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ON THE LOCUS OF THE NUCLEOLUS.

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Abstract In this paper we prove some properties of the locus of the nucleolus of a set C that can be the core of a cooperative game. We show that there are two subsets of C, the kernel and the least core, only dependent on the set C that contain this locus. From this fact we derive that for a set C which is the core of *convex game* the locus consists of one point. We also give a necessary and sufficient condition that the locus consists of one point.

1. Introduction

It was during the Oberwolfach Conference in January 1989 that Michael Maschler forwarded the following question. If the core of a cooperative game is given what can be said about the position of the nucleolus? In this paper we will give a partial answer to that question. We prove that every set C that can be the core of a cooperative game contains two subsets, the kernel and the least core of C, only dependent on C, which contain all nucleoli of games with core C. In special cases, when the intersection of the kernel and the least core consists of one point we can infer that 'the core determines the nucleolus'.

Let N be a finite player set and C be a (nonempty) compact convex set in R^N. For every coalition $T \subset N$ we define $a_T := \min \{(e_T, x) \mid x \in C\}$. We call the set C *a pre-core* if

$$C = \{x \in R^N \mid (e_N, x) = a_N,\ (e_T, x) \geq a_T \text{ for all } T \subset N\}.$$

It is immediately clear that the game v_C defined by $v_C(S) = a_S$ for all $S \subset N$ has core C and that the game v_C is an exact game (i.e. for every coalition S there is a core element $x \in C$ with $x(S) := \sum_{i \in S} x_i = v_C(S)$ [Schmeidler (1972)]). As one can see from the definition the exact game v_C is uniquely determined by the set C and C is conversely the core of v_C:

There is a one-to-one correspondence between pre-cores and exact games.

The key concept of this paper will be the *locus of the nucleolus* in C. If C is a pre-core,the *locus of the nucleolus* $\mathcal{L}oc\,\mathcal{N}uc\,(C)$ is the set of points $x \in C$ which can be the nucleolus of a cooperative game with core C.

The following example from [Maschler, Peleg and Shapley (1979)] shows that $\mathcal{L}oc\,\mathcal{N}uc\,(C)$ may consist of more than one point i.e. the core of a game does not determine the nucleolus of a game.

Example 1: Let C be the set $\{(s, 1-s, s, 1-s) \mid 1/4 \leq s \leq 1\}$. For $0 \leq t \leq 1/4$ we define the games v_t by: $v_t(N) = 2$, $v_t(123) = 1+t$, $v_t(S) = 1$ for $S = (124)$, (134), (234), (12), (23), (34) and (14). Further $v_t(13) = 1/2$ and $v_t(S) = 0$ for all other coalitions.

It is an easy exercise to prove that the core of all the games v_t is C and that the nucleolus of v_t corresponds to $s = 1/2(1+t)$. These points are in the set $\mathcal{L}oc\,\mathcal{N}uc\,(C)$. The reason that the nucleolus "moves" when t increases is that coalition $S = (123)$ is redundant for the determination of the core (as long as $t \leq 1/4$) but is influencing the position of the nucleolus.

Conclusion: The locus of the nucleolus of a pre-core C may consist of more than one point.

In the next section we will introduce two subsets of C, the *kernel* and the *least core* of C which satisfy the properties:

(1) These sets contain the locus of the nucleolus $\mathcal{L}oc\mathcal{N}uc\,(C)$.

(2) They are only dependent on the pre-core C.

2. The kernel and the least core of a pre-core C

The first proposition is well known (cf. [Maschler, Peleg and Shapley (1979)]) and we give it here for completeness only.

If (N, v) is a cooperative game, the *kernel* $\mathcal{K}(v)$ *of* v is the set of points $x \in \mathcal{I}(v)$ for which
$\max\{v(S) - x(S) \mid i \in S \subset N\backslash j\} := s_{ij}(x) > s_{ji}(x) := \max\{v(S) - x(S) \mid j \in S \subset N\backslash i\}$
implies $x_j = v(j)$ [Davis and Maschler (1965)].

