μ and variance-covariance matrix Ω (positive definite), and A is symmetric. The subroutine CUM calculates the first s cumulants and moments of Q. An auxiliary subroutine, PARINT, is also presented and works out all possible partitions of a given integer.

Keywords: Quadratic forms; Moments; Cumulants; Calculation of expectations; Partitions of an integer.

1 Introduction

In this paper we consider quadratic forms, $Q = x'Ax$, where x is normally distributed with mean μ and variance-covariance matrix Ω (positive definite), and A is symmetric. The subroutine CUM calculates the first s cumulants and moments of Q. The routine is based on theory developed by Magnus (1978, 1979) for the central case where $\mu = 0$, and Magnus (1986, Lemmas 2 and 3) for the general case.

An auxiliary subroutine, PARINT, is of independent interest. PARINT works out all possible partitions of a given integer M. That is, all possible vectors $(n_1, n_2, ..., n_M)$ are obtained such that for non-negative integers $n_1, ...n_M$:

$$n_1 + 2n_2 + ... + Mn_M = M.$$

0943-4062/93/1/39-45 \$2.50

Some background, motivation and examples for both routines is provided in a companion paper Magnus and Pesaran (1992a). A related subroutine, QRMOM, considers ratios of quadratic forms and is presented in Magnus and Pesaran (1992b). The full Fortran 77 code of CUM and PARINT (and QRMOM) is given in Magnus and Pesaran (1992c).

Section 2 describes CUM, section 3 PARINT and section 4 introduces the Fortran 77 code.

2 The Subroutine CUM

2.1 Description and Purpose

The subroutine CUM calculates the first s cumulants and moments of the quadratic form $x'Ax$, where x is an $n \times 1$ vector of normally distributed variables with some mean μ and a positive definite (hence non-singular) variance-covariance matrix Ω and A is an $n \times n$ symmetric matrix. The subroutine CUM uses the auxiliary subroutine PARINT (described in section 3), which is of independent interest.

2.2 Parameter Statements

The following parameters have been set in subroutines CUM and PARINT:

ISPAR = 77 & ISDIM=12	The dimensions of the array ISPRTN (ISPAR× ISDIM) where all possible partitions for a particular s are stored. This two dimensional array is set up by subroutine PARINT.
MAXMOM=24	The maximum of s allowed.

The above parameter values are sufficient for $s \leq 12$. If $12 < s \leq 24$ is to be calculated, then both ISPAR and ISDIM should be increased. ISDIM should be at least equal to s and ISPAR should be set at least equal to the number of all possible partitions of integer s. For a table containing the partitions for integers up to 100 see Hall (1986, p. 38).

Working out cumulants and moments for $s > 24$ is also possible. In that case not only ISPAR and ISDIM should be changed in the parameter statements, but also MAXMOM. In that case the subroutine PARINT should also be adapted (see section 3.2).

2.3 Common Statements

The use of the subroutine CUM does not involve any common statements.

2.4 Structure

SUBROUTINE CUM(IMOM, N, LS, A, IEMU, EMU, VARLOW, RKUM, RMOM, WORK1, WORK2, WORK3, VEC, IFAULT)

Formal Parameters

IMOM	integer	input:	=0 if only cumulants are required =1 if both cumulants and moments are required
N	integer	input:	number of observations n
LS	integer	input:	order of the highest cumulant or moment required
		output:	unchanged unless LS > M where M=MIN(MAXMOM,ISDIM) in which case LS is set equal to M
A	real array of dimension at least N × (N+1))/2	input:	symmetric matrix in the quadratic form $x'Ax$. Only the lower part of A is stored as: $a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}$ etc.
IEMU	integer	input:	$= 0$ if $\mu = 0$ $\neq 0$ if $\mu \neq 0$
EMU	real array of dimension at least N	input:	vector μ. Values required only if IEMU $\neq 0$ though storage should be allocated to it
VARLOW	real array of dimension at least N × (N+1)/2	input:	$= L$ where $\Omega = LL'$, L lower triangular
RKUM	real array of dimension at least LS	output:	the required cumulants with i-th cumulant stored in RKUM(i)
RMOM	real array of dimension at least LS	output:	the required moments with i-th moment stored as RMOM(i)

