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SIMPLICIAL VARIABLE DIMENSION ALGORITHMS FOR SOLVING THE NONLINEAR COMPLEMENTARITY PROBLEM ON A PRODUCT OF UNIT SIMPLICES USING A GENERAL LABELLING*†

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This paper deals with the nonlinear complementarity problem on the product space of unit simplices, S. A simplicial variable dimension algorithm developed by van der Laan and Talman for proper labellings of S is extended to the case of general labellings. General labellings allow a more natural description of the complementarity problem on the boundary of S. A distinctive feature of the new algorithm is that lower dimensional simplicial movement can occur both on the boundary and in the interior of S. In contrast, the van der Laan and Talman algorithm for proper labellings of S allows lower dimensional simplicial movement only in the interior of S. Computational experiments confirm the usefulness of general labellings for solving nonlinear complementarity problems.

1. Introduction. This paper concerns the nonlinear complementarity problem (NLCP) on the product space of unit simplices. Let I^k denote the index set $\{1, \ldots, k\}$. Let $S = S^{n_1} \times \cdots \times S^{n_N}$ denote the product of N unit simplices $S^k = \{y \in R_+^{k+1} | \sum_{i=1}^{k+1} y_i = 1\}$, $k = n_1, \ldots, n_N$. Let $n = \sum_{j=1}^{N} n_j$ and $m = \sum_{j=1}^{N}(n_j + 1) = n + N$. An element $x \in S$ will be denoted $(x_1, \ldots, x_N)$ with $x_j = (x_{j1}, \ldots, x_{j(n_j+1)})' \in S^{n_j}$, $j \in I^N$. Similar notation will be used to describe elements of $R^{n_1+1} \times \cdots \times R^{n_N+1} \subset R^m$. The datum for an NLCP on S is a continuous function $z: S \to R^m$, $x = (x_1, \ldots, x_N) \to z(x) = (z_1(x), \ldots, z_N(x))$ satisfying $x_j' z_j(x) = 0$ for all $x \in S$ and $j \in I^N$. Given such a function the NLCP on S consists in finding a point $x \in S$ such that $z(x) \leqslant 0$. Such a point x is complementary to $z(x)$, that is, $x_{ji} z_{ji}(x) = 0$ for all $j \in I^N$ and $i \in I^{n_j+1}$.

This problem arises quite naturally in different fields such as economics, nonlinear programming, and game theory. An example of an NLCP on a single unit simplex is the problem of computing an equilibrium price vector in a pure exchange economy. In this example the datum $z(\cdot)$ is a market excess demand function. The complementarity condition $x'z(x) = 0$ is known as the Walras law (see, e.g., [10]). An example of an NLCP defined over a product of unit simplices arises in game theory when computing a Nash equilibrium for a noncooperative multiperson game. Each player contributes a strategy simplex to the simplicial product S. Examples of such NLCPs will be presented in the computational section of the paper. Also discussed in that section are NLCPs arising in nonlinear programming.

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The NLCP on S is equivalent to the well-known Brouwer fixed point problem on S (see, e.g., [10]). The NLCP formulation adopted in this paper emphasizes that we are especially concerned with the possibility of solutions, or fixed points, on the boundary of S. Nash equilibria for noncooperative games, for example, typically lie on the boundary of S. Similarly, NLCPs arising in nonlinear programming have solutions lying on the boundary of S unless all constraints are binding at the optimal solution. This paper presents a simplicial algorithm for solving the NLCP.

Scarf [10] first introduced a simplicial algorithm for solving the NLCP on a single unit simplex, S^n. His algorithm is based on a subdivision (or triangulation) of S^n into n-dimensional simplices, or n-simplices, and on a labelling assigning to each point in S^n an integer from the set I^{n+1}. A simplex of the subdivision is said to be completely labelled if its $n+1$ vertices jointly bear all labels in I^{n+1}. The labelling is constructed in such a way that a completely labelled simplex yields an approximate solution for the NLCP. The reader will verify that an example of such labelling is

$$l(x) = \min\{i \in I^{n+1} | z_i(x) = \max_{h \in I^{n+1}} z_h(x)\}, \qquad x \in S^n. \tag{1.1}$$

To ensure the convergence of the Scarf algorithm to a completely labelled simplex, the labelling must satisfy a condition on the boundary of S^n, called properness. A labelling is said to be (Scarf-)proper if each point $x = (x_1, \ldots, x_{n+1})'$ on the boundary of S^n is given a label i for which $x_i = 0$. Some simplicial algorithms alternatively require the labelling to be Sperner-proper. A labelling is Sperner-proper if each point x carries a label i for which $x_i > 0$. For simplicity we cast our discussion in terms of Scarf-proper labellings and refer to Freund [2] for details on Sperner-proper labellings.

An example of a rule which assigns unique and proper labels to points on the boundary is

$$l(x) = \min\{i \in I^{n+1} | x_i = 0 \text{ and } x_{i-1 (\text{mod } n+1)} > 0\}, \qquad x \in \partial S^n, \tag{1.2}$$

with $\partial S^n = \{x \in S^n | x_i = 0 \text{ for some } i \in I^{n+1}\}$. Note that this rule assigns boundary points artificial labels, namely labels that bear no relation to the datum of the NLCP. The reader will also verify that even when the labelling is given by (1.2) on the boundary of S^n, a completely labelled simplex still yields an approximate solution for the NLCP.

The Scarf algorithm uses the information provided by the labelling to generate a path of adjacent simplices terminating at a completely labelled simplex. Two simplices are said to be adjacent if they share all but one vertex. A typical position of the algorithm is an n-simplex whose $n+1$ vertices jointly carry all labels in I^n, one label being duplicated. A typical step of the algorithm consists in deleting one of the vertices with the duplicated label and moving to an adjacent simplex. The latter simplex shares n vertices carrying all labels in I^n with the former simplex. The simplex differs from the previous simplex in exactly one vertex. The algorithm terminates if the new vertex carries the missing label, $n+1$. If not, the new vertex shares its label with one other vertex of the simplex and a new position is reached. The algorithm proceeds by deleting the old vertex with the duplicated label.

The Scarf simplicial path terminates with a completely labelled simplex if the labelling satisfies (1.2). The accuracy of the associated approximate solution for the NLCP generally increases with the grid size (or mesh) of the subdivision. If a given approximate solution is found to be of insufficient accuracy, the subdivision needs to be refined. Eaves [1] first presented a simplicial algorithm which continuously refines the subdivision. Another approach to computing approximate solutions of increasing accuracy is the use of a restart algorithm. A restart algorithm is an algorithm which can be initiated at an arbitrary grid point. Successive restarts with subdivisions having decreasing grid sizes yield increasingly more accurate solutions. Every restart is initiated close to the most accurate solution computed so far.

Merrill [4] first introduced a restart algorithm for complementarity or fixed point problems on R^n. Kuhn and MacKinnon [3] later proposed a similar algorithm for complementarity problems on S^n. The restart possibility is obtained by introducing an additional dimension. The simplex S^n is embedded into the set $S^n \times [0,1]$. The latter set is subdivided into $(n+1)$-simplices having vertices on the "real" level, $S^n \times \{0\}$, or on the "artificial" level, $S^n \times \{1\}$. Such a subdivision is easily obtained by mapping $S^n \times [0,1]$ into a subdivided subset of S^{n+1}. Vertices on the real level are labelled according to (1.1) and (1.2). Vertices on the level $S^n \times \{1\}$ are artificially labelled so as to enable a restart at the desired location and to ensure the finite convergence of the algorithm. The path generated by the algorithm consists of adjacent $(n+1)$-simplices whose common n-subsimpices bear all labels in I^{n+1}. The path terminates whenever the algorithm finds a completely labelled n-simplex on the real level.

Lüthi [8] avoids the artificial labelling on the boundary of S^n and uses labelling rule (1.1) on the whole simplex. The resulting labelling is not necessarily proper so that the existence of a completely labelled n-simplex is no longer guaranteed. However, Lüthi observed that when all vertices in a subdivision of S^n are labelled according to (1.1) lower dimensional simplices on the boundary of S^n yield approximate solutions for the NLCP if they satisfy the so-called completeness condition. A t-simplex $(0 \leqslant t \leqslant n)$ is called complete if for each index $i \in I^{n+1}$ one of the vertices of the simplex carries label i or the simplex lies on the boundary $x_i = 0$ of S^n. As mentioned previously, solutions for the NLCP often lie on the boundary. When this is the case, complete simplices contained in a boundary generally provide more accurate solutions than the completely labelled n-simplices generated by simplicial algorithms using proper labellings. To find a complete simplex, Lüthi extends Merrill's restart algorithm to the case of a general labelling of S^n. The new algorithm allows lower dimensional simplicial movement on the boundary of $S^n \times [0,1]$, which is typically faster than a movement in the same direction using full dimensional simplices.

Van der Laan and Talman [5] bypass the introduction of an artificial level and the embedding of S^n into $S^n \times [0,1]$. Their algorithm, which assumes a proper labelling of S^n, directly generates a path of adjacent simplices of varying dimension in S^n. This path starts at an arbitrary grid point representing a 0-simplex, and terminates at a completely labelled n-simplex. The attractiveness of this restart method lies in the fact that movements with simplices of varying dimension in S^n are typically faster than movements with full dimensional simplices in S^{n+1}.

Finally, in [6] van der Laan and Talman extend their restart algorithm on S^n to deal with complementarity problems defined on a product of N unit simplices, $S = S^{n_1} \times \cdots \times S^{n_N}$. The algorithm assumes a proper labelling of S. A labelling of S is a function assigning to each point of S an integer in the set $I = \{(j,k) \mid j \in I^N$ and $k \in I^{n_j+1}\}$. A labelling of S is said to be proper if each point on the boundary of S carries a label (j,k) for which $x_{jk} = 0$. A rule which assigns proper labels to points on the boundary is

$$l(x) = \text{lexicomin}\{(j,k) \in I \mid x_{jk} = 0 \text{ and } x_{jk'} > 0 \text{ with } k' = k - 1 (\text{mod } n_j + 1)\},$$
$$x \in \partial S, \tag{1.3}$$

where $\partial S = \{x \in S \mid x_{jk} = 0$ for some $j \in I^N$ and $k \in I^{n_j+1}\}$. Given a simplicial subdivision and a proper labelling of S, the van der Laan and Talman algorithm for solving the NLCP on S determines a simplex whose vertices jointly carry the labels

$(j,1),\ldots,(j,n_j+1)$ for some $j \in I^N$. Such a simplex is said to be completely labelled. A completely labelled simplex in a subdivision of S is easily seen to yield approximate solutions for the NLCP if points in the interior of S are labelled according to the rule

$$l(x) = \text{lexicomin}\{(j,h) \in I \mid z_{jh}(x) = \max_{(i,k)\in I} z_{ik}(x)\}. \tag{1.4}$$

Note that this rule extends labelling rule (1.1) to a product of simplices.

