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SOLVING THE NONLINEAR COMPLEMENTARITY PROBLEM WITH LOWER AND UPPER BOUNDS

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Abstract: In order to solve the nonlinear complementarity problem with lower and upper bounds, a simplicial variable dimension restart algorithm is introduced. The algorithm subdivides the set on which the problem is defined into simplices and generates from an arbitrarily chosen starting point a piecewise linear path of points leading to an approximate solution. When the accuracy is not sufficient the algorithm can be restarted at the approximate solution with a finer simplicial subdivision. The piecewise linear path generated by the algorithm is followed by a sequence of adjacent simplices of varying dimension. The path can be interpreted as the path of solutions of the nonlinear complementarity problem with parametrized bounds.

1. Introduction.

This paper is concerned with the development of a simplicial algorithm for finding an approximate solution for the nonlinear complementarity problem with lower and upper bounds. The problem is defined as follows.

Given two vectors a and b in R^n with $a_i < b_i$ for all $i \in \{1,\dots,n\}$ and a continuous function $f:C^n \to R^n$, with C^n defined as $C^n = \{x \in R^n | \; a \leq x \leq b\}$, find an $x^* \in C^n$ such that for all $i \in \{1,\dots,n\}$

$$\begin{aligned} f_i(x^*) &\leq 0 \text{ if } a_i = x_i^* \\ f_i(x^*) &= 0 \text{ if } a_i < x_i^* < b_i \\ f_i(x^*) &\geq 0 \text{ if } \quad x_i^* = b_i. \end{aligned} \tag{1.1}$$

This problem is also known as the generalized nonlinear complementarity problem (GNLCP) and it is frequently met in economic problems.

The GNLCP encloses many well-known problems in the field of mathematical programming. Among these problems we mention the nonlinear complementarity problem (NLCP) and the generalized linear complementarity problem (GLCP). The NLCP can be seen as a special case of (1.1) by taking $a_i = 0$ and $b_i = +\infty$ for all $i \in \{1,...,n\}$. For an algorithm solving the NLCP we refer to (2). The GLCP can be seen as a special case of (1.1) by assuming f to be linear. An algorithm solving the GLCP can be found in (3). Our algorithm is a natural alternative for the simplicial algorithm developed by van der Laan and Talman in (4).

The paper is organized as follows. Section 2 introduces the path of points the algorithm follows approximately. The steps of the algorithm are described in section 3. To approximate the path described in section 2 the algorithm makes use of a simplicial subdivision of C^n. In section 4 we present an appropriate simplicial subdivision of C^n.

2. The path to be approximated by the algorithm.

The path of points in C^n which the algorithm will approximately follow starts in an arbitrarily chosen point $v \in C^n$. With respect to this point v we make the following assumption.

Assumption 2.1 (nondegeneracy at v): In the starting point v there does not exist an $i \in \{1,...,n\}$ such that $f_i(v) = 0$.

Starting in v the algorithm follows approximately a path of points x in C^n such that for some ρ, $0 \leq \rho \leq 1$, x solves the GNLCP on $C^n_\rho := (1-\rho)\{v\} + \rho C^n$ with respect to f, i.e., for all $i \in \{1,...,n\}$

$$\begin{aligned} f_i(x) &\leq 0 \text{ if } (1-\rho)v_i + \rho a_i = x_i \\ f_i(x) &= 0 \text{ if } (1-\rho)v_i + \rho a_i < x_i < (1-\rho)v_i + \rho b_i \end{aligned} \qquad (2.1)$$

$f_i(x) \geq 0$ if $\qquad x_i = (1-\rho)v_i + \rho b_i$.

Under some regularity and nondegeneracy conditions the set of points x being a solution of (2.1) for some ρ, $0 \leq \rho \leq 1$, form piecewise smooth curves. Each of these curves is either a loop or a path with two end points. One of these paths, say P, has v as an end point for $\rho = 0$. All other end points of paths in C^n are solutions to (1.1). The algorithm follows approximately the path P from v to its other end point.

