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Cooperation, compensation and transition

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To my family

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Contents

Chapter 1

Introduction

1.1 Introduction and overview

Cooperation plays a fundamental role in fostering developments of human society. The study of cooperation is thus meaningful. Like economic agents mainly have competitive behavior and cooperative behavior, game theory, a mathematical theory that designs and uses tools to study interactions among decision makers, deals with both models of competition and cooperation as well. Especially, the cooperative side of the body of the theory involves issues like how coalitions of players form, how they behave, and in particular how they bargain over their joint choice of actions, and how they solve the problem of sharing the joint gains of cooperation. Especially this last issue is vital to strengthen cooperation itself. The reason is simple, without a well justified sharing rule, cooperation can hardly form or continue. Cooperative game theory comprises many different models, the most popular of which is the model of transferable utility games introduced by von Neumann and Morgenstern (1944). One can think of a transferable utility game as a surplus sharing problem in which an amount of money (as the net maximal profit for the group of players) is to be divided and where one abstracts from the fact that the players involved might put different values on the monetary payoffs they may receive.

Compensation is a specific aspect of surplus sharing problems providing incentives for agents to sacrifice their own interests for a moment to obtain higher payoffs for the coalition as a whole. In such a way, the compensation problem can be analysed within a cooperative game theoretic framework. In the real world, providing a reasonable amount of compensation to such players is part of a practical strategy for getting the necessary resources to attain efficiency for the whole group. However, compensation is not explicitly studied in the existing literature and sometimes completely neglected. For instance, the classical work on the issue of externality concentrates on the efficiency problem but cannot serve as the (normative) answer to the question how to compensate the agents who are negatively affected by the others.

Since cooperation can be facilitated by providing compensations to players, one can apply this idea to practice. In a political economy perspective, transition by compensation becomes a prominent approach because a transition process can be viewed as a kind of cooperation of the whole society and compensating agents to buy their support for the reforms helps to realize the process while keeping social stability.

Therefore, the three topics of cooperation, compensation, and transition are well linked to each other, and will be treated in three corresponding parts (six chapters besides this chapter) of this monograph.

The first part (chapters 2 and 3) focuses on solution concepts for cooperative games. A new solution concept for transferable utility games, the consensus value, is introduced in chapter 2. Chapter 3 generalizes it to partition function form games.

Chapter 2, based on the paper "The consensus value: a new solution concept for cooperative games" (Ju, Borm and Ruys (2004)), introduces the consensus value. We characterize the consensus value as the unique function that satisfies efficiency, symmetry, the quasi dummy property and additivity. By means of the transfer property, a second characterization is provided. By defining the stand-alone reduced game, a recursive formula for the value is established. In addition, it is shown that this value is the average of the Shapley value and the equal surplus solution. We also discuss a possible generalization. Furthermore, we apply the consensus value to the issue of merger incentives in network industries with essential facilities.

A generalization of the consensus value to the class of partition function form games is studied in chapter 3 that is a chapter based on the paper "The consensus value for games in partition function form" (Ju (2004)). The concepts and axioms, related to the consensus value (Ju, Borm and Ruys (2004)), are extended. This value is characterized as the unique function that satisfies efficiency, complete symmetry, the quasi-null player property and additivity. By means of the transfer property, a second characterization is provided. Moreover, it is shown that this value satisfies individual rationality under a certain condition, and well balances the tradeoff between coalitional effects and externality effects. By modifying the stand-alone reduced game, a recursive formula for the value is established as well. A further generalization of the consensus value is discussed. Moreover, two applications of the consensus value are given: one is for oligopoly games in partition function form and the other is about participation incentives in free-rider situations.

The second part (chapters 4 and 5) is devoted to the theme of compensation while it also studies new models of cooperation. Chapter 4 develops a general framework about cooperation and compensation, the so-called project-allocation situations, and further investigates the applications of the consensus value to problems of compensating losses and sharing surpluses. Chapter 5 constructs a new model to study externalities and the associated compensation problem. Specific solution concepts are proposed.