In this paper we generalize the van der Laan and Talman restart algorithm on S to the case of a general labelling of S. A Merrill-type restart algorithm (i.e. a restart algorithm defined with the help of an artificial dimension) extending Lüthi's algorithm to a product of unit simplices is not known. The new algorithm presents the variable dimension features of the van der Laan and Talman algorithm on S and of Lüthi's algorithm on S^n. Lower dimensional movement is possible both in the interior of S and on the boundary of S. As indicated earlier, both factors favorably influence the efficiency of the computation. This is confirmed in the numerical experiments reported at the end of the paper.

The paper contains some theoretical results. First, we introduce the notion of a complete simplex in a labelled simplicial subdivision of S. A simplex σ is said to be complete if there exists an index $j \in I^N$ such that for each $h \in I^{n_j+1}$ either σ lies on the boundary $x_{jh} = 0$ of S, or σ has a vertex carrying label (j,h). Our main theoretical result is a lemma stating the existence of a complete simplex in an arbitrarily labelled simplicial subdivision of S. The lemma is proven constructively using the simplicial restart algorithm presented in this paper. The lemma was stated independently by Freund [2]. The algorithm presented in Freund's constructive proof of the lemma can be considered a special case of our algorithm, where the initial grid point is chosen to be a vertex of S. In our theoretical section, we further show how, when the labelling is given by (1.4), complete simplices yield approximate solutions for the NLCP. A by-product of our arguments is an intersection theorem for closed subsets of S that generalizes an earlier result on S^n by Scarf [10]. See also [2].

The paper is organized as follows. §2 contains our theoretical results. §3 reviews the van der Laan and Talman algorithm for determining a completely labelled simplex in a properly labelled simplicial subdivision of S^n. §4 extends the latter algorithm to the case of a general labelling of S. §5 discusses the introduction of vector labels in the algorithm. As is well known, see, e.g. [10], vector labels generally yield more accurate approximate solutions than integer labels. §6 contains our computational results.

2. Theoretical results. Let G denote a simplicial subdivision of S. Let $l: S \to I$ be an integer labelling of S. We define the label and index sets of a t-simplex and then state when the simplex is complete. Notice that the index set of a t-simplex identifies the faces of S containing the t-simplex.

2.1. DEFINITION. Let $\tau = \tau(x^1,\ldots,x^{t+1})$ be a t-simplex in G, $0 \leqslant t \leqslant n = \sum_{j=1}^N n_j$. $L(\tau) = \{l(x^1),\ldots,l(x^{t+1})\}$ is the label set of τ. $I(\tau) = \{(j,h) \in I \mid x_{jh} = 0$ for all $x \in \tau\}$ is the index set of τ.

2.2. DEFINITION. Let τ be a simplex in G. τ is complete if there exists an index $j \in I^N$ such that $\{(j,1),\ldots,(j,n_j+1)\} \subset L(\tau) \cup I(\tau)$. τ is then said to be j-complete. τ is said to be (j-) completely labelled if $\{(j,1),\ldots,(j,n_j+1)\} \subset L(\tau)$.

Definition 2.2 generalizes to a product of unit simplices the concepts of complete and completely labelled simplices in a simplicial subdivision of a single unit simplex.

The following lemma states the existence in a labelled subdivision of S of a complete simplex. It will be proven constructively in §4.

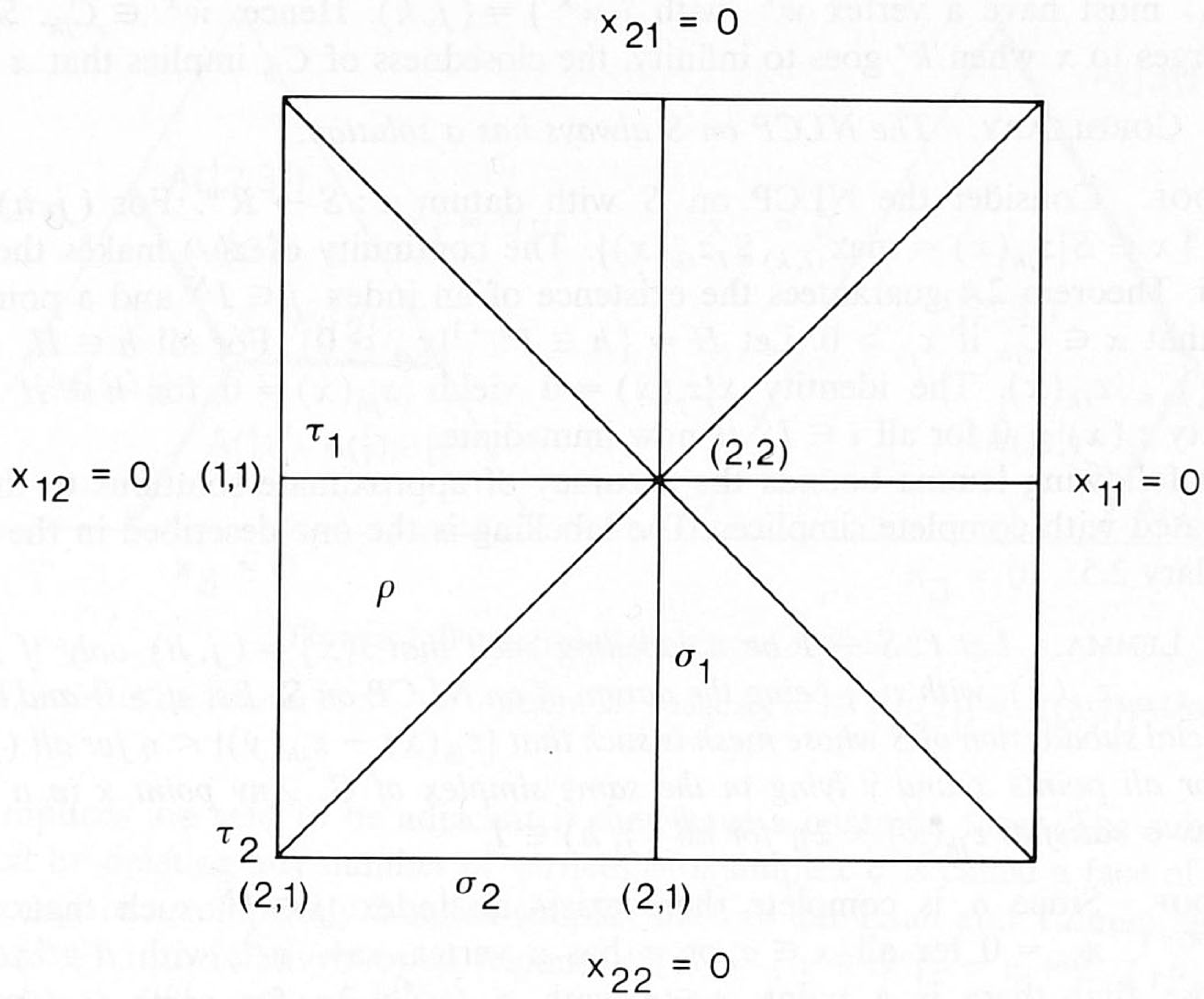

FIGURE 1. Complete Simplices on S ($N = 2$, $n_1 = n_2 = 1$).

2.3. LEMMA (Generalized simplicial Scarf lemma on S). *A generally labelled simplicial subdivision of S contains a complete simplex.*

The lemma is illustrated in Figure 1. The labels of the vertices appear in parentheses. The 1-simplices σ_1 and σ_2 are 2-complete. The 0-simplex τ_1 is 1-complete and τ_2 is 2-complete. The 2-simplex ρ is 2-complete. Observe that ρ has a 1-complete face but is not 1-complete.

One easily observes that if the labelling is proper and given by (1.3) on the boundary of S, then every complete simplex is necessarily completely labelled. The same holds if the labelling is proper, without necessarily satisfying (1.3), and if the simplicial subdivision contains no simplex that intersects all faces $x_{jh} = 0$ of S, $h \in I^{n_j+1}$, for some $j \in I^N$. This last result is known as the simplicial Scarf lemma (see, e.g., [10]). We thus have shown that the simplicial Scarf lemma on S follows as a corollary of Lemma 2.3. Another corollary is the following intersection theorem for labelled subsets of S. See also [2].

2.4. THEOREM (Generalized Scarf lemma on S). *Let $C = \{C_{jh} | (j, h) \in I\}$ be a collection of possibly empty closed subsets of S that cover S, i.e., $\cup_{(j,h) \in I} C_{jh} = S$. Then there exist a point $x \in S$ and an index $j \in I^N$ such that, for each $h \in I^{n_j+1}$, $x \in C_{jh}$ or $x_{jh} = 0$.*

PROOF. The proof follows from Lemma 2.3 and a limit argument. First, let $\{G_k | k = 0, 1, \dots\}$ be a sequence of triangulations whose grid sizes go to 0 as k goes to infinity. An example of a triangulation that can be refined arbitrarily will be given in §4. Let $l\colon S \to I$ be a labelling of S verifying $l(x) = (j, h)$ only if $x \in C_{jh}$. Lemma 2.3 states that each G_k contains at least one complete simplex, say σ_k. There thus is an index $j \in I^N$ and a subsequence of j-complete simplices, $\{\sigma_{k'} | k' = 0, 1, \dots\}$ converging to a point $x \in S$. Suppose that $x_{jh} > 0$ for some $(j, h) \in I$. For sufficiently large

k', $\sigma_{k'}$ must have a vertex $w^{k'}$ with $l(w^{k'}) = (j, h)$. Hence, $w^{k'} \in C_{jh}$. Since $w^{k'}$ converges to x when k' goes to infinity, the closedness of C_{jh} implies that $x \in C_{jh}$.

2.5. COROLLARY. *The NLCP on S always has a solution.*

PROOF. Consider the NLCP on S with datum $z: S \to R^m$. For $(j, h) \in I$, let $C_{jh} = \{x \in S | z_{jh}(x) = \max_{(i,k) \in I} z_{ik}(x)\}$. The continuity of $z(\cdot)$ makes the sets C_{jh} closed. Theorem 2.4 guarantees the existence of an index $j \in I^N$ and a point $x \in S$ such that $x \in C_{jh}$ if $x_{jh} > 0$. Let $H = \{h \in I^{n_j+1} | x_{jh} > 0\}$. For all $h \in H$, $z_{jh}(x) = \max_{(i,k) \in I} z_{ik}(x)$. The identity $x_j' z_j(x) = 0$ yields $z_{jh}(x) = 0$ for $h \in H$. The inequality $z_i(x) \leqslant 0$ for all $i \in I^N$ is now immediate.