The following proposition shows that for a game (N, v) with nonempty core C the intersection of C with the kernel of (N, v) is only dependent on C and contains the nucleolus of (N, v).

Proposition 1. *If (N, v) is a cooperative game and C is the non-empty core of v then $\mathcal{K}(v) \cap C$ is determined by C only (and contains the nucleolus of v).*

Proof: We shall prove that the set $\mathcal{K}(v) \cap C$ consists of the points $x \in C$ for which

$$(2.1) \qquad \max\{t \in R \mid x + t e_i - t e_j \in C\} + \min\{t \in R \mid x + t e_i - t e_j \in C\} = 0 \text{ for all } i \neq j.$$

Let $x \in C$. The point $x_t := x + t e_i - t e_j$ is in the core C if and only if $x(S) - t \geq v(S)$ for all coalitions S with $j \in S \subset N\backslash i$ and $x(S) \geq v(S) + t$ for coalitions S with $i \in S \subset N\backslash j$. The other constraints are not in danger. Therefore, the largest t for which $x_t \in C$ is $-s_{ji}(x)$ and the lowest t for which $x_t \in C$ is $s_{ij}(x)$. Hence, we find:

condition (2.1) is equivalent with $s_{ij}(x) = s_{ji}(x)$ for all $i \neq j$.

Finally we must prove that $x \in \mathcal{K}(v) \cap C$ implies that $s_{ij}(x) = s_{ji}(x)$ for all $i \neq j$. If not, then e.g. $s_{ij}(x) > s_{ji}(x)$. This gives $x_j = v(j)$ and $s_{ji}(x) \geq 0$. Then $s_{ij}(x) > 0$ and $x \notin C$. QED

If we define the *kernel of the pre-core* C as the set given by condition (2.1) (notation: $\mathcal{K}(C)$) we find that the kernel of C contains the nucleolus of every game (N, v) with core C i.e.

$$\mathcal{L}oc\mathcal{N}uc\,(C) \subset \mathcal{K}(C).$$

Before we can define the least core of a pre-core C, we have to introduce some terminology. Let C be a pre-core and (N, v_C) *the* exact game with core C. Then we can distinguish three kinds of coalitions: A coalition S is *tight* if $(e_S, x) = v_C(S)$ for all $x \in C$. A coalition S will be called *essential* if $C \cap \{x \mid (e_S, x) = v_C(S)\}$ has codimension 1 in the core C (determines a facet of C). Coalitions which are neither tight nor essential are called *redundant*. Notice that if C is full dimensional, only the grand coalition is tight and each facet is determined by exactly one essential coalition. This means that the values of redundant coalitions can be decreased without changing the core of the game. In the example of the first section the coalitions $S = N$, $\{12\}$, $\{23\}$, $\{34\}$ and $\{14\}$ are tight and e.g. the coalition $S = \{24\}$ is essential for the boundary point ("facet") $s = 1$. The exact game v_C gives to the coalitions (123) and (134) the value 5/4 and to the coalitions (1) and (3) the value 1/4. The other coalitions have the same values as in the example. Notice that in this example all coalitions are tight or essential.

Let C be a pre-core and $\mathcal{S}$ the family of coalitions which are tight or essential. Let $E: C \to R^{\mathcal{S}}$ be the map $x \to \{E_S(x)\}_{S \in \mathcal{S}}$ where $E_S(x) = v_C(S) - x(S)$. As usual in the definition of a nucleolus, $\theta: R^{\mathcal{S}} \to R^s$, $s := |\mathcal{S}|$ is the map that orders the coordinates in a weakly decreasing order and $\preceq_{lex}$ is the lexicographic ordering on R^s. The *least core* of C is the set of points in C where the pointwise maximum $\bigvee_{S \in \mathcal{S}} E_S$ of the excess functions $\{E_S\}_{S \in \mathcal{S}}$ takes its minimum value. The *nucleolus* of C are the points of C where $\theta \circ E$ takes its lexicographic minimum.