By introducing the notions of projects and shares, chapter 4 (based on the paper "Compensating losses and sharing surplus in project-allocation situations" by Ju, Ruys and Borm (2004)) studies a class of economic environments, the so-called project-allocation situations, in which society may profit from cooperation, e.g., by reallocating the initial shares of projects among agents. This chapter mainly focuses on the associated issues of compensation of losses and surplus sharing arising from the reallocation of projects. For this purpose, we construct and analyse an associated project-allocation game and a related system of games that explicitly models the underlying cooperative process. We further argue that the consensus value fits well in the framework of project-allocation situations.

Differing from the classical literature (Pigou (1920), Coase (1960), Arrow (1970)) and the relatively recent studies (cf. Varian (1994)) that associate the externality problem with efficiency, chapter 5 stresses the compensation problem and the normative compensation rules in the context of externality. By taking players' absolute stand-alone situations into consideration, this chapter constructs a new game-theoretic framework: primeval games. It shows that primeval games well capture the features of inter-individual externalities and help to analyse the associated compensation problems. This chapter introduces several solution concepts which may serve as benchmarks for solving such problems. Firstly, the Shapley value is generalized to this framework and a modified Shapley value is obtained. By taking a bilateral perspective on the consequences of externalities, the consensus value for primeval games is defined. Characterizations of the two solution concepts are provided. A generalization of the consensus value is discussed. Moreover, that chapter suggests a more context-specific solution concept, the primeval value, which seems more appropriate for this class of games. Special properties of this value are studied. Finally, possible connections between this framework and classical cooperative games are discussed.

The third part (chapters 6 and 7) is an application of the first two parts into the study of some compensation issues in transition economies. Chapter 6 analyses the gradualist transition reform in China and points out that China actually adopted an approach of "experimentation" for the transition process. Chapter 7 proposes to adopt from now on an alternative approach, namely "transition by compensation".

Chapter 6, based on the paper "The river, the stones and the gradualist reform in China's banking sector" (Ju (2003)), analyses that as a miniature of the whole economic reform and transition in China, the reforms in the banking sector well reflect the basic ideas and features of China's transition model: gradualism in process, experimentalism in method, pragmatism in attitude, and evolutionism in nature. By briefly retrospecting and discussing the reform measures and process in the banking sector, chapter 6 provides a window to look into the unique philosophy behind China's reform and transition and summarizes the "transition by experimentation" approach. Meanwhile, a practical role that the chapter can play is to provide the background (knowledge) for chapter 7 which argues that the experimentation approach might not be appropriate for the new situations in the transition process and proposes an alternative approach, namely, "transition by compensation", based on the analysis of the compensation problems in China's transition period.

Chapter 7 is devoted to the policy implications of the theoretical studies in the previous chapters to China's transition reforms. Generally, the transition of an economy from one state to another can hardly be implemented without proper compensation of parties that have to bear losses or give up some privileges, outdated rights, or established interests. We argue that the approach of "transition by compensation" is particularly suitable for the transition process in China at this moment. Based on solution concepts for cooperative games, we propose fair compensation rules and analyse the necessary elements to establish an effective compensation system. The analysis focuses on a study of the demolition and eviction in China but implications are derived for the whole economy.

1.2 Preliminaries on cooperative game theory

Game theory is a mathematical framework for modelling and analysing conflict situations that involve decision makers or agents, called the players of the game, with possibly diverging interests.

The foundation of game theory was laid by von Neumann (1928). However, it was not until the publication of the seminal book "Theory of Games and Economic Behavior" by von Neumann and Morgenstern (1944) that game theory received widespread attention. Since then, game theory has evolved into an important tool for modelling and analysing in economics and other social sciences as well as in evolutionary biology.

Cooperative game theory concentrates on cooperative behavior by analysing the negotiation process within a group of individuals in establishing a contract on a joint plan of activities, including an allocation of the correspondingly generated revenues. In particular, the possible levels of collaboration and the revenues of each possible *coalition* (a subgroup of cooperating players) are taken into account so as to allow for a better comparison of each player's role and impact within the group as a whole, and to settle on a compromise allocation in an objectively justifiable way.