The following lemma bounds the accuracy of approximate solutions to the NLCP associated with complete simplices. The labelling is the one described in the proof of Corollary 2.5.

2.6. LEMMA. *Let $l: S \to I$ be a labelling such that $l(x) = (j, h)$ only if $z_{jh}(x) = \max_{(i,k) \in I} z_{ik}(x)$, with $z(\cdot)$ being the datum of an NLCP on S. Let $\eta > 0$ and let G be a simplicial subdivision of S whose mesh is such that $|z_{jh}(x) - z_{jh}(y)| < \eta$ for all $(j, h) \in I$ and for all points x and y lying in the same simplex of G. Any point x in a complete simplex σ satisfies $z_{jh}(x) < 2\eta$ for all $(j, h) \in I$.*

PROOF. Since σ is complete there exists an index $j \in I^N$ such that, for each $h \in I^{n_j+1}$, $x_{jh} = 0$ for all $x \in \sigma$ or σ has a vertex, say, y^{jh}, with $l(y^{jh}) = (j, h)$. Suppose that there is a point $x \in \sigma$ with $z_{ik}(x) \geqslant 2\eta$ for some $(i, k) \in I$. Let $H = \{h \in I^{n_j+1} | x_{jh} > 0\}$. Consider an arbitrary index $h \in H$. σ has a vertex, y^{jh}, with $l(y^{jh}) = (j, h)$. Since both x and y^{jh} lie in σ, $z_{ik}(y^{jh}) > \eta$. The labelling of y^{jh} then implies $z_{jh}(y^{jh}) \geqslant z_{ik}(y^{jh}) > \eta$. Hence $z_{jh}(x) > 0$ for all $h \in H$. This clearly contradicts $x_j' z_j(x) = 0$ and proves Lemma 2.6.

3. The van der Laan and Talman algorithm for proper labellings of S^n. In this section we review the van der Laan and Talman algorithm for finding a completely labelled simplex in a simplicial subdivision of a single properly labelled unit simplex, S^n. The algorithm first subdivides S^n into n-simplices according to the Q triangulation. The grid points of this triangulation form the set $\{y \in S^n | y_i = m_i/d,\ m_i \text{ integer},\ i \in I^{n+1}\}$, with d being a positive integer. The grid size of this triangulation is $1/d$. To define the triangulation further, let

$$Q = [q(1), \ldots, q(n+1)] = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & & -1 \\ -1 & 0 & 0 & \cdots & 1 \end{bmatrix} \in R^{(n+1)\times(n+1)}.$$

An n-simplex of the Q triangulation, $\sigma = \sigma(y^1, \ldots, y^{n+1})$, is the convex hull of $n + 1$ grid points, $y^1, \ldots, y^{n+1}$, satisfying $y^{i+1} = y^i + q(\pi_i)/d$, $i = 1, \ldots, n$, with $\pi = (\pi_1, \ldots, \pi_{n+1})$ being a permutation of I^{n+1}. Since $y^{n+1} = y^i + q(\pi_{n+1})/d$, an n-simplex σ has $n + 1$ representations in terms of a leading vertex, y^1, and a permutation, π.

A t-simplex $\sigma = \sigma(y^1, \ldots, y^{t+1})$, $0 \leqslant t \leqslant n$, in a simplicial subdivision is the convex hull of $t + 1$ vertices, $y^1, \ldots, y^{t+1}$, of an n-simplex in the subdivision. The $(t-1)$-simplex obtained by deleting a vertex of σ is called a facet of σ. More specifically, the facet $\tau(y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^{t+1})$ of σ is called the facet of σ opposite vertex y^i.

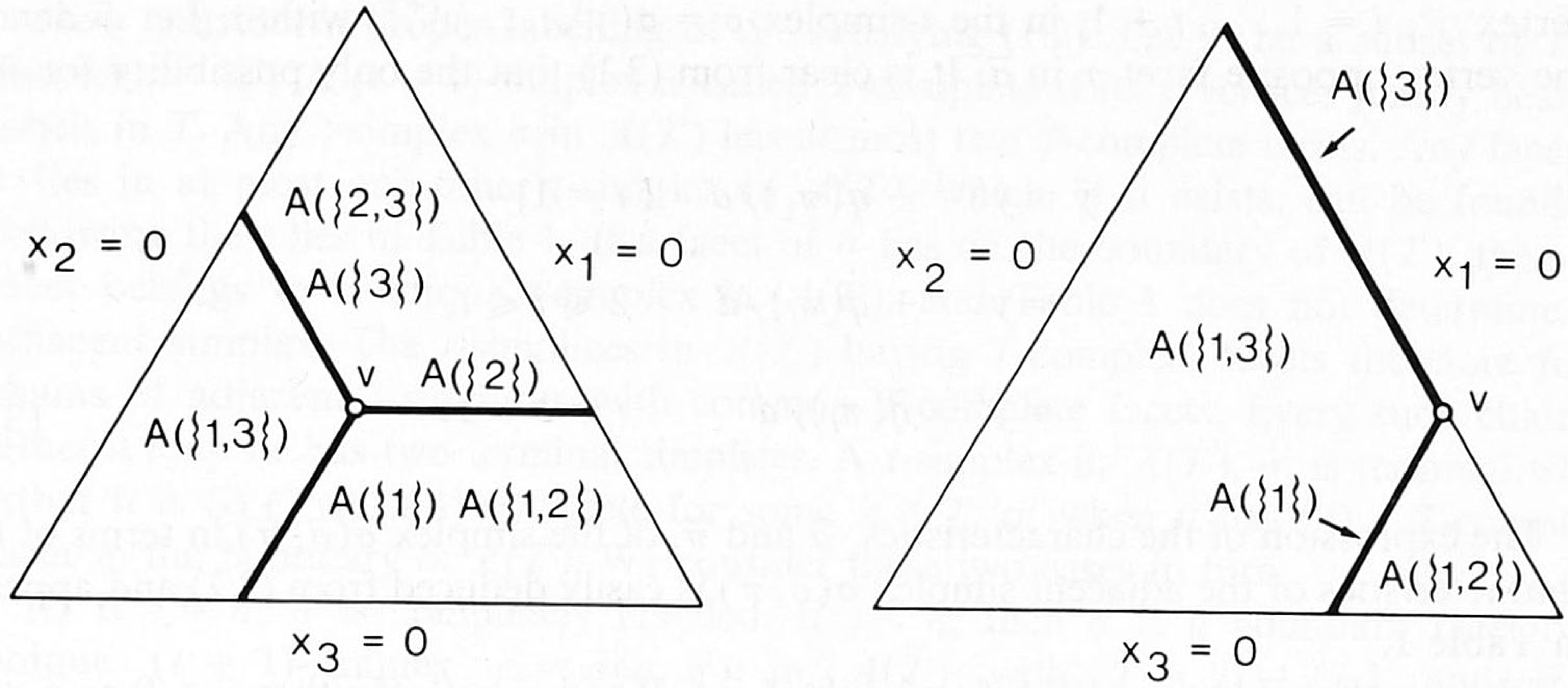

FIGURE 2. Illustration of the regions $A(T)$ in S^2.
a. v lies in the interior of S^2. b. v lies on the boundary of S^2 $[A(\{2\}) = A(\{2,3\}) = \varnothing]$

Two simplices are said to be adjacent if they have a common facet. The subsimplex obtained by deleting any number of vertices of a simplex σ is called a face of σ.

To compute a completely labelled simplex the van der Laan and Talman algorithm partitions S^n into relatively open regions $A^o(T) = \{y \in S^n | y = v + \sum_{j\in T}\alpha_j q(j), \alpha_j > 0\}$, where v is an arbitrary grid point and where T is a subset of I^{n+1} of cardinality at most n ($|T| \leqslant n$). Let $A(T)$ denote the closure of $A^o(T)$. The regions $A(T)$ are illustrated in Figure 2. Observe that $A(\varnothing) = A^o(\varnothing) = \{v\}$ and that $A(T)$ may be empty if the initial point v lies on the boundary of S^n.

Each nonempty region $A(T)$ is a t-dimensional convex, compact subset of S^n with $t = |T|$. The Q triangulation subdivides each nonempty $A(T)$ into t-simplices. Each such t-simplex is uniquely characterized by a nonnegative integer vector $a = (a_1, \ldots, a_{n+1})'$ satisfying $a_j = 0$ for $j \notin T$, and by a permutation π of T. Such a simplex is then denoted $\sigma(a, \pi)$. The $t+1$ vertices of the t-simplex $\sigma(a, \pi)$ in $A(T)$ are

$$
\begin{aligned}
y^i &= v + \sum_{j\in T} a_j q(j)/d, \quad i = 1, \\
&= y^{i-1} + q(\pi_{i-1})/d, \quad i = 2, \ldots, t+1.
\end{aligned}
\tag{3.1}
$$

Each n-simplex σ in S^n lies in a unique region $A(T)$. Hence, although σ has $n+1$ different representations in terms of a leading vertex and a permutation of I^{n+1}, its representation $\sigma = \sigma(a, \pi)$ is unique.

An important property of the given simplicial subdivision of each region $A(T)$ is that any facet of a t-simplex in $A(T)$ is the facet of at most one other t-simplex in $A(T)$ ($t = |T|$). We now identify the t-simplex $\bar{\sigma}$ in $A(T)$ sharing the facet τ opposite

TABLE 1

i	$\bar{a} = (\bar{a}_1, \ldots, \bar{a}_{n+1})'$		$\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_t)$
1	$\bar{a}_h = a_h + 1$	for $h = \pi_1$	$(\pi_2, \ldots, \pi_t, \pi_1)$
	$= a_h$	otherwise	
$2, \ldots, t$	$\bar{a}_h = a_h$	for $h = 1, \ldots, n+1$	$(\pi_1, \ldots, \pi_{i-2}, \pi_i, \pi_{i-1}, \ldots, \pi_t)$
$t+1$	$\bar{a}_h = a_h - 1$	for $h = \pi_t$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$
	$= a_h$	otherwise	

vertex y^i, $i = 1,\ldots, t+1$, in the t-simplex $\sigma = \sigma(y^1,\ldots, y^{t+1})$ with σ. Let $\bar{y}$ denote the vertex opposite facet τ in $\bar{\sigma}$. It is clear from (3.1) that the only possibility for $\bar{y}$ is

$$\begin{aligned} \bar{y} &= y^{t+1} + q(\pi_1)/d \quad \text{if } i = 1, \\ &= y^{i-1} + q(\pi_i)/d \quad 2 \leqslant i \leqslant t, \\ &= y^1 - q(\pi_t)/d \quad i = t+1. \end{aligned} \tag{3.2}$$

The expression of the characteristics, $\bar{a}$ and $\bar{\pi}$, of the simplex $\bar{\sigma}(\bar{a}, \bar{\pi})$ in terms of the characteristics of the adjacent simplex $\sigma(a, \pi)$ is easily deduced from (3.2) and appears in Table 1.