$$\mathcal{LC}(C) := \{x \in C \mid \bigvee_{S \in \mathcal{S}} E_S(x) \leq \bigvee_{S \in \mathcal{S}} E_S(y) \text{ for all } y \in C\}$$

and

$$\mathcal{N}(C) := \{x \in C \mid \theta \circ E(x) \preceq_{lex} \theta \circ E(y) \text{ for all } y \in C\}.$$

Since C is compact $\mathcal{LC}(C)$ and $\mathcal{N}(C)$ are non-empty. As the excess function E_S, $S \in \mathcal{S}$ are constant on $\mathcal{N}(C)$ and the linear inequalities $(e_S, x) \geq a_S$ for tight or essential coalitions S determine the bounded set C, the nucleolus of C consists of one point (cf. Schmeidler (1969) and Potters and Tijs (1991)). Note that all the concepts needed to define the least core and the nucleolus of the pre-core C are uniquely determined by C itself.

In the second proposition we will prove that the least core of every game with core C (cf. [Maschler, Peleg and Shapley (1979)] and [Maschler, Potters and Tijs (1991)]) coincides with $\mathcal{LC}(C)$. As the nucleolus of a game is always a point of the least core of a game we find that also the set $\mathcal{LC}(C)$ is a subset of C which contains the locus of the nucleolus $\mathcal{L}oc\,\mathcal{N}uc\,(C)$.

Proposition 2. *The least core of each game (N, v) with core C equals $\mathcal{LC}(C)$ and is therefore not dependent on v.*

Proof: We may assume that C is full-dimensional. Otherwise $C = \mathcal{LC}(C)$ and there is nothing to prove. The least core of a game (N, v) consists of the optimal points x of the linear program

(2.2) $\max t$ under the conditions $t - x(S) \leq -v(S)$, for all $S \subset N$, $x(N) = v(N)$.

Let t be a real number in the segment $[0, \hat{t}]$ where $\hat{t}$ is the maximal value of t in the linear program (2.2) determining the least core. Let us consider the linear program

(2.3) $\min x(T)$ under the conditions $x(S) \geq v(S) + t$ for all $S \in \mathcal{S}$ and $x(N) = v(N)$

where T is any redundant coalition for C. We will prove that the minimal value of this program is at least $v(T) + t$. In that case the optimal points of the first program (2.2) do not change if we replace the constraints $t - x(S) \leq -v(S)$, $S \subset N$ by $t - x(S) \leq -v(S)$, $S \in \mathcal{S}$ i.e. $\mathcal{LC}(v) = \mathcal{LC}(C)$. The dual program of the linear program (2.3) is

(2.4) $\max \sum_{S \in \mathcal{S}} y_S (v(S) + t) + z\, v(N)$ with $y_S \geq 0$, $S \in \mathcal{S}$, $z \in R$, $\sum_{S \in \mathcal{S}} y_S e_S + z e_N = e_T$.

For $t = 0$ the maximal value of (2.4) (= the minimal value of (2.3)) is at least $v(T)$ (as feasible points of (2.3) are in $C = Core(v)$). If we take an optimal solution $(\hat{y}_S, S \in \mathcal{S}, \hat{z})$ for $t = 0$ then this point is also feasible for $t > 0$ and the value of the goal function is

$$\sum_{S \in \mathcal{S}} \hat{y}_S v(S) + \hat{z}\, v(N) + (\sum_{S \in \mathcal{S}} \hat{y}_S)\, t.$$

We are left to prove that $\sum_{S \in \mathcal{S}} y_S e_S + z e_N = e_T$ with $y_S \geq 0$, $S \in \mathcal{S}$ implies that $\sum_{S \in \mathcal{S}} y_S \geq 1$. By taking $j \notin T$ we find that $z \leq 0$ and if we take $i \in T$ we find $\sum_{S \in \mathcal{S}} y_S \geq \sum_{S \in \mathcal{S}: i \in S} y_S + z = 1$.
QED

Summarizing the two preceding propositions we have

(2.5) $\mathcal{L}oc\,\mathcal{N}uc\,(C) \subset \mathcal{K}(C) \cap \mathcal{LC}(C)$.