A systematic description of the outcomes that may emerge in a family of games is called a *solution*. In cooperative game theory, there is no single solution concept dominating the field as much as the Nash equilibrium for non-cooperative game theory (Nash (1951), for a survey, see van Damme (2000)), although the core and the Shapley value are frequently considered as such. Solution concepts in cooperative game theory formulate requirements regarding payoffs. In general, cooperative solutions suggest how the total value of the grand coalition can be split among all the players in a satisfactory way.

Below we provide preliminaries on cooperative game theory. For a comprehensive treatment of game theory in general, we refer to Myerson (1991).

Cooperative game theory primarily deals with joint payoffs that can be obtained by groups of players if they coordinate their actions. In some situations these joint profits are freely transferable among the players (i.e., there exist no restrictions on the division of joint profits among the players) whereas in other situations they are not.

A *cooperative game with transferable utility*, or *TU game* (in characteristic function form), is a pair (N, v), where N denotes the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the *characteristic function*, assigning to every coalition $S \subset N$ of players a value, or *worth*, $v(S)$, representing the total payoff to this coalition of players when they cooperate. N is called the *grand coalition*. By convention, $v(\emptyset) = 0$. Here, the term of transferable utility (cf. Friedman (1977, 1986)) means that for each $S \subset N$, the scalar value $v(S)$ can be freely apportioned among the members of S. We denote the class of all TU games with player set N by TU^N. Where no confusion can arise, we sometimes denote a game (N, v) by v.

A game (N, v) is called *superadditive* if for every pair of disjoint coalitions the value of the union of the coalitions is at least the sum of the values of the two coalitions separately:

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \subset N$ with $S \cap T = \emptyset$.

A game (N, v) is called *subadditive* if

$$v(S \cup T) \leq v(S) + v(T)$$

for all $S, T \subset N$ with $S \cap T = \emptyset$.

A *solution (concept)* for TU^N is a map f that assigns to each game $v \in TU^N$ a subset $f(v)$ of $\mathbb{R}^N$. If for every game $v \in TU^N$ the set $f(v)$ is a singleton, then we call f a *one-point solution (concept)*.

Cooperative game theory usually analyses adequate divisions of the worth of the grand coalition. Two well-known requirements of an allocation $x \in R^N$ for a game $v \in TU^N$ are

(i) *Efficiency:* $\sum_{i \in N} x_i = v(N)$;

(ii) *Individual rationality:* $x_i \geq v(\{i\})$ for all $i \in N$.

Allocations satisfying (i) and (ii) are called *imputations*.

The *core* (cf. Ransmeier (1942) and Gillies (1959)) $Core(v)$ of a game $v \in TU^N$ is defined by

$$Core(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}$$

So, core elements are imputations which are stable against coalitional deviations: no coalition has an incentive to split off from the grand coalition, since for each coalition S what it is allocated in total according to x (i.e., $\sum_{i \in S} x_i$) is at least what it can obtain by forming S (i.e., $v(S)$). However, the core of a cooperative game can be empty.

Given a game $v \in TU^N$, for any coalition $S \subset N$ and any player $i \in N \backslash S$, we call $v(S \cup \{i\}) - v(S)$ the *marginal contribution* of player i to coalition S.

Let $\Pi(N)$ denote the collection of orderings or permutations on N. The *marginal vector* $m^\sigma(v)$ of a game $v \in TU^N$ corresponding to the ordering $\sigma \in \Pi(N)$ is defined by

$$m^\sigma_{\sigma(k)}(v) = v(\{\sigma(1), ..., \sigma(k)\}) - v(\{\sigma(1), ..., \sigma(k-1)\})$$

for all $k \in \{1, ..., |N|\}$.

The *Shapley value* (cf. Shapley (1953)) of a game $v \in TU^N$, $\Phi(v)$, is defined as the average of the marginal vectors, i.e.

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

This formula for the Shapley value can be reformulated into

$$\Phi_i(v) = \sum_{S \subset N \backslash \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S))$$

for all $v \in TU^N$ and $i \in N$.