Observe that $\bar{\sigma}$ lies in $A(T)$ if and only if $\bar{y} \geqslant 0$ and $\bar{a} \geqslant 0$. If either $\bar{y} \ngeqslant 0$ or $\bar{a} \ngeqslant 0$, the facet τ lies on the boundary of $A(T)$ and there is no t-simplex in $A(T)$ adjacent with σ through facet τ.

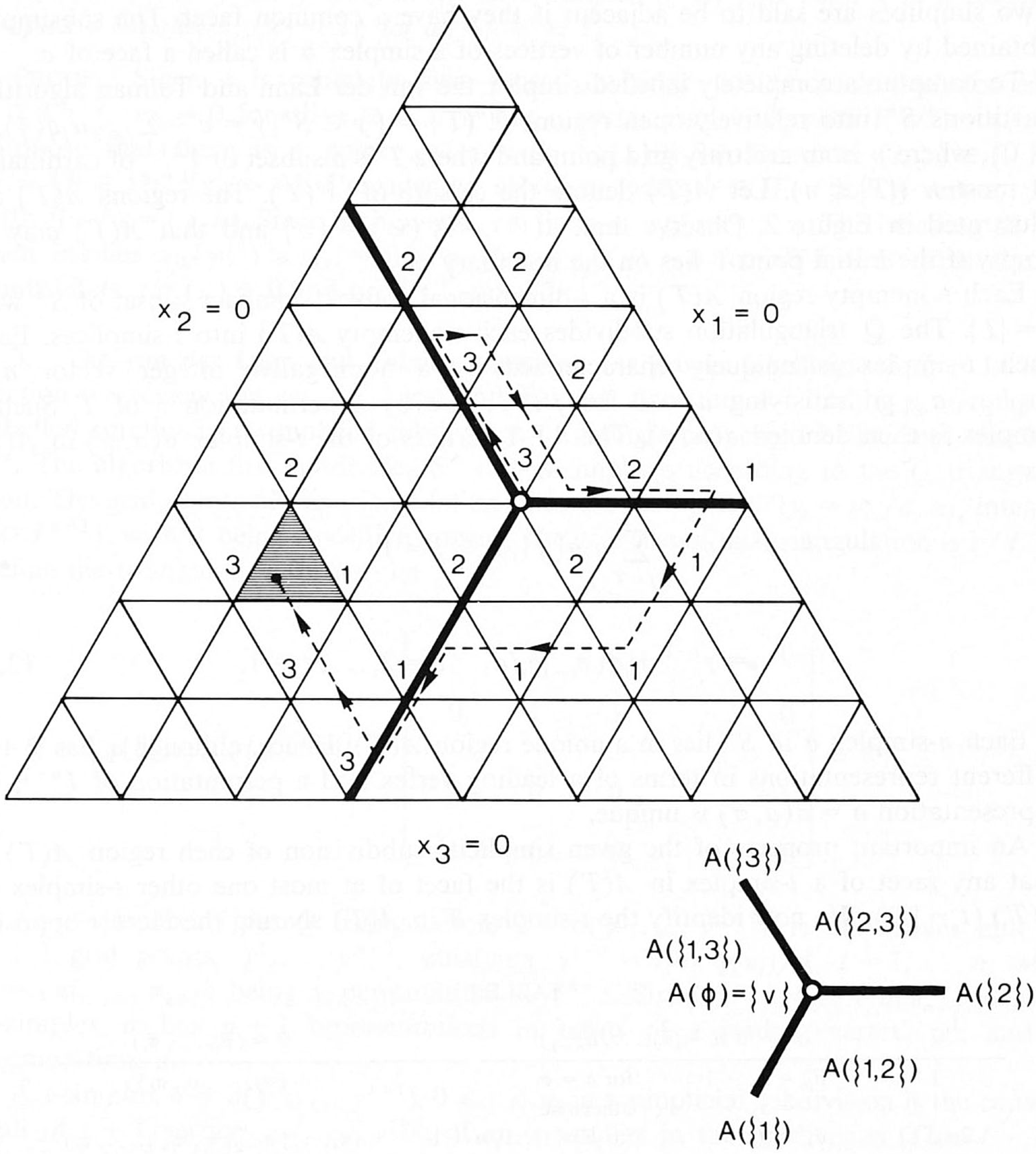

FIGURE 3. Illustration of the van der Laan and Talman algorithm for determining a completely labelled simplex in a properly labelled subdivision of S^2.

Now consider a proper labelling of S^n satisfying (1.2). Let T be a subset of I^{n+1} and let $t = |T|$. A $(t-1)$-simplex is called T-complete if its t vertices jointly bear all labels in T. Any t-simplex σ in $A(T)$ has at most two T-complete facets. Any facet of σ lies in at most one other t-simplex in $A(T)$, which, if it exists, can be found by following the rules in Table 1. If a facet of σ lies on the boundary of $A(T)$, then the facet belongs to a unique t-simplex in $A(T)$, and Table 1 does not determine an adjacent simplex. The t-simplices in $A(T)$ having T-complete facets therefore form chains of adjacent t-simplices with common T-complete facets. Every such chain is either a loop or has two terminal simplices. A t-simplex in $A(T)$, σ, is terminal when either it is (i) $(T \cup \{k\})$-complete for some $k \notin T$, or when it has (ii) a T-complete facet in the boundary of $A(T)$. We consider these two cases in turn.

(i) If $t = n$, σ is completely labelled. If $t < n$, then σ is a boundary facet of a unique $(t+1)$-simplex $\bar{\sigma} = \bar{\sigma}(a, \bar{\pi})$ in $A(\bar{T})$ with $\bar{T} = T \cup \{k\}$ and $\bar{\pi} = (\pi_1, \ldots, \pi_t, k)$. $\bar{\sigma}$ is a terminal simplex in a chain of $(t+1)$-simplices in $A(T)$ having common $\bar{T}$-complete facets.

(ii) The properness condition (1.2) implies that the T-complete facet, τ, lies in $A(\bar{T})$ with $\bar{T} = T \setminus \{k\}$, $k = \pi_t$. If $t > 1$, then τ is a terminal $(t-1)$-simplex in a chain of $(t-1)$-simplices in $A(\bar{T})$ having common $\bar{T}$-complete facets. If $t = 1$, then $\bar{T} = \varnothing$ and $\tau = \{v\}$. Notice that $T = \{l(v)\}$ for this to occur.

A simplex could present both features (i) and (ii), in which case the chain reduces to this single simplex. We nevertheless think of such one-simplex chain as having two terminal simplices to account for the two distinct termination criteria present in the simplex.

Except for the terminal simplex $\{v\}$, each terminal simplex of a chain is either completely labelled, or uniquely determines a terminal simplex of a new chain. Chains can thus be linked yielding loops, or paths with two terminal simplices. Except for the simplex $\{v\}$, each terminal simplex on a path is completely labelled. There thus is a unique path which has $\{v\}$ as terminal simplex. The van der Laan and Talman algorithm follows this path to the other terminal simplex which will be completely labelled. The algorithm is illustrated in Figure 3.

4. An algorithm for general labellings on S. The algorithm presented in the previous section was extended by van der Laan and Talman [6] to find a completely labelled simplex in a properly labelled triangulation of a product of unit simplices, S. This section generalizes the latter algorithm to general labellings of S.

The key to the new algorithm is to understand the extension of the regions $A(\cdot)$, as defined in the van der Laan and Talman algorithm, into larger regions $\tilde{A}(\cdot)$. Indeed, having defined a triangulation of these extended regions $\tilde{A}(\cdot)$, the new algorithm can be described quite simply as operating in the regions $\tilde{A}(\cdot)$ in the same fashion as the original algorithm operates in the regions $A(\cdot)$.

To motivate our modification of the original algorithm, we examine the case of a single unit simplex $S = S^n$ with $n = 2$. First, consider the nonproper labelling $l(y) = j$ for all $y \in S^2$, with $j \in I^3$. The unique complete simplex in any simplicial subdivision of S^2 is the 0-simplex $\{e(j)\}$, where $e(j)$ is the jth unit vector. To reach this complete 0-simplex from the starting 0-simplex $\{v\}$, it seems desirable to extend $A(\{j\})$ into a 1-dimensional piecewise linear region $\tilde{A}(\{j\})$ having extreme points v and $e(j)$. A natural extension of the regions $A(\{j\})$, $j \in I^3$, appears in Figure 4. For this 2-dimensional example, the regions $\tilde{A}(\{i, j\})$ coincide with the original regions $A(\{i, j\})$. Next, consider a general labelling of S^2, like the one indicated in Figure 4. The previous hint that the new algorithm operates in the by now familiar way in the regions $\tilde{A}(\cdot)$ should enable the reader to replicate the simplicial path generated by the new algorithm in its solution of the problem depicted in Figure 4.