By increasing ρ from 0 the path P leaves v in the direction pointing towards the corner point z of C^n where $z_i = b_i$ if $f_i(v) > 0$ and $z_i = a_i$ if $f_i(v) < 0$ for all $i \in \{1,\ldots,n\}$. If along the path P at a point $x = (1-\rho)v + \rho z$, with ρ between 0 and 1 and z a point in the boundary of C^n, $f_j(x)$ becomes zero for some $j \in \{1,\ldots,n\}$ while $z_j = a_j$ (or b_j), then either x solves (1.1) or the path continues by increasing x_j from $(1-\rho)v_j + \rho a_j$ (decreasing x_j from $(1-\rho)v_j + \rho b_j$). If at a point x on P, x_j becomes equal to $(1-\rho)v_j + \rho a_j$ (or $(1-\rho)v_j + \rho b_j$) for a $j \in \{i \mid f_i(x) = 0\}$, then the path P continues by decreasing (increasing) $f_j(x)$ from zero. Finally, if at a point x on P, ρ becomes equal to 1, then, because $C_1^n = C^n$ and hence the conditions in (2.1) reduce to (1.1), the point x is a solution to the GNLCP in (1.1) and thereby an end point of the path P in C^n. In this way the path P leads from v to a solution of (1.1).

3. The algorithm.

The algorithm approximately follows the path P described in section 2 by generating a piecewise linear (p.l.) path $\bar{P}$ connecting v with an approximate solution $\bar{x}$ of (1.1). For a description of this p.l. path we approximate the function f by a p.l. approximation F.

To define a p.l. approximation F of f we need to subdivide C^n into simplices. So, let G^n be a triangulation or simplicial subdivision of C^n. For an appropriate simplicial subdivision of C^n we refer the interested reader to section 4.

Definition 3.1: The p.l. approximation F of f with respect to the simplicial subdivision G^n of C^n at a point $x \in C^n$ is given by

$$F(x) = \sum_{i=1}^{n+1} \lambda_i f(y^i) \tag{3.1}$$

where the convex hull $\sigma(y^1,\ldots,y^{n+1})$ of $y^1,\ldots,y^{n+1}$ in C^n is an n-dimensional or n-simplex in G^n containing x and where $\lambda_1,\ldots,\lambda_{n+1} \geq 0$ are such that $x = \sum_{i=1}^{n+1} \lambda_i y^i$ and $\sum_{i=1}^{n+1} \lambda_i = 1$.

The results obtained in section 2 with respect to f can also be applied to the p.l. approximation F of f. In particular, there exists a p.l. path $\bar{P}$ of points in C^n connecting v and a solution to (1.1) with respect to F. For each point x on the path $\bar{P}$ there exists a ρ between 0 and 1 such that for all $i \in \{1,\ldots,n\}$

$$\begin{aligned} &F_i(x) \leq 0 \text{ if } (1-\rho)v_i + \rho a_i = x_i \\ &F_i(x) = 0 \text{ if } (1-\rho)v_i + \rho a_i < x_i < (1-\rho)v_i + \rho b_i \\ &F_i(x) \geq 0 \text{ if } x_i = (1-\rho)v_i + \rho b_i. \end{aligned} \tag{3.2}$$

Notice that in (3.2) the sign pattern of F(x) plays a very important role. Therefore we introduce the notion of a sign vector.

Definition 3.2: A vector $s \in \mathbb{R}^n$ is a sign vector when for all $i \in \{1,\ldots,n\}$ we have that $s_i \in \{-1,0,+1\}$.

Now, let for each sign vector s the set $C^n(s)$ be defined by

$$C^n(s) = \{x \in C^n \mid \text{for all } i,\ x_i = a_i \text{ if } s_i = -1 \text{ and } x_i = b_i \text{ if } s_i = +1\}. \tag{3.3}$$

If $v \in C^n(s)$ we define $A(s) = \emptyset$, otherwise A(s) is the convex hull of v and $C^n(s)$, i.e.,

$$A(s) = \{x \in C^n \mid \text{for some } \rho,\ 0 \leq \rho \leq 1, \text{ and for all } i$$
$$\begin{aligned} (1-\rho)v_i + \rho a_i &= x_i && \text{if } s_i = -1 \\ (1-\rho)v_i + \rho a_i \leq x_i &\leq (1-\rho)v_i + \rho b_i && \text{if } s_i = 0 \\ x_i &= (1-\rho)v_i + \rho b_i && \text{if } s_i = +1\}. \end{aligned} \tag{3.4}$$

Clearly, $x \in \bar{P}$ satisfies $x \in A(s)$ with s the sign vector such that $s = \text{sgn}(F(x))$.