This in fact provides a probabilistic interpretation of the Shapley value for a player as his expected marginal contribution (cf. Myerson (1991)).

It is readily seen that the Shapley value satisfies the following properties:

- *Efficiency:* For all $v \in TU^N$, $\sum_{i \in N} \Phi_i(v) = v(N)$;
- *Symmetry:* For all $v \in TU^N$ and for any two players $i, j \in N$ such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subset N \backslash \{i, j\}$, we have $\Phi_i(v) = \Phi_j(v)$;
- *Dummy:* For all $v \in TU^N$ and a player $i \in N$ such that $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subset N \backslash \{i\}$, we have $\Phi_i(v) = v(\{i\})$;
- *Additivity:* $\Phi(v_1 + v_2) = \Phi(v_1) + \Phi(v_2)$ for all $v_1, v_2 \in TU^N$, where $v_1 + v_2$ is defined by $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subset N$.

Theorem 1.2.1 *(Shapley (1953)) The Shapley value Φ is the unique one-point solution concept on TU^N that satisfies efficiency, symmetry, dummy, and additivity.*

Chapter 2

The consensus value: a new solution concept for cooperative games

2.1 Introduction

In this chapter[1], we study the problem of sharing the joint gains of cooperation and try to find a solution concept which can not only be axiomatically characterized but is also constructive (based on an explicit process of sharing gains of cooperation). Following a simple and natural way of generalizing the standard solution for 2-person games into n-person cases, we obtain a new solution concept for TU games: *the consensus value.*

The consensus value is related to two well established solution concepts: the equal surplus solution (cf. Moulin (2003)) and the Shapley value (Shapley (1953)).

The equal surplus solution, also known as the CIS-value (Center of Imputation Set value, cf. Driessen and Funaki (1991), van den Brink and Funaki (2004)), assigns to every player her individual value, and distributes the remainder of the value of the grand coalition equally among all players. Thus, the equal surplus solution is a central solution concept in terms of egalitarianism. Moreover, it is particularly useful for a class of games where the only possible final outcomes are either the complete cooperation of all players or the complete breakdown of cooperation[2]. However, since the equal surplus solution rules out the consideration on partial cooperation, it fails to explain the interaction between coalitions and leaves the evolution process from com-

[1]This chapter is based on Ju, Borm and Ruys (2004).

[2]Another possible interpretation could be that we only have information about the two extreme ends of a game: the individual values and the value of the grand coalition; or simply when we do not care about partial cooperation.

plete breakdown to complete cooperation as a blackbox. Consequently, this solution concept seems insufficient for general n-person cooperative games but could well serve as a specific benchmark.

The Shapley value, on the other hand, takes all coalitional values into account and somehow corresponds to the players' expected marginal contributions. Moreover, the Shapley value is characterized as the unique function that satisfies efficiency, symmetry, the dummy property and additivity. Although the Shapley value serves as the central solution concept for TU games, there is still critique. For instance, Luce and Raiffa (1957) criticize the efficiency postulate and the additivity postulate. A recent critique on the efficiency postulate can be found in Maskin (2003). In this chapter, the justification of the dummy property is considered. Generally speaking, there may exist two extreme opinions about the gain of a dummy player. From the individualist point of view, we do get the classical dummy property requiring that a dummy player obtains exactly her own value $v(\{i\})$. However, from the egalitarian or collectivistic perspective, one can argue that all members of a society including dummies should share the joint surplus equally among them. This distinction opens up the possibility to relax this postulate. In this spirit we introduce and discuss a so-called *quasi dummy property*.

In addition, in our opinion, the constructive interpretation of the Shapley value, i.e., the marginal contribution approach, is not so convincing. Here, the terminology of "marginal contribution" is somewhat misleading. In fact, the marginal contribution is jointly created by the existing coalition of players and the entrant, but not by the entrant solely. Following this reasoning, it seems too much to give a later entrant the whole marginal value in superadditive games. Similarly, this rule is hard to implement if the marginal contribution is less than the entrant's individual value if the loss is caused by the interaction between the entrant and the incumbents. Of course, those aspects are smoothed out in some sense by taking the average of the marginal contributions over all different orders.