Corollary *If C is the core of a convex game then $\mathcal{L}oc\mathcal{N}uc(C) = \{\mathcal{N}(C)\}$.*

Proof: If C is the core of a convex game v then $v_C = v$ is convex [Schmeidler (1972)]. For a convex game the kernel consists of one point, the nucleolus of the game [Maschler,Peleg and Shapley (1972)]. This means that $\mathcal{K}(C)$ consists of one point and the same holds for the locus of the nucleolus by (2.5). QED

Let (N, v) be an arbitrary balanced game and define a second game $(N, \bar{v})$ by

$$\bar{v}(S) := \max(v(S), v(N) - M_v(N \backslash S)).$$

The vector M_v is the vector of marginals for the grand coalition i.e. $M_v(i) = v(N) - v(N \backslash i)$ for all $i \in N$.

The games (N, v) and $(N, \bar{v})$ have the same core.

This can be seen as follows. As $\bar{v} \geq v$ and $\bar{v}(N) = v(N)$ we find $Core(\bar{v}) \subset Core(v)$. If, however, $x \in Core(v)$ we have $x(S) \geq v(S)$ for all $S \subset N$ and $x_i \leq M_v(i)$ for all $i \in N$. From the last inequalities follows that $x(S) = x(N) - x(N \backslash S) \geq v(N) - M_v(N \backslash S)$. Therefore, $x(S) \geq \bar{v}(S)$ for every coalition $S \subset N$. QED

If (N, v) is a balanced game and $(N, \bar{v})$ is *convex* then the locus of the nucleolus of the core of (N, v) consists of one point, the nucleolus of (N, v). This follows from the corollary and the preceding remark. In the literature there are two classes of games satisfying the condition that $(N, \bar{v})$ is convex namely the class of *k-convex games* [Driessen (1985)] and the class of *clan games* [Potters, Poos, Tijs and Muto (1989)].

(1) K-convex games. Let k be a natural number $1 \leq k \leq n$. According to [Driessen (1985)] a cooperative game (N, v) is *k-convex* if $v S) \leq v(N) - M_v(N \backslash S)$ for all coalitions S with $|S| \geq k$ and the game v_k defined by $v_k(S) = \bar{v}(S)$ for $|S| \geq k$ and $v_k(S) = v(S)$ for $|S| < k$ is convex. If (N, v) is a k-convex game then $v \leq v_k \leq \bar{v}$ and because v and $\bar{v}$ have the same core, say C, also the convex game v_k has core C. Therefore, if C is the core of a k-convex game then C is also the core of a convex game. Consequentially, the locus of the nucleolus consists of one point.

(2) Clan games. A game (N, v) is a *clan game* if $v \geq 0$, $M_v \geq 0$ and there is a nonempty coalition Cl (the clan) such that $v(S) = 0$ if $S \not\supset Cl$ (the clan condition) and $v(S) \leq v(N) - M_v(N \backslash S)$ if $S \supset Cl$ (the union condition) [Potters et al. (1989)].

In a clan game the clan members have a strong incentive to stick together. Otherwise a positive result is not possible. The non-clan members have no possibility to obtain a positive result by their own but they can block a part of the profit to be made by the clan members and this blocking power is larger if they form a union before they start the negotiations with the clan.

For a clan game (N, v) with clan Cl the game $\bar{v}$ has the values

$$\bar{v}(S) = \begin{cases} 0 & \text{if } S \not\supset Cl \\ v(N) - M_v(N \backslash S) & \text{if } S \supset Cl \end{cases}$$

The first line follows from $M_v \geq 0$ and $M_v(i) = v(N)$ if $i \in Cl$; the second line follows from the union condition. It is not difficult to see that $\bar{v}$ is a convex game: For $S, T \subset N$ with $Cl \subset S \cap T$ we have $\bar{v}(S) + \bar{v}(T) = \bar{v}(S \cup T) + \bar{v}(S \cap T)$. If S or T doesn't contain the clan the convexity condition follows from the monotonicity of $\bar{v}$. Hence, we can apply the corollary and find that the locus of the core of a clan game consists of one point too.