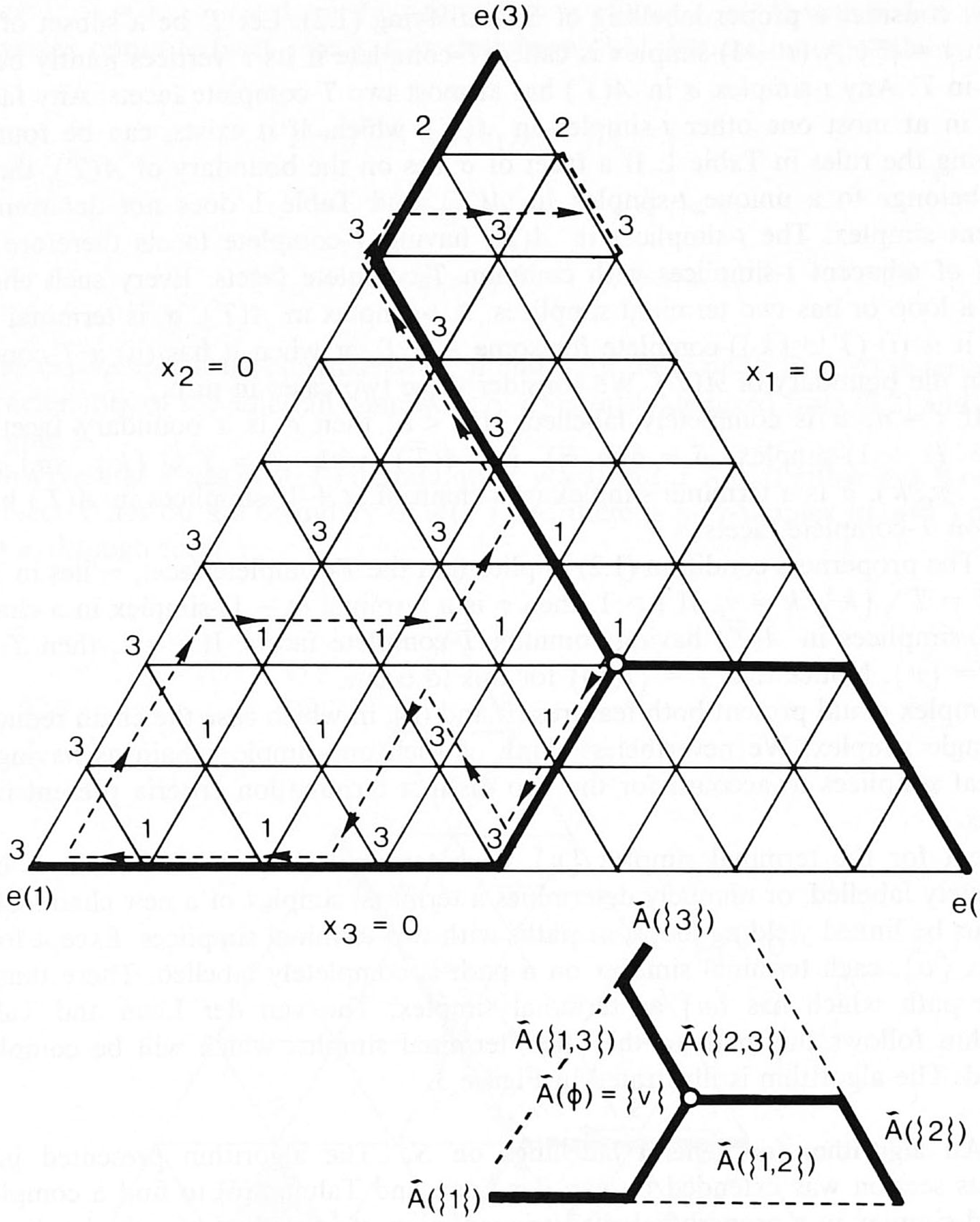

FIGURE 4

We now turn to the general case, $S = S^{n_1} \times \cdots \times S^{n_N}$. We first briefly review the subdivision of S into simplices according to the Q triangulation [6, 11]. The subdivision of a product of unit simplices is a straightforward generalization of the subdivision of a single unit simplex. The grid points of the Q triangulation of S form the set $G = \{ y \in S | y_{jh} = m_{jh}/d_j,\ m_{jh} \text{ integer}, (j, h) \in I\}$, with $(d_1, \ldots, d_N)$ being a vector of positive integers. The vector $(1/d_1, \ldots, 1/d_N)$ is the grid size vector of the triangulation. To specify the triangulation further, we generalize the matrix Q to be an $m \times m$ block diagonal matrix

$$Q = \begin{bmatrix} Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & Q_N \end{bmatrix}.$$

Each block Q_j is the $(n_j + 1) \times (n_j + 1)$ matrix used in § 3 for a Q triangulation of the unit simplex S^{n_j}. Let $j \in I^N$, $h \in I^{n_j+1}$, and let $g = \sum_{i=1}^{j-1}(n_i + 1) + h$. The gth column of Q is denoted $q(j, h)$. Hence $q(j, h) = e(j, h) - e(j, h')$ where $h' = h - 1 (\text{mod } n_j + 1)$ and where $e(j, k)$ denotes an m-dimensional unit vector having a $+1$ is position $\sum_{i=1}^{j-1}(n_i + 1) + k$. A leading vertex and a permutation of n columns of Q define an n-simplex of the Q triangulation of S, just as in the case $N = 1$ discussed in §3. The n columns of Q must, of course, be linearly independent. Hence, for each $j \in I^N$, the n chosen columns of Q contain, for each $j \in I^N$, exactly n_j columns of Q_j.

Given an arbitrary grid point $v \in G$, the van der Laan and Talman algorithm partitions S into relatively open regions $A^o(T) = \{y \in S | y = v + \sum_{(j,h) \in T} \alpha_{jh} q(j, h), \alpha_{jh} > 0\}$, where T is a subset of I such that the columns of Q with indices in T are linearly independent, i.e. $|T \cap I(j)| \leqslant n_j$ for each $j \in I^N$ with $I(j) = \{(j, h) | h \in I^{n_j+1}\}$. Such a subset T of I will be said to be independent. Let $A(T)$ denote the closure of $A^o(T)$.

Each nonempty region $A(T)$ is a t-dimensional, compact, convex subset of S, $t = |T|$. The Q triangulation subdivides each nonempty $A(T)$ into t-simplices. Each such t-simplex is uniquely characterized by a nonnegative vector $a = (a_1, \ldots, a_N) \in R^{n_1+1} \times \cdots \times R^{n_N+1}$ satisfying $a_{jh} = 0$ for $(j, h) \notin T$, and by a permutation π of T. The simplex is then devoted $\sigma(a, \pi)$. The $t + 1$ vertices of $\sigma(a, \pi)$ are

$$y^i = v + \sum_{(j,h) \in T} a_{jh} D^{-1} q(j, h), \qquad i = 1,$$

$$= y^{i-1} + D^{-1} q(\pi_{i-1}), \qquad i = 2, \ldots, t + 1,$$

where D is the $m \times m$ diagonal matrix whose (j, h)th diagonal element is equal to d_j, $(j, h) \in I$.

We now come to the main construction required for the extension of the van der Laan and Talman algorithm for proper labellings on S to general labellings. We extend the region $A(T)$ into the region $\tilde{A}(T) = \cup B(T, U)$, where the union is taken over all subsets U of I such that $T \cap U = \varnothing$ and $T \cup U$ independent, and where $B(T,U) = A(T \cup U) \cap \{x \in S | x_{jh} = 0 \text{ for all } (j, h) \in U\}$. Each nonempty $\tilde{A}(T)$ consists of a union of $|T|$-dimensional convex pieces.

Each nonempty $B(T, U)$ is subdivided by the Q triangulation into t-simplices, $t = |T|$. To describe these simplices we need to introduce some notation. With $(j, h) \in I$, let $p(j, h)$ denote the first index of the sequence $(j, h - 1), (j, h - 2), \ldots, (j, 1), (j, n_j + 1), \ldots, (j, h)$ not in U. Let $b(j, h)$ be such that $p(j, h) = (j, b(j, h))$ and let $c = b(j, h) + 1 \ (\text{mod } n_j + 1)$.

The set $P(j, h)$ is defined as follows:

$$P(j, h) = \{(j, c), (j, c + 1), \ldots, (j, n_j + 1), (j, 1), \ldots, (j, h)\} \qquad \text{if } c > h,$$

$$= \{(j, c), (j, c + 1), \ldots, (j, h)\} \qquad \text{if } c \leqslant h.$$

A t-simplex in $B(T, U)$ is characterized by a quadruple (T, U, a, π) where $a \in R^{n_1+1} \times \cdots \times R^{n_N+1}$ is a nonnegative vector satisfying $a_{jh} = 0$ for $(j, h) \notin T \cup U$, and where π is a permutation of the t elements of T. The simplex is then denoted $\sigma(T, U, a, \pi)$. The $t + 1$ vertices of $\sigma(T, U, a, \pi)$ are grid points in S satisfying

$$\text{(i)} \quad y^1 = v + \sum_{(j,h) \in T \cup U} a_{jh} D^{-1} q(j, h),$$

$$\text{(ii)} \quad y^i = y^{i-1} + D^{-1} r(\pi_{i-1}), \quad i = 2, \ldots, t + 1, \tag{4.1}$$

$$\text{(iii)} \quad U \subset \{(j, h) \in I | y^1_{jh} = 0\} \subset T \cup U,$$

with $r(j, h) = \sum_{(j,k) \in P(j,h)} q(j, k) = e(j, h) - e(j, b(j, h))$. Observe that (4.1.iii) identifies U as the index set of the facets of S containing $\sigma(T, U, a, \pi)$. The definition of the directions $r(\cdot)$ indicates that $\sigma(T, U, a, \pi)$ also is a face of a simplex in $A(T \cup U)$.

An important property of our simplicial subdivision of $B(T, U)$ is that any facet of a t-simplex in $B(T, U)$ is the facet of at most one other t-simplex $B(T, U)$, $t = |T|$. We identify the t-simplex $\bar{\sigma}$ in $B(T, U)$ sharing the facet τ opposite vertex y^i, $i \in I^{t+1}$, in the t-simplex $\sigma = \sigma(y^1, \ldots, y^{t+1})$ with σ. Let $\bar{y}$ denote the vertex opposite facet τ in $\bar{\sigma}$. It is clear from (4.1) that the only possibility for $\bar{y}$ is

$$\begin{aligned} \bar{y} &= y^{t+1} + D^{-1}r(\pi_1) \quad \text{if} \quad i = 1, \\ &= y^{i-1} + D^{-1}r(\pi_i) \qquad 2 \leqslant i \leqslant t, \\ &= y^1 - D^{-1}r(\pi_t) \qquad i = t+1. \end{aligned} \tag{4.2}$$

The expression of the characteristics $\bar{a}$ and $\bar{\pi}$ of the simplex $\bar{\sigma}(T, U, \bar{a}, \bar{\pi})$ in terms of the characteristics of $\sigma(T, U, a, \pi)$ is easily deduced from (4.2) and appears in Table 2.

The reader will observe that $\bar{\sigma}$ lies in $B(T, U)$ if and only if $\bar{y} \geqslant 0$ and $\bar{a} \geqslant 0$. If either $\bar{y}$ or $\bar{a}$ fail to be nonnegative, the facet τ lies on the boundary of $B(T, U)$ and there is no t-simplex in $B(T, U)$ adjacent with σ through facet τ.

It is immediate from (4.2) that $\bar{y} \not\geqslant 0$ only if

$$y^{t+1}_{p(\pi_1)} = 0 \quad \text{and} \quad i = 1,$$

$$y^{i-1}_{p(\pi_i)} = 0 \quad \text{and} \quad 2 \leqslant i \leqslant t,$$

$$y^1_{\pi_t} = 0 \quad \text{and} \quad i = t+1.$$

Furthermore, Table 2 indicates that $\bar{a} \not\geqslant 0$ only if $i = t + 1$ and $a_{jc} = 0$ with $c = b(j, h) + 1 \pmod{n_j + 1}$ and $(j, h) = \pi_t$.

We examine these cases in turn.