The simplicial subdivision G^n of C^n has to be such that it triangulates each nonempty subset A(s) into t-simplices where t, the dimension of A(s), is equal to $|I^0(s)|+1$ with $I^0(s) := \{i \in \{1,\dots,n\} | s_i = 0\}$ (see section 4 for an appropriate simplicial subdivision). So, if $x \in A(s)$, then there are a t-simplex $\sigma(y^1,\dots,y^{t+1})$ in A(s) and numbers $\lambda_1,\dots,\lambda_{t+1} \geq 0$ such that $x = \sum_{i=1}^{t+1} \lambda_i y^i$ and $\sum_{i=1}^{t+1} \lambda_i = 1$.

On the other hand, if $\text{sgn}(F(x)) = s$, then there exist $\mu_h \geq 0$, $h \notin I^0(s)$, such that $F(x) = \sum_{h \notin I^0(s)} \mu_h s_h e(h)$, where e(h) is the n-dimensional unit vector with $e_i(h) = 1$ if $h = i$. Hence, if x lies on the path $\bar{P}$, then for some sign vector s there is a t-simplex $\sigma(y^1,\dots,y^{t+1})$ in A(s) such that the system of linear equations given by

$$\sum_{i=1}^{t+1} \lambda_i \begin{bmatrix} f(y^i) \\ 1 \end{bmatrix} - \sum_{h \notin I^0(s)} \mu_h s_h \begin{bmatrix} e(h) \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix} \tag{3.5}$$

has a nonnegative solution $\lambda_i^* \geq 0$, $i = 1,\dots,t+1$, $\mu_h^* \geq 0$, $h \notin I^0(s)$, with $x = \sum_{i=1}^{t+1} \lambda_i^* y^i$. The vector $\underline{0}$ in (3.5) denotes the n-vector of zeros.

System (3.5) is a system of n+1 equations with n+2 unknowns leaving us with one degree of freedom. So, assuming nondegeneracy, a line segment of solutions to (3.5) exists which can be followed by making a linear programming pivot step in (3.5). This line segment corresponds to the linear piece of $\bar{P}$ in σ defined by the points $x = \sum_{i=1}^{t+1} \lambda_i y^i$.

In an end point of a line segment of solutions to (3.5) either $\lambda_p = 0$ for some $p \in \{1,\dots,t+1\}$ or $\mu_j = 0$ for some $j \notin I^0(s)$. If at an end point, $\lambda_p = 0$ for some $p \in \{1,\dots,t+1\}$, then the point $\bar{x} = \sum_{i \neq p}$

$\lambda_i y^i$ lies in the facet τ of σ opposite the vertex y^p. The facet τ is either also a facet of exactly one other t-simplex, say $\bar{\sigma}$, in A(s) or τ lies in the boundary of A(s).

Suppose $\bar{\sigma}$ exists. Then, in order to continue the path $\bar{P}$ in A(s), a pivot step is made in (3.5) with the column $[f(\bar{y})^\top,1]^\top$ corresponding to the unique vertex $\bar{y}$ of $\bar{\sigma}$ not contained in τ. The algorithm is continued by repeating the procedure described.

Suppose $\bar{\sigma}$ does not exist and hence τ lies in the boundary of A(s). If τ lies in $C^n(s)$, then the algorithm has found a point $\bar{x} \in C^n(s)$ with sign vector s equal to $\mathrm{sgn}(F(\bar{x}))$ so that $\bar{x}$ is an approximate solution for (1.1). Otherwise, τ is a (t-1)-simplex in $A(\bar{s})$ where $\bar{s}$ is a sign vector such that $\bar{s}_\ell \neq 0$ for some $\ell \in I^0(s)$ while $\bar{s}_i = s_i$ for all $i \neq \ell$. Then the algorithm continues in $A(\bar{s})$ by pivoting the column $[\bar{s}_\ell e(\ell)^\top,0]^\top$ into (3.5).