Comment: For k-convex games the game v_k is a convex game with the same core as (N, v). A convex game is exact and therefore, $v_k = v_C$, the unique exact game with core $C = Core(v)$. For clan games the game $\bar{v}$ is convex and therefore $\bar{v} = v_C$.

3. Some other properties of the locus of the nucleolus.

Let C be a pre-core of full dimension (only the grand coalition is tight) and let $\mathcal{S}$ and $\mathcal{R}$ be the families of essential and redundant coalitions. The set of games (N, v) with core C can be described by the (in)equalities $v(S) = v_C(S)$ for $S \in \mathcal{S}$ and $v(R) \leq v_C(R)$ for $R \in \mathcal{R}$. Let $\Delta := \max_{S \in \mathcal{S}} (v_C(N) - v_C(S) - v_C(N \backslash S))$. In the next proposition we prove that $\mathcal{N}(C)$ is the nucleolus of (N, v) if $v(R) < v_C(R) - \Delta$ for all redundant coalitions $R \in \mathcal{R}$ i.e. if the redundant coalitions have small values they have no longer influence on the position of the nucleolus.

Proposition 3. *If C is a pre-core of full dimension and (N, v) is a game with core C of which all redundant coalitions R have values smaller than $v_C(R) - \Delta$, then $\mathcal{N}(C) = Nu(v)$.*

Proof: The proof is based on two observations:

(1) If $v(R) < v_C(R) - \Delta$, then $v(R) - x(R) < v_C(S) - x(S)$ for all essential coalitions $S \in \mathcal{S}$ and all core elements $x \in C$.

(2) The positive cone generated by $\{e_S\}_{S \in \mathcal{S}}$ and $-e_N$ contains all points of R^N.

Proof of (1): Let $x \in C$ and $S \in \mathcal{S}$. Then

$$\Delta \geq v_C(N) - v_C(S) - v_C(N \backslash S) \geq x(S) - v_C(S).$$

because $v_C(N) = x(N)$ and $x(N \backslash S) \geq v_C(N \backslash S)$. For a redundant coalition $R \in \mathcal{R}$ we have

$$v(R) - x(R) < v_C(R) - x(R) - \Delta \leq -\Delta \leq v_C(S) - x(S).$$

Proof of (2): As C is compact and as it is determined by the linear relations

$$x(S) \geq v_C(S) \quad \text{for} \quad S \in \mathcal{S} \quad \text{and} \quad x(N) = v_C(N)$$

each linear function $x \to (c, x)$ has a minimum value on C and the dual linear program is feasible for every vector $c \in R^N$. This means that $\sum_{S \in \mathcal{S}} y_S e_S + z e_N = c$ is solvable with $y_S \geq 0$ for $S \in \mathcal{S}$ and $z \in R$. Therefore, every point of R^N is a nonnegative linear combination of $\{e_S\}_{S \in \mathcal{S}}$ and $\{e_N, -e_N\}$. We can skip the vector e_N since the family $\mathcal{S}$ contains a balanced family (cf. [Maschler, Potters and Tijs (1991)]).

Let $x_0 \in C$ be the nucleolus of (N, v) and suppose that $x_0 \prec_{lex} \mathcal{N}(C)$. If we order the excesses of x_0 and $\mathcal{N}(C)$ in a weakly decreasing order, in both sequences the excesses of essential coalitions precede the excesses of the redundant coalitions (by observation (1)). From the definition of $\mathcal{N}(C)$ we infer that $x_0(S) = \mathcal{N}(C)(S)$ for all $S \in \mathcal{S}$. Then $x_0 = \mathcal{N}(C)$ by observation (2). QED

Corollary $\mathcal{N}(C) \in \mathcal{L}oc\mathcal{N}uc(C)$.

In the following proposition we give a necessary and sufficient condition that $\mathcal{L}oc\mathcal{N}uc(C)$ consists of the point $\mathcal{N}(C)$ only.