Although basically we follow the same line as the Shapley value to study the problem of sharing gains of cooperation, i.e., using an average serial method, we propose to modify the allocation of marginal contributions by a method which is based on the standard solution for 2-person games. Given an ordering of players, we take a bilateral perspective[3] and consider that any surplus is the joint contribution between an existing

[3] This is the key feature of the constructive sharing procedure of the consensus value. One may find an alternative interpretation of the Shapley value from Maskin (2003) that is based on a sequential bargaining approach. However, it is still a unilateral perspective.

coalition of players (i.e., the incumbents) and an entrant. By taking the incumbents as one party and the entrant as a second party, the standard solution for 2-person games can be applied all the way with consensus. That is, all the joint surpluses are always equally split between the corresponding two parties. Since no specific ordering is pre-determined, we average over all possible permutations. Such a constructive process is helpful to solve practical problems. In chapter 4, we apply it to study loss compensation and surplus sharing problems in project-allocation situations.

We characterize the consensus value as the unique one-point solution concept for TU games that satisfies efficiency, symmetry, the quasi dummy property and additivity. By means of the transfer property, an alternative characterization for the consensus value is provided. We also establish a recursive formula for the consensus value by defining a stand-alone reduced game. Moreover, surprisingly, we find that the consensus value is the average of the Shapley value and the equal surplus solution. Furthermore, by introducing a share parameter on the splitting of joint surpluses, we obtain a generalization of the consensus value. In particular, the Shapley value and the equal surplus solution are the two polar cases of these generalized consensus values.

In section 2, we formally define the consensus value and establish a recursive formula. As an illustration we consider glove games. In section 3, we characterize the consensus value in an axiomatic way and discuss the properties under consideration. Moreover, it is shown that the consensus value is the average of the Shapley value and the equal surplus solution. An alternative characterization using the transfer property is provided. We then discuss a possible generalization of the consensus value in section 4. Section 5 further extends the idea of defining the consensus value by means of an alternative approach to share surpluses, i.e., based on the comparison between coalitions' sizes, which results in another solution concept. The final section applies the consensus value to the issue of merger incentives, which is based on a model of network industries with essential facilities (cf. Jeon (2003)).

2.2 The consensus value

Let us consider an arbitrary 2-person cooperative TU game with player set $N = \{1, 2\}$ and characteristic function v determined by the values: $v(\{1\})$, $v(\{2\})$ and $v(\{1, 2\})$. A reasonable solution is that player 1 gets

$$v(\{1\}) + \frac{v(\{1,2\}) - v(\{1\}) - v(\{2\})}{2}$$

and player 2 gets

$$v(\{2\}) + \frac{v(\{1,2\}) - v(\{2\}) - v(\{1\})}{2}.$$

That is, the (net) surplus generated by the cooperation between player 1 and 2, $v(\{1,2\}) - v(\{1\}) - v(\{2\})$, is equally shared between the two players. This solution is called the *standard* solution for 2-person cooperative games.

Now, we provide a generalization of the standard solution for 2-person games into n-person cases. It follows the following line of reasoning.

Consider a 4-person game (N, v) with player set $N = \{1, 2, 3, 4\}$. Assume we have the order $(1, 2, 3, 4)$: player 1 shows up first, player 2 second, then player 3, and finally player 4. When player 2 joins player 1, we in fact have a 2-person situation, and following the principles of the standard solution, the surplus $v(\{1,2\}) - v(\{1\}) - v(\{2\})$ will be equally split among them.

Next player 3 enters the scene, who would like to cooperate with player 1 and 2. Because coalition $\{1, 2\}$ has already been formed before she enters the game, player 3 will actually cooperate with the existing coalition $\{1, 2\}$ instead of simply cooperating with 1 and 2 individually. If $\{1, 2\}$ agrees to cooperate with 3 as well, the coalitional value $v(\{1, 2, 3\})$ will be generated. Now, the question is how to share it?