If at an end point of solutions to (3.5), μ_j is zero for some $j \notin I^0(s)$, then at $\bar{x} = \sum_{i=1}^{t+1} \lambda_i y^i$ we have $F_j(\bar{x}) = s_j\mu_j = 0$. Let $\bar{s}$ be a sign vector such that $\bar{s}_j = 0$ and $\bar{s}_h = s_h$ for $h \neq j$. Suppose that $A(\bar{s}) = \emptyset$. Then $\bar{x}$ lies in $C^n(\bar{s})$ whereas $\mathrm{sgn}(F(\bar{x})) = \bar{s}$. Hence, $\bar{x}$ is an approximate solution to (1.1). Otherwise, if $A(\bar{s}) \neq \emptyset$, then there is exactly one (t+1)-simplex $\bar{\sigma}$ in $A(\bar{s})$ having σ as a facet. Now the algorithm continues by pivoting the column $[f(\bar{y})^\top,1]^\top$ into (3.5), where $\bar{y}$ is the vertex of $\bar{\sigma}$ not contained in σ.

Now we have described how the algorithm proceeds along the path $\bar{P}$ in the different subsets A(s) of C^n, we still have to describe the initialization of the algorithm at v. At v the system (3.5) becomes

$$\lambda_1\begin{bmatrix} f(v) \\ 1 \end{bmatrix} - \sum_{h=1}^{n} s_h^0\mu_h\begin{bmatrix} e(h) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.6}$$

having a unique solution $\lambda_1 = 1$, $\mu_h = s_h^0 f_h(v) > 0$, $h \in \{1,\ldots,n\}$, where $s^0 = \mathrm{sgn}(f(v))$. If $A(s^0) = \emptyset$, then $v \in C^n(s^0)$ and the algorithm stops with an exact solution at v. Otherwise, the starting point v is a facet of a unique 1-simplex $\sigma(y^1,y^2)$ in $A(s^0)$ with $y^1 = v$. The algorithm then pivots the column $[f(y^2)^\top,1]^\top$ into (3.6).

Since all steps are unique, returning to v is impossible, and the number of simplices is finite, the algorithm terminates within a finite number of steps with an approximate solution $\bar{x}$ of (1.1). The accuracy of the approximation $f(\bar{x})$ can be measured by the smallest $\varepsilon > 0$ for which for all $i \in \{1,\dots,n\}$

$$\begin{array}{lll} f_i(\bar{x}) \leq \varepsilon & \text{if } a_i = \bar{x}_i \\ -\varepsilon \leq f_i(\bar{x}) \leq \varepsilon & \text{if } a_i < \bar{x}_i < b_i \\ -\varepsilon \leq f_i(\bar{x}) & \text{if } \bar{x}_i = b_i . \end{array} \tag{3.7}$$

If $f(\bar{x})$ is not accurate enough, i.e. if ε is too large, the algorithm is repeated being started at $v = \bar{x}$ with a finer simplicial subdivision of C^n. This in the hope to find a more accurate approximation within a relative small number of steps. In this way, within a finite number of steps an approximate solution with any accuracy can be found.

4. A simplicial subdivision of C^n.

In order to triangulate C^n one can use any simplicial subdivision. The only restriction one has to pose on the triangulation of C^n to underly the algorithm described in section 3 is that it has to triangulate all nonempty subsets $A(s)$. A triangulation that perfectly fits into this framework is the V-triangulation of the product space of unit simplices developed in (1). In this section we adapt the V-triangulation to a triangulation of C^n.