(i) $i = 1$: $y^{t+1}_{p(\pi_1)} = 0$. Let $\pi_1 = (j, h)$. The facet τ of σ lies in the face $x_{p(\pi_1)} = 0$ of S. If $|(T \cup U) \cap I(j)| = n_j$ then τ does not belong to any other region $B(\cdot)$; if $|(T \cup U) \cap I(j)| < n_j$ then τ is a facet of another simplex $\bar{\sigma}$ in $B(T, \bar{U})$ with

TABLE 2

i	$\bar{a}$		$\bar{\pi}$
1	$\bar{a}_{jh} = a_{jh} + 1$	$(j, h) \in P(\pi_1)$	$(\pi_2, \ldots, \pi_t, \pi_1)$
	$= a_{jh}$	otherwise	
$2, \ldots, t$	$\bar{a}_{jh} = a_{jh}$		$(\pi_1, \ldots, \pi_{i-2}, \pi_i, \pi_{i-1}, \ldots, \pi_{i+1}, \ldots, \pi_t)$
$t + 1$	$\bar{a}_{jh} = a_{jh} - 1$	$(j, h) \in P(\pi_t)$	$(\pi_t, \pi_1, \ldots, \pi_{t-1})$
	$= a_{jh}$	otherwise	

$\bar{U} = U \cup \{p(\pi_1)\}$. Having updated U, the characteristics $\bar{a}$ and $\bar{\pi}$ of the simplex $\bar{\sigma} = \bar{\sigma}(T, \bar{U}, \bar{a}, \bar{\pi})$ are as given in Table 2.

(ii) $2 \leqslant i \leqslant t$: $y^{i-1}_{p(\pi_i)} = 0$. Note that for this case to occur $\pi_{i-1} = p(\pi_i)$. τ is a $(t-1)$-simplex in $B(\bar{T}, \bar{U})$ with $\bar{T} = T \setminus \{\pi_{i-1}\}$ and $\bar{U} = U \cup \{\pi_{i-1}\}$.

(iii) $i = t+1$ and $y^1_{\pi_t} = 0$. τ is $(t-1)$-simplex in $B(\bar{T}, \bar{U})$ with $\bar{T} = T \setminus \{\pi_t\}$ and $\bar{U} = U \cup \{\pi_t\}$.

(iv) $i = t+1$, $y^1_{\pi_t} > 0$, and $a_{jc} = 0$ where $c = b(j, h) + 1 \pmod{n_j + 1}$, $(j, h) = \pi_t$. Let (j, k) be the index of the last zero element in the nondecreasing sequence $(a_{jc}, a_{j(c+1)}, \ldots, a_{jh})$. If $k = h$ then τ is a $(t-1)$-simplex in $B(\bar{T}, U)$ with $\bar{T} = T \setminus \{\pi_t\}$. If $k \neq h$ then $\bar{U} = U \setminus \{(j, k)\}$ and τ is a facet of a t-simplex $\bar{\sigma}$ in $B(T, \bar{U})$. The characteristics $\bar{a}$ and $\bar{\pi}$ of $\bar{\sigma} = \bar{\sigma}(T, \bar{U}, \bar{a}, \bar{\pi})$ are as given in Table 2.

We summarize our examination of the above four cases. If a facet τ of a t-simplex in $B(T, U)$ lies on the boundary of $B(T, U)$, then one and only one of the following holds:

(i) facet τ does not lie in any other region $B(\cdot)$;

(ii) facet τ is also a facet of a t-simplex in exactly one other region $B(T, \bar{U})$ with $\bar{U}$ differing from U in a single element;

(iii) facet τ is itself a $(t-1)$-simplex in exactly one other region $B(\bar{T}, \bar{U})$ with $|\bar{T}| = |T| - 1$ and $\bar{U}$ differing from U in at most one element.

Now consider a general labelling $l: S \to I$ and the problem of determining a complete simplex. Let T be a nonempty, independent subset of I and let $t = |T|$. The t-simplices in $B(T, U)$ having T-complete facets form chains of adjacent t-simplices with common T-complete facets. Each chain is either a loop, or has two terminal simplices. A t-simplex belonging to such a simplicial chain in $B(T, U)$ is terminal when (i) it has a T-complete facet on the boundary of $B(T, U)$, or when (ii) it has only one T-complete facet. As in §3, we continue to say that even a single simplex chain has two terminal simplices, thereby identifying that there are two ways in which the simplex is a terminal simplex. In the next paragraph, we show that, except for the starting simplex $\{v\}$, each terminal simplex either is complete, or uniquely determines a terminal simplex in another region $B(\bar{T}, \bar{U})$, where the sets $\bar{T}$ and $\bar{U}$ differ from T and U respectively in at most one element. The 0-simplex $\{v\}$ is a terminal simplex of a single simplicial chain, the chain of 1-simplices in $\tilde{A}(\{l(v)\}$ having $\{l(v)\}$-complete facets. Simplices in $\tilde{A}(T)$ having T-complete facets can, for varying T, be linked into chains of adjacent simplices of varying dimension. Each chain which is not a loop has two terminal simplices. The 0-simplex $\{v\}$ is the only terminal simplex that may not be complete. Our simplicial algorithm determines a complete simplex by following the chain for which $\{v\}$ is a terminal simplex to its other terminal simplex.

We now indicate that a terminal simplex of a chain of adjacent t-simplices in $B(T, U)$ having T-complete facets, $t = |T|$, either is complete, or uniquely determines a terminal simplex of a similar chain in a region $B(\bar{T}, \bar{U})$ with $\bar{T}$ and $\bar{U}$ differing from T and U respectively in at most one element. Let $\sigma = \sigma(T, U, a, \pi)$ be a terminal simplex in $B(T, U)$ having a T-complete facet τ. Let $y^1, \ldots, y^{t+1}$ be the vertices of σ and let τ be the facet opposite y^i in σ, $i \in I^{t+1}$. σ is terminal because (i) τ lies on the boundary of $B(T, U)$, or because (ii) σ is $\bar{T}$-complete, with $\bar{T} = T \cup \{(j, h)\}$, $(j, h) \notin T$. We examine both cases in turn.

(i) τ lies on the boundary of $B(T, U)$. There is no simplex $\bar{\sigma}$ in $B(T, U)$ adjacent with σ through facet τ. The preceding discussion of boundary facets of t-simplices in $B(T, U)$ yields the following four subcases.

(a) $i = 1$: $y^{t+1}_{p(\pi_1)} = 0$. Let $\pi_1 = (j, h)$. τ is either complete (if $|(T \cup U) \cap I(j)| = n_j$) or is a T-complete facet of a terminal simplex $\bar{\sigma} = \bar{\sigma}(T, \bar{U}, \bar{a}, \bar{\pi})$ in $B(T, \bar{U})$ with $\bar{U} = U \cup \{p(\pi_1)\}$ (if $|(T \cup U) \cap I(j)| < n_j$). Having updated U, the characteristics $\bar{a}$ and $\bar{\pi}$ are as given in Table 2.

(b) $2 \leqslant i \leqslant t$: $y^{i-1}_{p(\pi_i)} = 0$. τ is a terminal simplex in $B(\bar{T}, \bar{U})$ with $\bar{T} = T \setminus \{\pi_{i-1}\}$ and $\bar{U} = U \cup \{\pi_{i-1}\}$.

(c) $i = t + 1$ and $y^1_{\pi_t} = 0$. τ is a terminal simplex in $B(\bar{T}, \bar{U})$ with $\bar{T} = T \setminus \{\pi_t\}$ and $\bar{U} = U \cup \{\pi_t\}$.

(d) $i = t + 1$ and $y^1_{\pi_t} > 0$, and $a_{jc} = 0$ where $c = b(j, h) + 1 \pmod{n_j + 1}$, $(j, h) = \pi_t$. If $a_{jh} = 0$ then τ is a terminal simplex in $B(\bar{T}, U)$ with $\bar{T} = T \setminus \{\pi_t\}$. If $a_{jh} > 0$, let (j, k) be the index of the last zero element in the sequence $(a_{jc}, a_{j(c+1)}, \ldots, a_{jh})$. Then $\bar{U} = U \setminus \{(j, k)\}$ and τ is a T-complete facet of a t-simplex $\bar{\sigma} = \bar{\sigma}(T, \bar{U}, \bar{a}, \bar{\pi})$ in $B(T, \bar{U})$. The characteristics $\bar{a}$ and $\bar{\pi}$ are given in Table 2.

(ii) σ is $\bar{T}$-complete for some $\bar{T} = T \cup \{(j, h)\}$ with $(j, h) \notin T$.

(a) $(j, h) \notin U$. σ is either complete (if $|(T \cup U) \cap I(j)| = n_j$) or a facet of a terminal simplex $\bar{\sigma} = \bar{\sigma}(\bar{T}, U, a, \bar{\pi})$ in $B(\bar{T}, U)$ (if $|(T \cup U) \cap I(j)| < n_j$), with $\bar{\pi} = (\pi_1, \ldots, \pi_t, (j, h))$.

(b) $(j, h) \in U$. σ is a facet of a terminal simplex $\bar{\pi}$ in $B(\bar{T}, \bar{U})$ with $\bar{U} = U \setminus \{(j, h)\}$. To specify the characteristic $\bar{\pi}$ of $\bar{\sigma} = \bar{\sigma}(\bar{T}, \bar{U}, a, \bar{\pi})$, let (j, k) be the first index in the sequence $(j, h + 1), \ldots, (j, n_j + 1), (j, 1), \ldots, (j, h)$ not in $\bar{U}$. If $(j, k) \in T$ then $(j, k) = \pi_i$ for some $i \in I^t$ and $\bar{\pi} = (\pi_1, \ldots, \pi_{i-1}, (j, h), \pi_i, \ldots, \pi_t)$. If $(j, k) \notin T$ then $\bar{\pi} = (\pi_1, \ldots, \pi_t, (j, h))$.

We have thus shown that, except for the terminal simplex $\{v\}$, every terminal simplex in $B(T, U)$ either is complete, or uniquely determines a terminal simplex in a different region $B(\bar{T}, \bar{U})$.

The algorithm, starting at the 0-simplex $\{v\}$, generates a simplicial chain going through various regions $\tilde{A}(T)$, where the simplices generated in $\tilde{A}(T)$ have T-complete facets. The steps followed by the algorithm in its simplicial movement can be described as follows.

Step 0 [Initialization]. Set $T = \varnothing$, $U = \{(j, h) \in I \mid v_{jh} = 0\}$, $a = 0$, $\pi = \varnothing$ and $\bar{y} = v$.

Step 1 [Computation of the label of the incoming vertex]. Compute $l(\bar{y})$. Let $l(\bar{y})$ be equal to (j, h). Proceed to one of the following subcases.

(a) $l(\bar{y}) \notin T \cup U$. If $|(T \cup U) \cap I(j)| = n_j$, then $\sigma(T, U, a, \pi)$ is j-complete; stop. If not, set $i = t + 1$ and go to Step 2.