Again, following the standard solution for 2-person games, one can argue that both parties should get half of the joint surplus $v(\{1, 2, 3\}) - v(\{1, 2\}) - v(\{3\})$ in addition to their individual values. The reason is simple: coalition $\{1, 2\}$ can be regarded as one player instead of two players because they have already formed a cooperating coalition. Internally, 1 and 2 will receive equal shares of the surplus because this part is obtained extra by the coalition $\{1, 2\}$ cooperating with coalition $\{3\}$.

In order to clearly illustrate this idea, we discuss the situation where a fourth player appears. Now, player 4 is collaborating with coalition $\{1, 2, 3\}$ as a whole since those three players have formed into one cooperating coalition before 4 enters. According to the standard solution for 2-person games, coalition $\{1, 2, 3\}$ gets

$$v(\{1,2,3\}) + \frac{v(\{1,2,3,4\} - v(\{1,2,3\}) - v(\{4\}))}{2}.$$

Since this amount is obtained by the cooperation between coalition $\{1, 2\}$ and player 3, these two parties will get equal share of the surplus. Thus, coalition $\{1, 2\}$ gets

$$v(\{1,2\}) + \frac{\left(v(\{1,2,3\}) + \frac{v(\{1,2,3,4\} - v(\{1,2,3\}) - v(\{4\}))}{2}\right) - v(\{1,2\}) - v(\{3\}))}{2}.$$

Finally, players 1 and 2 share this amount in the same fashion.

One can also tell this story in the reverse way, yielding the same outcome in terms of surplus sharing. Here, a 3-person case suffices to show the idea. Initially, three players cooperate with each other and $v(\{1,2,3\})$ is obtained. We now consider players leaving the existing coalition one by one in the opposite order $(3,2,1)$. So, player 3 leaves first. By the standard solution for 2-person games, player 3 should get half of the joint surplus plus her individual payoff, i.e.,

$$v(\{3\}) + \frac{v(\{1,2,3\}) - v(\{3\}) - v(\{1,2\})}{2},$$

as 1 and 2 remain as one cooperating coalition $\{1,2\}$. Thus, the value left for coalition $\{1,2\}$, which we call the *standardized remainder* (the value left for the corresponding remaining coalition) for $\{1,2\}$, is

$$v(\{1,2\}) + \frac{v(\{1,2,3\}) - v(\{1,2\}) - v(\{3\})}{2}.$$

In the same fashion, the standardized remainder for $\{1\}$ will be

$$v(\{1\}) + \frac{\left(v(\{1,2\}) + \frac{v(\{1,2,3\}) - v(\{1,2\}) - v(\{3\})}{2}\right) - v(\{1\}) - v(\{2\})}{2},$$

when player 2 leaves the coalition $\{1,2\}$ next.

Extending this argument to an n-person case, we obtain a general method, which can be understood as a *standardized remainder rule* since we take the later entrant (or earlier leaver) and all her pre-entrants (or post-leavers) as two parties and apply the standard solution for 2-person games all the way. Furthermore, since no ordering is pre-determined for a TU game, we will average over all possible orderings.

Formal definitions are provided below. For an ordering $\sigma \in \Pi(N)$ and $k \in \{1,2,...,|N|\}$ we define $S_k^\sigma = \{\sigma(1),\sigma(2),...,\sigma(k)\} \subset N$ and $S_0^\sigma = \emptyset$. Let $v \in TU^N$. Recursively, we define

$$r(S_k^\sigma) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S_k^\sigma) + \frac{r(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\})}{2} & \text{if } k \in \{1,...,|N|-1\}, \end{cases}$$

where $r(S_k^\sigma)$ is the *standardized remainder*[4] for coalition S_k^σ: the value left for S_k^σ after allocating surpluses to earlier leavers $N \backslash S_k^\sigma$.

[4]Obviously, the standardized remainder not only depends on S_k^σ but also on σ and v. Since no confusion can arise and for notational simplicity, we use $r(S_k^\sigma)$.