Let C be a full dimensional pre-core and be $\bar{v}_C$ the game with values $\bar{v}_C(S) = v_C(S)$ if $S = N$ or S is essential and $\bar{v}_C(R) = v_C(R) - \Delta$ if R is redundant. Let $x_0 = \mathcal{N}(C)$. Then x_0 is the nucleolus (and the prenucleolus) of $\bar{v}_C$. Let $\bar{\mathcal{B}}_t$ be the families of coalitions defined by

$$\{S \subset N \mid \bar{v}_C(S) - x_0(S) \geq t\}.$$

These families $\bar{\mathcal{B}}_t$ are balanced (as far as they are not empty) [Sobolev (1975)]. Let $[\bar{\mathcal{B}}_t]$ be the linear subspace of R^N generated by $\{e_S \mid S \in \bar{\mathcal{B}}_t\}$. For each redundant coalition R we define $t_R :=$ $v_C(R) - x_0(R)$. Using these notations we can formulate the following propsition:

Proposition 4. *If C is a pre-core of full dimension then* $\mathcal{L}oc\mathcal{N}uc(C) = \{\mathcal{N}(C)\}$ *if and only if* $e_R \in [\bar{\mathcal{B}}_{t_R}]$ *for every redundant coalition R.*

Proof: Let (N, v) be a game with core C and let $\mathcal{B}_t$ be the families of coalitions $\{S \subset N \mid v(S) - x_0(S) \geq t\}$. We prove that these families are balanced. For every redundant coalition $R \in \mathcal{B}_t$ we have $t_R \geq t$ (because $v_C(R) \geq v(R)$) and therefore, $\bar{\mathcal{B}}_{t_R} \subset \bar{\mathcal{B}}_t$ and $\mathcal{B}_{t_R} \subset \mathcal{B}_t$. Let $e_R = \sum_{S \in \bar{\mathcal{B}}_{t_R}} y_S^R e_S$ with $y_S^R \in R$. As $\bar{\mathcal{B}}_t$ is balanced we have $\sum_{T \in \bar{\mathcal{B}}_t} x_T e_T = e_N$ with positive coefficients x_T for all $T \in \bar{\mathcal{B}}_t$. Then

$$\sum_{T \in \bar{\mathcal{B}}_t} x_T e_T + \delta \sum_{R \in \mathcal{B}_t} e_R - \delta \sum_{R \in \mathcal{B}_t} \sum_{S \in \bar{\mathcal{B}}_{t_R}} y_S^R e_S = e_N.$$

For small but positive δ this gives the balancedness of $\mathcal{B}_t$. Then $\mathcal{N}(C) = x_0$ is the (pre)nucleolus of every game (N, v) with core C.

Conversely, if $e_R \notin [\mathcal{B}_{t_R}]$ we construct a game (N, v) with core C and nucleolus $\neq \mathcal{N}(C)$. Let $v(S) = v_C(S)$ if $S = N, R$ or S is essential and $v(S) = \bar{v}_C(S)$ if S is redundant $\neq R$. If $x_0 = \mathcal{N}(C)$ is the nucleolus of (N, v) then in particular $\mathcal{B}_{t_R} = \bar{\mathcal{B}}_{t_R} \cup \{R\}$ is balanced. The equality holds because $\bar{\mathcal{B}}_{t_R}$ does not contain *all* essential coalitions and therefore no redundant coalition $\neq R$ (cf. the proof of proposition 3). This means that the equation $\sum_{S \in \bar{\mathcal{B}}_{t_R}} \hat{y}_S e_S = e_N$ as well as the equation $\sum_{S \in \mathcal{B}_{t_R}} y_S e_S + z e_R = e_N$ have a solution with positive coefficients. This means that $z e_R \in [\bar{\mathcal{B}}_{t_R}]$ with $z > 0$. This is in contradiction with the assumption about e_R. QED

Comment: This proposition is no longer true if C is not full dimensional. In the example of section 1 every coalition is tight or essential. Therefore, one side of the equivalence is an empty condition but we have seen that the locus of the nucleolus consists of more than one point.

Finally we prove that the locus is a path-connected subset of C.