To describe the triangulation we first subdivide each nonempty $A(s)$ into subsets $A(s,\gamma(T))$ with $\gamma(T) = (\gamma_1,\dots,\gamma_{t-1})$, $t = |I^0(s)|+1$, a permutation of the t-1 elements of a set T such that for all $j \in I^0(s)$ either j or -j belongs to T. If we define the projection $p(s)$ of v on $C^n(s)$ as the vector with elements

$$p_h(s) = \begin{cases} a_h & \text{if } s_h = -1 \\ b_h & \text{if } s_h = +1 \\ v_h & \text{if } s_h = 0 \end{cases} \qquad h \in \{1,\dots,n\}, \tag{4.1}$$

then $A(s,\gamma(T))$ is defined as the convex hull of v and the projections $p(s^h)$, $h \in \{1,\dots,t\}$, where

$$s^h = s + \sum_{j=h}^{t-1} e(\gamma_h), \qquad h \in \{1,\dots,t\}, \tag{4.2}$$

with for all i, $e_i(\gamma_j) = +1$ if $\gamma_j = i$, $e_i(\gamma_j) = -1$ if $\gamma_j = -i$, and $e_i(\gamma_j) = 0$ otherwise. Notice that $s^t = s$ and that $p(s^1)$ is a vertex of C^n.

For some positive integer m, each nonempty $A(s,\gamma(T))$ is now triangulated into t-simplices $\sigma(y^1,\pi)$ with vertices $y^1,\dots,y^{t+1}$ in C^n such that

i) $y^1 = v + \sum_{k=1}^{t} a(k)m^{-1}q(k)$ with integers $a(k)$ satisfying $0 \le a(t) \le \dots \le a(1) \le m-1$;

ii) $\pi = (\pi_1,\dots,\pi_t)$ is a permutation of the elements of $\{1,\dots,t\}$ such that for all $i \in \{1,\dots,t-1\}$ holds : $p > p'$ if $\pi_{p'} = i$, $\pi_p = i+1$, and $a(\pi_{p'}) = a(\pi_p)$;

iii) $y^{i+1} = y^i + m^{-1}q(\pi_i)$, $i = 1,\dots,t$,

where $q(1) = p(s^1) - v$ and

$$q(k) = p(s^k) - p(s^{k-1}), \qquad k = 2,\dots,t. \tag{4.3}$$

If we denote this triangulation by $G_m^n(s,\gamma(T))$, then the set $A(s)$ is triangulated by the union $G_m^n(s)$ of $G_m^n(s,\gamma(T))$ over all $\gamma(T)$. Moreover, C^n is triangulated by the union G_m^n of $G_m^n(s)$ over all s, m^{-1} being the grid size.

In section 3 we described how to follow the path $\bar{P}$ through C^n by making pivot steps in the system of equations (3.5) with respect to a sequence of adjacent simplices σ in $A(s)$ for varying sign vectors s. After having introduced a specific triangulation of C^n we will now describe how, given the parameters y^1, π, and $a(h)$, for $h = 1,\dots,t$, of a t-simplex σ, the parameters of a simplex $\bar{\sigma}$ adjacent to σ are obtained.

The movement from a t-simplex $\sigma(y^1,\pi)$ in $A(s,\gamma(T))$ to an adjacent simplex $\bar{\sigma}(\bar{y}^1,\bar{\pi})$ is called a replacement step when $\bar{\sigma}(\bar{y}^1,\bar{\pi})$ is also a t-simplex in $A(s,\gamma(T))$. Making a replacement step we replace the vertex y^p, $p \in \{1,\ldots,t+1\}$, of σ opposite the common facet τ of σ and $\bar{\sigma}$ by the vertex $\bar{y}$ of $\bar{\sigma}$ not belonging to τ. The possibilities are listed in Table 1, where $a_h = a(h)$, $h = 1,\ldots,t$, and $a_h = 0$, $h = t+1,\ldots,n$.

Table 1: Replacement step.