We then construct the *individual standardized remainder vector* $s^\sigma(v)$, which corresponds to the situation where the players leave the game one by one in the order $(\sigma(|N|), \sigma(|N|-1), ..., \sigma(1))$ and assign to each player $\sigma(k)$, besides her individual payoff $v(\{\sigma(k)\})$, half of the net surplus from the standardized remainder $r(S_k^\sigma)$. Formally,

$$s^\sigma_{\sigma(k)}(v) = \begin{cases} v(\{\sigma(k)\}) + \frac{r(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})}{2} & \text{if } k \in \{2, ..., |N|\} \\ r(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 2.2.1 *For every $v \in TU^N$, the consensus value $\gamma(v)$ is defined as the average of the individual standardized remainder vectors, i.e.,*

$$\gamma(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s^\sigma(v).$$

Hence, the consensus value can be interpreted as the expected individual standardized remainder a player can get by participating in coalitions.

Following the process of obtaining the consensus value, a more descriptive name for this solution concept could be the *average serial standardized remainder* value[5].

Example 2.2.2 *Consider the 3-person TU game described below.*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	10	0	0	18	23	0	30

With $\sigma : \{1,2,3\} \longrightarrow N$ defined by $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(3) = 3$, which is shortly denoted by $\sigma = (2\ 1\ 3)$, we get

$$\begin{aligned} s_3^\sigma(v)(= s^\sigma_{\sigma(3)}(v)) &= v(\{3\}) + \frac{1}{2}(v(N) - v(\{1,2\}) - v(\{3\})) = 6, \\ s_1^\sigma(v)(= s^\sigma_{\sigma(2)}(v)) &= v(\{1\}) + \frac{1}{2}(r(\{1,2\}) - v(\{2\}) - v(\{1\})) = 17, \\ s_2^\sigma(v)(= s^\sigma_{\sigma(1)}(v)) &= r(\{2\}) = v(\{2\}) + \frac{1}{2}(r(\{1,2\}) - v(\{2\}) - v(\{1\})) = 7. \end{aligned}$$

[5]In the same spirit, an alternative name for the Shapley value could be the average serial marginal contribution value.

All individual standardized remainder vectors[6] *are given by*

σ	s_1^σ	s_2^σ	s_3^σ
(123)	17	7	6
(132)	$18\frac{1}{4}$	$3\frac{1}{2}$	$8\frac{1}{4}$
(213)	17	7	6
(231)	20	5	5
(312)	$18\frac{1}{4}$	$3\frac{1}{2}$	$8\frac{1}{4}$
(321)	20	5	5

Hence, $\gamma(v) = \left(18\frac{5}{12}, 5\frac{1}{6}, 6\frac{5}{12}\right)$ *whereas the Shapley value of this game is given by* $\Phi(v) = \left(20\frac{1}{6}, 3\frac{2}{3}, 6\frac{1}{6}\right)$.

To further illustrate the consensus value, we consider glove games.

Example 2.2.3 *(A glove game)*

Let $N = \{1, 2, 3\}$ *be the set of players. Player 1 has one left hand glove. Player 2 and 3 have one right hand glove each. A single glove is worth nothing. A (left-right) pair is worth 1 Euro. The corresponding TU game* (N, v) *is determined by the values:* $v(\{i\}) = 0$ *for all* i *in* N, $v(\{2, 3\}) = 0$, *and* $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 1$.

One can readily check that the consensus value of this game is $\gamma(v) = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

In the more general case where $|N| > 3$ *but there is still only one left hand glove player while all the others have one right hand glove each, the consensus value yields that the left hand glove player gets* $\frac{1}{2}$ *and each right hand glove player gets* $\frac{1}{2(|N|-1)}$.

Next it is shown that the consensus value satisfies the basic property of relative invariance with respect to strategic equivalence.

Lemma 2.2.4 *The consensus value* γ *is relative invariant with respect to strategic equivalence, i.e. for* $\alpha > 0$, $\beta \in \mathbb{R}^N$ *and* $v \in TU^N$, *we have*

$$\gamma_i(\alpha v + \beta) = \alpha\gamma_i(v) + \beta_i$$

for all $i \in N$, *where* $\alpha v := \alpha v(S)$ *for all* $S \subset N$.