(b) $l(\bar{y}) \in T$. Determine the vertex y^i of $\sigma(T, U, a, \pi)$ for which $l(y^i) = l(\bar{y})$ and $y^i \neq \bar{y}$. Go to Step 3.

(c) $l(\bar{y}) \in U$. Identify the first element, say (j, k), in the sequence $(j, h + 1), \ldots, (j, n_j + 1), (j, 1), \ldots, (j, h)$ not belonging to U. If $(j, k) \notin T$, set $i = t + 1$. If $(j, k) \in T$, let i be the index such that $\pi_i = (j, k)$. Proceed to Step 2.

Step 2 [Increase in the number of elements of T with possible decrease in the number of elements of U]. Set

$$\bar{T} = T \cup \{l(\bar{y})\},$$

$$\bar{U} = U \qquad \text{if } l(\bar{y}) \notin U,$$

$$= U \setminus \{l(\bar{y})\} \qquad l(\bar{y}) \in U,$$

$$\bar{\pi} = (\pi_1, \ldots, \pi_{i-1}, l(\bar{y})) \qquad \text{if } i = t + 1,$$

$$= (\pi_1, \ldots, \pi_{i-1}, l(\bar{y}), \pi_i, \ldots, \pi_t) \quad 1 \leqslant i \leqslant t.$$

Let $\bar{y} = y^i + D^{-1} r(\bar{\pi}_i)$. Set $T = \bar{T}$, $U = \bar{U}$, $\pi = \bar{\pi}$, and return to step 1.

Step 3 [Replacement of y^i in $\sigma(T, U, a, \pi)$]. As long as the conditions below are not satisfied, a and π are updated as in Table 2. Let $\bar{y}$ be the new vertex of $\sigma(T, U, \bar{a}, \bar{\pi})$, as given in (4.2). Set $a = \bar{a}$, $\pi = \bar{\pi}$, and return to Step 1. The exceptions to the above rule are:

(a) $i = 1$ and $y^{t+1}_{p(\pi_1)} = 0$. [Increase in the number of elements of U.] Let π_1 be equal to (j, h). Set $\bar{U} = U \cup \{p(\pi_1)\}$. If $|(T \cup U) \cap I(j)| = n_j$ then $\bar{\sigma}(T, \bar{U}, a, \pi)$ is complete; stop. If not then the updates $\bar{a}$ and $\bar{\pi}$ are computed as in Table 2 and let $\bar{y} = y^{t+1} + D^{-1} r(\bar{\pi}_t)$. Set $U = \bar{U}$, $a = \bar{a}$, $\pi = \bar{\pi}$, and return to Step 1.

(b) $2 \leqslant i \leqslant t$ and $y^{i-1}_{p(\pi_i)} = 0$. Go to Step 4.

(c) $i = t + 1$ and $a_{jc} = 0$, where $\pi_t = (j, h)$ and $c = b(j, h) + 1 \pmod{n_j + 1}$. Go to Step 4 if $y^1_{\pi_t} = 0$. If $y^1_{\pi_t} > 0$, let (j, k) be the index of the last zero element in the sequence $(a_{jc}, a_{j(c+1)}, \ldots, a_{jh})$. If $(j, k) = \pi_t$ proceed to Step 4. If $(j, k) \neq \pi_t$ go to Step 5.

Step 4 [Decrease in the number of elements of T with possible increase in the number of elements of U]. Set

$$\bar{T} = T \setminus \{\pi_{i-1}\},$$

$$\begin{aligned} \bar{U} &= U \cup \{\pi_{i-1}\} \quad \text{if } y^1_{\pi_{i-1}} = 0, \\ &= U \qquad\qquad\quad\; y^1_{\pi_{i-1}} > 0, \end{aligned}$$

$$\bar{\pi} = (\pi_1, \ldots, \pi_{i-2}, \pi_i, \ldots, \pi_t).$$

Let $\bar{y}^i$ be the vertex in $\bar{\sigma}(\bar{T}, \bar{U}, a, \bar{\pi})$ with $l(\bar{y}^i) = \pi_{i-1}$. Set $T = \bar{T}$, $U = \bar{U}$, $\pi = \bar{\pi}$ and $i = i$, and return to Step 3.

Step 5 [Decrease in the number of elements of U]. Set

$$\bar{U} = U \setminus \{(j, k)\},$$

$$\begin{aligned} \bar{a}_{jh'} &= a_{jh'} - 1 \quad \text{for } (j, h') \in \bar{P}(j, h), \\ &= a_{jh'} \qquad\quad \text{otherwise}, \end{aligned}$$

$$\bar{\pi} = (\pi_t, \pi_1, \ldots, \pi_{t-1}).$$

Let $\bar{y} = y^1 - D^{-1} r(\bar{\pi}_1)$. Set $U = \bar{U}$, $a = \bar{a}$, $\pi = \bar{\pi}$, and return to Step 1.

This concludes the description of the steps followed by the algorithm in its simplicial movement. The possible number of different simplices visited by the algorithms being finite, the algorithm will terminate finitely with a complete simplex unless it returns to a previously visited simplex. The first simplex visited twice would be reachable in three different ways from adjacent positions [7]. This is impossible as we have shown that any simplex of a simplicial chain is adjacent to at most two simplices of the same chain, with the 0-simplex $\{v\}$ being adjacent to at most one simplex of the same chain. Another well-known way to prove the absence of cycling is to verify that the algorithm's steps satisfy the 'reversibility property.' To explain this property, assume, that after initializing the algorithm at a position associated with a simplex σ, the steps of the algorithm generate the vertex $\bar{y}$ and a new position associated with the simplex $\bar{\sigma}$. The algorithm obeys the 'reversibility property' if simplicial movement in the 'reverse direction' (i.e. through the facet of $\bar{\sigma}$ opposite $\bar{y}$) returns the algorithm to σ. If satisfied, this property clearly eliminates any possibility of returning to a previously visited position, except the initial one. But the latter is excluded as well, because the starting position can be reached (and left) in only one way.

The convergence of the algorithm to a complete simplex constructively proves Lemma 2.3 for the Q triangulation. The proof for an arbitrary triangulation of S is similar. Given a vertex v of S, one defines 1-dimensional simplicial paths, intersecting only in v, from v to each of the vertices of S. These paths are then used to define the regions $A(T)$, and then the extended regions $\tilde{A}(T)$, just as we did for the Q triangulation. For more details, we refer to [2], where v is chosen to be one of the vertices of S.

5. Vector labelling. Approximate solutions to the NLCP on S can also be computed using vector labelling. The extension from integer to vector labels is standard (see, e.g., [10], [6]). We highlight differences that arise with [6] because of the improperness of the labelling.

To approximate solutions for the NLCP on S with datum $z: S \to R^m$ using vector labelling, every point in S is associated with a vector according to the rule $l: S \to R^m$, $x \to l(x) = z(x) + Me$, with $e = (1, \dots, 1)' \in R^m$. The positive scalar M is chosen large enough such that $-Me$ is a vector that bounds the function $z(\cdot)$ from below, i.e. $z(x) + Me > 0$ for all $x \in S$. The continuity of $z(\cdot)$ ensures the existence of such a scalar M.

Given the vector labelling $l(\cdot)$, a t-simplex $\sigma = \sigma(y^1, \dots, y^{t+1})$ lying on the boundary $\partial S(U) = \{x \in S | x_{jh} = 0 \text{ for } (j, h) \in U\}$ of S is said to be complete if the linear system

$$\sum_{i=1}^{t+1} l(y^i)\lambda_i + \sum_{(j,h) \in I} e(j, h)\mu_{jh} = Me \tag{5.1}$$

has a nonnegative solution λ_i, μ_{jh} verifying, for some $j \in I^N$, $\mu_{jh} > 0$ only if $(j, h) \in U$. Simplex σ is then also said to be j-complete. The following lemma states that a complete simplex yields an approximate solution of the NLCP on S.

5.2. LEMMA. *Let $z: S \to R^m$ be the datum of an NLCP on S. Let $l: S \to R^m$, $x \to l(x) = z(x) + Me$ be the associated vector labelling of S. Let $\eta > 0$ and let G be a simplicial subdivision of S whose mesh is such that $|z_{jh}(x) - z_{jh}(y)| < \eta$ for all $(j, h) \in I$ whenever x and y lie in the same simplex. Finally, let $\sigma = \sigma(y^1, \dots, y^{t+1})$ be a complete simplex in the boundary $\partial S(U)$ of S, $U \subset I$. Any point $y \in \sigma$ satisfies $z_{jh}(y) < 2\eta$ for all $(j, h) \in I$.*

PROOF. Let λ_i and μ_{jh} denote the weights associated with the complete simplex σ and satisfying (5.1). Let $j \in I^N$ be such that $\mu_{jh} > 0$ only if $(j, h) \in U$. Further, let $\lambda = \sum_{i=1}^{t+1}\lambda_i$ and $U(j) = I(j) \cap U$.

We first claim that $\lambda > 0$. If not $\mu_{jh} = M$ for all $(j, h) \in I(j)$ and, hence, $U(j) = I(j)$. This is impossible as no simplex in a simplicial subdivision of S^j can be contained in all facets of S^j.

Next, observe that $\mu_{jh} = 0$ for $(j, h) \in \overline{U}(j)$, where $\overline{U}(j) = I(j) \setminus U$. Equation (5.1) can be rewritten

$$\sum_{i=1}^{t+1} \left(z_{jh}(y^i) + M\right)\lambda_i = M \quad \text{for } (j, h) \in \overline{U}(j). \tag{5.2}$$

Simple manipulations yield

$$\sum_{i=1}^{t+1}\left(z_{jh}(y^i)-z_{jh}(y)\right)\lambda_i/\lambda+z_{jh}(y)=M(1-\lambda)/\lambda \quad \text{for } (j,h)\in\bar{U}(j). \quad (5.3)$$

Observe that, since $y\in\sigma$, $y_{jh}=0$ for $(j,h)\in U(j)$ and $\Sigma_{(j,h)\in\bar{U}(j)}y_{jh}=1$. Hence $0=(y_j)'z_j(y)=\Sigma_{(j,h)\in\bar{U}(j)}y_{jh}z_{jh}(y)$. Summation of the above equations, after multiplication with the weights y_{jh}, yields

$$\sum_{(j,h)\in\bar{U}(j)}y_{jh}\sum_{i=1}^{t+1}\left(z_{jh}(y^i)-z_{jh}(y)\right)\lambda_i/\lambda=M(1-\lambda/\lambda).$$

The inequality $M|1-\lambda/\lambda|<\eta$ easily follows as the vectors $y, y^1,\ldots,y^{t+1}$ all lie in σ. Equation (5.3) then yields $z_{jh}(y)<2\eta$ for $(j,h)\in\bar{U}(j)$. The validity of the inequality for any $(j,h)\in I$ follows from (5.1) in the same manner if we first recall that the weights μ_{jh} are nonnegative. ■

To use the algorithm of §4 with vector labelling we need to extend the notion of T-completeness. A $(t-1)$-simplex $\tau=\tau(y^1,\ldots,y^t)$ in $\tilde{A}(T)$, $t=|T|$, is said to be T-complete if the system of linear equations

$$\sum_{i=1}^{t}l(y^i)\lambda_i+\sum_{(j,h)\in I}e(j,h)\mu_{jh}=Me \quad (5.4)$$

has a nonnegative solution λ_i, μ_{jh} with $\mu_{jh}>0$ only if $(j,h)\notin T$. We assume, without loss of generality, that linear system (5.4) is nondegenerate. The linear system, when feasible, then has a unique solution.