WORK1	real array of dimension at least N × (N+1)/2		work space
WORK2	real array of dimension at least N × (N+1)/2		work space
WORK3	real array of dimension at least N × (N+1)/2		work space
VEC	real array of dimension at least N		work space
IFAULT	integer	output:	a fault indicator where: 0: no error 1: N $\leq$ 1 2: LS < 1 3: diagonal elements of L not all positive 4: L can't be inverted 5: ISPAR in the parameter statement is too small

2.5 Auxiliary Algorithms

CUM calls the following function and subroutine:

FUNCTION INX(I,J):	picks out the appropriate element of a symmetric matrix stored in lower triangular form
SUBROUTINE PARINT:	constructs the matrix containing all partitions of an integer

2.6 Constants

The DATA statement in CUM sets EPS = 1.0-11 as a small number. Any number with an absolute value below EPS will be treated as zero.

2.7 Precision

The version of the routines listed below is in double precision (Real*8). In order to change the program to single precision the following changes should be made:

(1) Change all IMPLICIT REAL*8 to IMPLICIT REAL*4

(2) Change the constants in the DATA statements to single precision versions.

(3) Change DABS to ABS in the subroutine CUM.

2.8 Time and Accuracy

The results for typical CPU times and accuracy of calculations are reported in Magnus and Pesaran (1992a). These calculations were carried out using the VAX 6330 at the London School of Economics.

3 The Subroutine PARINT

3.1 Description and Purpose

The subroutine PARINT works out all possible partitions of an integer M. That is, all possible vectors $(n_1, n_2, ..., n_M)$ are obtained such that for non-negative integers $n_1, ...n_M$:

$$n_1 + 2n_2 + ... + Mn_M = M.$$

The routine also works out the total number of partitions MR. The resulting vectors are stored as an MRxM two-dimensional array MPARTS. Thus each row of MPARTS will contain a possible set of $(n_1, ..., n_M)$.

Partitioning M in such a way is useful in a variety of situations where moments of random variables or functions of random variables are required. For example, see Kendall and Stuart (1977, Section 3.14), Hoque, Magnus and Pesaran (1988), Magnus and Pesaran (1989, 1991), and Magnus (1990). PARINT is used in the subroutines CUM and QRMOM, see Magnus and Pesaran (1992a,b).

3.2 Parameter Statement

A parameter statement has been used to set up MAXMOM = 24. If partitions for numbers bigger than 24 are required this parameter statement should be modified. In a

DATA statement the vector NUM has been set up to contain the number of all possible partitions of numbers up to 24. If a MAXMOM bigger than 24 is specified, the DATA statement for NUM should be extended accordingly. For a table containing the partitions for integers up to 100 see Hall (1986, p. 38).

3.3 Structure

SUBROUTINE PARINT (M, MPARTS, MRDIM, MDIM, MR, IFAIL)

Formal Parameters

M	integer	input:	integer for which all partitions are required
MPARTS	integer array of dimension MRDIM × (at least M)	output:	each row will contain a possible partition $n_1, n_2, ..., n_M$
MRDIM	integer	input:	first dimension of MPARTS
MDIM	integer	input:	second dimension of MPARTS
MR	integer	output:	number of all possible partitions found
IFAIL	integer	output:	fault indicator =0: no error =1: M < 1 or M > MAXMOM or M > MDIM =2: first dimension of MPARTS is not adequate

3.4 Accuracy and Precision

All calculations are carried out using integer arithmetic and are exact.

4 Fortran 77 Code

In Magnus and Pesaran (1992c) the complete Fortran 77 code of the following three algorithms is presented:

SUBROUTINE CUM

SUBROUTINE PARINT

FUNCTION INX(I,J)

A diskette containing the code is available upon request from the authors.

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