	$\bar{y}^1$	$\bar{\pi}$	$\bar{a}$
$p = 1$	$y^1+m^{-1}q(\pi_1)$	$(\pi_2,\ldots,\pi_t,\pi_1)$	$a+e(\pi_1)$
$1 < p < t+1$	y^1	$(\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\pi_{p+1},\ldots,\pi_t)$	a
$p = t+1$	$y^1-m^{-1}q(\pi_t)$	$(\pi_t,\pi_1,\ldots,\pi_{t-1})$	$a-e(\pi_t)$

In case the replacement step with respect to y^p cannot be performed, the facet τ of $\sigma(y^1,\pi)$ opposite y^p lies in the boundary of $A(s,\gamma(T))$. Lemma 4.1 describes when τ lies in the boundary of $A(s,\gamma(T))$.

Lemma 4.1: Let $\sigma(y^1,\pi)$ be a t-simplex in $G^n(s,\gamma(T))$ and τ the facet of σ opposite vertex y^p, $1 \leq p \leq t+1$. Then τ lies in the boundary of $A(s,\gamma(T))$ if and only if one of the following cases holds:

1) $p = 1$, $\pi_1 = 1$, and $a(\pi_1) = m-1$;
2) $1 < p < t+1$, $\pi_{p-1} = i$ and $\pi_p = i+1$ for some $i \in \{1,\ldots,t-1\}$, and $a(\pi_{p-1}) = a(\pi_p)$;
3) $p = t+1$, $\pi_t = t$, and $a(\pi_t) = 0$.

In case 1 of Lemma 4.1, τ lies in $C^n(s)$. In case 2 and when $i = 1$, σ shares τ with an adjacent t-simplex $\bar{\sigma}(y^1,\bar{\pi})$ in $A(s,\gamma(\bar{T}))$ where $\bar{T} = T \setminus \{\gamma_1\} \cup \{-\gamma_1\}$, $\gamma(\bar{T}) = (-\gamma_1,\gamma_2,\ldots,\gamma_{t-1})$, and $\bar{\pi} = (\pi_1,\ldots,\pi_{p-2},\pi_p,$

$\pi_{p-1},\pi_{p+1},\ldots,\pi_t)$. Otherwise in case 2, $\sigma(y^1,\pi)$ shares τ with an adjacent t-simplex $\bar{\sigma}(y^1,\bar{\pi})$ in $A(s,\bar{\gamma}(T))$ where $\bar{\gamma}(T) = (\gamma_1,\ldots,\gamma_{i-2},\gamma_i,\gamma_{i-1},\gamma_{i+1},\ldots,\gamma_{t-1})$ and $\bar{\pi} = (\pi_1,\ldots,\pi_{p-2},\pi_p,\pi_{p-1},\pi_{p+1},\ldots,\pi_t)$. Case 3 and when $I^0(s) \neq \emptyset$ represents the case where the facet τ opposite the vertex y^{t+1} of σ is the (t-1)-simplex $\bar{\sigma}(y^1,\bar{\pi})$ in $A(\bar{s},\bar{\gamma}(\bar{T}))$ where $\bar{s} = s + e(\gamma_{t-1})$, $\bar{T} = T \setminus \{\gamma_{t-1}\}$, $\bar{\gamma}(\bar{T}) = (\gamma_1,\ldots,\gamma_{t-2})$ and $\bar{\pi} = (\pi_1,\ldots,\pi_{t-1})$. Otherwise in case 3, we have that $t = 1$ and $a(1) = 0$ which means that $\tau = \{v\}$.

Finally, a t-simplex $\sigma(y^1,\pi)$ in $A(s,\gamma(T))$ is a facet of exactly one (t+1)-simplex $\bar{\sigma}$ in a nonempty $A(\bar{s})$ where $\bar{s}_k = 0$ for some $k \notin I^0(s)$ and $\bar{s}_i = s_i$ for all other $i \in \{1,\ldots,n\}$. More precisely, let $h = +k$ if $s_k = +1$ and $h = -k$ if $s_k = -1$, then $\bar{\sigma}$ is the (t+1)-simplex $\bar{\sigma}(y^1,\bar{\pi})$ in $A(\bar{s},\bar{\gamma}(\bar{T}))$ where $\bar{T} = T \cup \{h\}$, $\bar{\gamma}(\bar{T}) = (\gamma_1,\ldots,\gamma_{t-1},h)$, and $\bar{\pi} = (\pi_1,\ldots,\pi_t,t+1)$.

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