Consider a t-simplex $\sigma=\sigma(y^1,\ldots,y^{t+1})$ having a T-complete facet $\tau=\tau(y^1,\ldots,y^t)$. A linear programming pivot step introducing column $l(y^{t+1})$ into the feasible basis of (5.4) eliminates from the basis either a column $l(y^j)$, $1\leqslant j\leqslant t$, or a column $e(j,h)$, $(j,h)\notin T$. The first occurrence identifies a second T-complete facet in σ, $\bar{\tau}=\bar{\tau}(y^1,\ldots,y^{j-1},y^{j+1},\ldots,y^{t+1})$. The second occurrence indicates that σ is $\bar{T}$-complete with $\bar{T}=T\cup\{(j,h)\}$. The fact that the labels are nonnegative vectors different from zero implies that the linear system is bounded. The nondegeneracy assumption then ensures that every linear programming pivot step identifies a unique column to be removed from the current basis.

As in the case of integer labels, T-complete facets of simplices in $\tilde{A}(T)$ can, for varying T, be linked to form simplicial chains. The algorithm again follows the unique simplicial chain having the starting 0-simplex $\{v\}$ of endpoint to its other terminal simplex, which will be complete. The vector labelling algorithm accomplishes this by alternate linear programming pivot steps and simplicial replacement steps, outlined in the preceding section. The linear programming step determines whether a vertex of the current simplex needs to be replaced, or whether a new vertex needs to be added to the current simplex because of the extension of the label set T. The reverse of the latter step is called for when, for some $(j,h)\in T$, a T-complete facet, τ, is reached in $\tilde{A}(\bar{T})$ with $\bar{T}=T\setminus\{(j,h)\}$. Let $\tau=\tau(y^1,\ldots,y^t)$ with $t=|T|$. The unit vector $e(j,h)$ is introduced into the current basis for linear system (5.4). A linear programming pivot step removes a label $l(y^j)$, $1\leqslant j\leqslant t$, from the basis. The algorithm then pursues its simplicial movement by leaving simplex τ through its facet opposite vertex y^j.

6. Applications and numerical results.

Application 1. We first consider the strictly convex quadratic programming problem with quadratic constraints (QPQC)

$$\begin{aligned} &\min \quad Q_{n+1}(x) \\ &\text{s.t.} \quad Q_i(x) \leqslant 0, \qquad i \in I^n, \qquad (6.1) \\ &\qquad x \in P, \end{aligned}$$

where each $Q_i: R^k \to R$, $x \to Q_i(x)$ is a strictly convex quadratic function and where P is a nonempty compact polyhedron in R^m. See Phan-huy-Hao [9].

Let $x: S^n \to P$, $u \to x(u)$ denote the unique solution of the convex quadratic programming problem (with linear constraints)

$$\min \sum_{i=1}^{n+1} u_i Q_i(x) \quad \text{s.t.} \quad x \in P.$$

The mapping $x(\cdot)$ is easily shown to be continuous. Further, define the sets C_i, $i \in I^{n+1}$, as follows:

$$\begin{aligned} C_i &= \{u \in S^n | Q_i(x) = \max_{j \in I^n} Q_j(x(u)) \geqslant 0\} \quad \text{for } i = 1, \ldots, n, \\ &= \{u \in S^n | Q_j(x(u)) \leqslant 0 \text{ for all } j \in I^n\} \qquad = n+1. \end{aligned}$$

Theorem 2.4 states the existence of a vector $u \in S^n$ such that $u_i = 0$ or $u \in C_i$, $i \in I^{n+1}$. The reader will easily verify that, if $u_{n+1} > 0$, then $x(u)$ is an optimal solution of QPQC. The positivity of u_{n+1} is ensured by a standard constraint qualification requiring the existence of a point $y \in P$ satisfying $Q_i(y) < 0$ for all $i \in I^n$.

To compute a solution of QPQC we use the algorithm with integer labelling. The labelling is the one used in the proof of Theorem 2.4, $l(u) = \min\{i \in I^{n+1} | u \in C_i\}$.

The data for the three QPQC's solved appear in [9]. Our computational results are given in Table 3. A comparison with the results in [9] is difficult because of the absence of a precise description in [9] of the algorithm used, and because of an apparent difficulty in solving the third problem. The reported number of iterations required in that problem to refine a grid of size 10^{-1} by a factor of 10 is 278. The number of

TABLE 3

Cumulative Number of Function Evaluations Required to Solve The Three QPQC's Given in [9] with Integer Labelling (n = dimension)

Grid size	Function evaluations (cumulative)		
(d)	Problem 1 ($n = 2$)	Problem 2 ($n = 2$)	Problem 3 ($n = 4$)
$5 \cdot 10^{-1}$	4	5	2
$5 \cdot 10^{-2}$	7	7	4
$5 \cdot 10^{-3}$	10	11	7
$5 \cdot 10^{-4}$	15	18	13

iterations required in [9] to solve the second problem are approximately three times ours. We also solved the three problems using vector labelling. The resulting number of function evaluations did not differ much from the number reported in Table 3.

Application 2. Our second application concerns the computation of equilibrium strategy vectors of a noncooperative N-person game. Let $n_j + 1$ be the number of pure strategies of player $j \in I^N$, and let K denote the product of the index sets I^{n_j+1}. A vector $k = (k_1, \ldots, k_N) \in K$ denotes the pure strategy vector in which player $j \in I^N$ plays pure strategy $k_j \in I^{n_j+1}$. Furthermore, let $a^j(k)$ be the loss to player j if k is played. We assume without loss of generality that $a^j(k) > 0$ for all $k \in K$ and all $j \in I^N$. The simplex S^{n_j} is the mixed strategy space for player j. The simplicial product $S = S^{n_1} \times \cdots \times S^{n_N}$ is the strategy space of the noncooperative game, i.e., if $x \in S$, then x_{jh} denotes the probability that player j uses pure strategy $h \in I^{n_j+1}$. The expected loss to player j if x is played is $p^j(x) = \sum_{k \in K} a^j(k) \prod_{i=1}^N x_{ik_i}$. The expected loss to player j if he plays pure strategy h rather than x_j, and the other players maintain their strategies as given by x, is

$$m_h^j(x) = \sum_{k \in K_{jh}} a^j(k) \prod_{\substack{i=1 \\ i \neq j}}^N x_{ik_i},$$

with $K_{jh} = \{k \in K | k_j = h\}$. Observe that for each x in S and for each $j \in I^N$, $p^j(x) = \sum_{h=1}^{n_j+1} x_{jh} m_h^j(x)$. A point $x \in S$ is an equilibrium strategy vector for this noncooperative game if, for each $j \in I^N$, $p^j(x) \leqslant m_h^j(x)$ for all $h \in I^{n_j+1}$.

The computation of an equilibrium strategy vector is equivalent to solving an NLCP. The datum for this NLCP is $z : S \to R^m$, $x \to z(x) = (z_{jh}(x)) = (p^j(x) - m_h^j(x))$. Corollary 2.5 then states that every noncooperative multiperson game has an equilibrium strategy vector.

The integer and vector labelling versions of the algorithm were applied to solve three noncooperative multiperson games. The data for these problems are given in Appendix. We also applied the integer and vector labelling algorithms for proper labellings described in [6]. Table 4 summarizes the numerical results obtained with both algorithms when computing an approximate equilibrium strategy with accuracy $\max_{(j,h) \in I} z_{jh}(x) < 10^{-8}$. Each restart refines the grid size by the factor two. Table 4 shows that the elimination of artificial labels can considerably shorten the computation.

TABLE 4

Cumulative number of function evaluations (and linear programming pivot steps when applicable) required to solve the three games described in the appendix up to accuracy $\max_{(j,h) \in I} z_{jh}(x) < 10^{-8}$.

Game	Labelling			
	Integer		Vector	
	Proper	General	Proper	General
1	289	282	158 (146)	162 (149)
2	981	218	1120 (1758)	20 (20)
3	376	133	682 (653)	99 (77)

The table compares the use of a proper labelling (this is the algorithm presented in [6]) and a general labelling. Results are presented for both integer and vector labelling.

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Appendix: Loss tables for noncooperative multiperson games

Game 1: $N = 3,\ n_j + 1 = 2.$

	(1,1)	(1,2)	(2,1)	(2,2)
(1,1)	1	2	8	5
(1,2)	8	8	2	2
(2,1)	4	2	2	1
(2,2)	2	6	1	3
(3,1)	4	1	4	2
(3,2)	8	8	2	1

The key to reading the above loss table is as follows. The entry in row (j, p) and column (q, r) is the loss to player j if he plays pure strategy p and his opponents, ordered in increasing value of their index, play their pure strategies q and r, respectively. For example, the entry in row $(3, p)$ and column (q, r) represents the value of $a^3(q, r, p)$.

Game 2: $N = 3,\ n_j + 1 = 3.$

	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	2	3	4	2	3	3	4	1	5
(1,2)	1	1	4	3	4	1	6	8	2
(1,3)	4	7	2	4	5	5	3	6	4
(2,1)	5	6	7	4	8	9	3	5	1
(2,2)	1	1	3	3	2	1	2	2	4
(2,3)	2	3	6	5	3	6	7	5	8
(3,1)	1	3	5	1	6	2	1	2	4
(3,2)	2	6	5	3	3	7	8	5	5
(3,3)	5	2	2	4	6	5	8	1	3

Game 3: $N = 4,\ n_j + 1 = 2.$

	(1,1,1)	(1,1,2)	(1,2,1)	(1,2,2)	(2,1,1)	(2,1,2)	(2,2,1)	(2,2,2)
(1,1)	3	3	4	2	3	3	4	1
(1,2)	4	1	4	3	1	1	6	8
(2,1)	4	6	2	4	5	3	3	6
(2,2)	5	2	7	4	8	6	3	5
(3,1)	1	6	3	3	3	3	3	2
(3,2)	2	2	6	5	4	6	1	5
(4,1)	6	3	5	1	3	2	3	2
(4,2)	2	6	5	3	4	7	1	5

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