Proposition 5. *For every pre-core C the locus of the nucleolus is path-connected.*

Proof: We prove that every point of the locus can be connected by a path to $Nu(v_C)$. Suppose that x is the nucleolus of a game (N, v) with core C. Then $v \leq v_C$ and all the games $v_t = tv + (1-t)v_C$, $t \in [0,1]$ have core C. As the nucleolus is a continuous function of the game v [Schmeidler (1969)] the path $t \in [0,1] \to Nu(v_t)$ is a (continuous) path connecting $Nu(v_C)$ with $Nu(v) = x$. QED

Open question: Is the locus of the nucleolus always a convex subset of C?

4. Some instructive examples and final remarks.

In this final section we give some examples answering some questions the reader may have. In the first example we show that the locus of the nucleolus may consist of one point also if the pre-core C is not the core of a convex game.

Example 2: Let C be the convex hull of the points $(2,0,2,0)$, $(1,1,2,0)$, $(0,2,1,1)$, $(1,1,0,2)$ and $(2,0,1,1)$. The values of v_C in the coalitions (124), (12), (14) and (1) are 2, 2, 1 and 0 respectively. Hence, the game v_C is not convex. Nevertheless it is easy to see that the point $(1,1,1,1)$ is the unique point in C where $s_{12} = s_{21}$ and $s_{14} = s_{41}$ with respect to v_C. This means that according to proposition 1 the point $(1,1,1,1)$ is the only point in $\mathcal{K}(C) = \mathcal{L}oc\mathcal{N}uc(C)$.

The second example is a five person game which exhibits an important phenomenon which makes the locus problem so interesting but also so difficult.

Example 3: Let C be the core of the following game:

				18					
		L	L	L	L	9			
8	L	L	4	L	L	8	4	L	8
L	L	L	6	L	6	L	L	L	L
		L	L	L	L	L			

The coalitions are ordered in lexicographic order and an "L" indicates that the value of this coalition is so low that it has no influence on the shape of the core and the position of the nucleolus. The facets of C are determined by the coalitions (123), (234), (345), (15), (24), (134) and (235). The coalition (2345) is redundant. It is not difficult to see that the least core of this game is the set

$$\{(t+2, 6-t, 2, t+2, 6-t) \mid 0 \leq t \leq 4\}.$$

The end points of the least core are determined by the coalitions (134) for $t = 0$ and (235) for $t = 4$. The hyperplane $x(2345) = 11$ has no points in common with the least core (cf. the proof of proposition 1). The excess of coalition (2345) has decreased faster than the excesses of the other coalitions. The excesses of the coalitions (123), (234), (345), (15) and (24) are constant on the least core. Therefore, the position of the nucleolus in the least core is determined by the coalitions (134), (235) and (2345) and one might expect that the last coalition has no influence. But in fact the excesses of those three coalitions are $-2-2t$, $2t-10$ and $-7+t$ respectively. The nucleolus corresponds to $t = 5/3$ and coalition (235) has a smaller excess in that point than (2345). So in the shrinking process first the excess of (2345) decreases faster than the excess of (235) but after reaching the least core the excess of (2345) decreases slower than the excess of (235). This phenomenon finds its origin in the equality

$$e_{\{2345\}} = 1/2\, e_{\{235\}} + 1/2\, e_{\{24\}} + 1/2\, e_{\{345\}}.$$

If the payoffs $x(S)$ to the coalitions $S = (235)$, (24) and (345) increase with δ the payoff to coalition (2345) increases with $3/2\,\delta$ (faster) but after reaching the least core the payoffs to (24) and (345) does not change anymore. If the payoff to (235) increase with δ the payoff to coalition (2345) increases with $1/2\,\delta$ (slower).

Summary: In this paper we investigated the locus of the nucleolus in a pre-core C. We proved that this locus consists of one point if the set C is the core of a convex game. Example 2 showed that this condition is not necessary. For full dimensional pre-cores we gave a necessary and sufficient condition that the locus consist of one point. Further we proved that the locus is a path-connected subset of C and a subset of the intersection of the kernel and the least core of C.

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