[6] *The fact that two permutations like* (123) *and* (213) *yield the same payoff vector only holds for the class of all* 3*-person TU games.*

Proof. Let $v \in TU^N$, $\sigma \in \Pi(N)$, $\alpha > 0$ and $\beta \in \mathbb{R}^N$.
Claim 1: $r^{\alpha v+\beta}(S_k^\sigma) = \alpha r^v(S_k^\sigma) + \sum_{i\in S_k^\sigma} \beta_i$ for all $k \in \{1, 2, ..., |N|\}$.
For $k = |N|$, this is obvious.
For $k \in \{1, ..., |N| - 1\}$, using induction we have

$$\begin{aligned}
& r^{\alpha v+\beta}(S_k^\sigma) \\
= & \left(\alpha v(S_k^\sigma) + \sum_{i\in S_k^\sigma} \beta_i\right) \\
+ & \frac{1}{2}\left(r^{\alpha v+\beta}(S_{k+1}^\sigma) - \left(\alpha v(S_k^\sigma) + \sum_{i\in S_k^\sigma} \beta_i\right) - \left(\alpha v(\{\sigma(k+1)\}) + \beta_{\sigma(k+1)}\right)\right) \\
= & \alpha\left(v(S_k^\sigma) + \frac{1}{2}\left(r^v(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\})\right)\right) + \sum_{i\in S_k^\sigma} \beta_i \\
= & \alpha r^v(S_k^\sigma) + \sum_{i\in S_k^\sigma} \beta_i.
\end{aligned}$$

Claim 2: $s_{\sigma(k)}^\sigma(\alpha v + \beta) = \alpha s_{\sigma(k)}^\sigma(v) + \beta_{\sigma(k)}$ for all $k \in \{1, 2, ..., |N|\}$.
For $k \in \{2, ..., |N|\}$, by the definition of individual standardized remainder and claim 1 we find

$$\begin{aligned}
& s_{\sigma(k)}^\sigma(\alpha v + \beta) \\
= & \left(\alpha v(\{\sigma(k)\}) + \beta_{\sigma(k)}\right) \\
+ & \frac{1}{2}\left(r^{\alpha v+\beta}(S_k^\sigma) - \left(\alpha v(S_{k-1}^\sigma) + \sum_{i\in S_{k-1}^\sigma} \beta_i\right) - \left(\alpha v(\{\sigma(k)\}) + \beta_{\sigma(k)}\right)\right) \\
= & \alpha\left(v(\{\sigma(k)\}) + \frac{1}{2}\left(r^v(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})\right)\right) + \beta_{\sigma(k)} \\
= & \alpha s_{\sigma(k)}^\sigma(v) + \beta_{\sigma(k)}.
\end{aligned}$$

For $k = 1$, this is obvious.
From claim 1 and claim 2 it immediately follows that

$$\gamma_i(\alpha v + \beta) = \alpha\gamma_i(v) + \beta_i$$

for all $i \in N$. ■

Lemma 2.2.4 will facilitate the discussion about the following example, a variation of glove games.

Example 2.2.5 *(A super player)*

For a game $v \in TU^N$, we call player $i \in N$ a super player *if for all $S \subset N$*

$$v(S) = \begin{cases} \sum_{j \in S} v(\{j\}) & \text{if } i \notin S \\ \sum_{j \in S} v(\{j\}) + \alpha & \text{if } i \in S, S \neq \{i\} \\ v(\{i\}) & \text{if } S = \{i\} \end{cases}$$

where $\alpha \in \mathbb{R}$.

Using the fact that the consensus value is relative invariant with respect to strategic equivalence, we can easily obtain that

$$\gamma_i(v) = \frac{1}{2}\left(v(N) - \sum_{k \in N} v(\{k\})\right) + v(\{i\}) = \frac{\alpha}{2} + v(\{i\})$$

and for all $j \in N \backslash \{i\}$

$$\gamma_j(v) = \frac{1}{2(|N|-1)}\left(v(N) - \sum_{k \in N} v(\{k\})\right) + v(\{j\}) = \frac{\alpha}{2(|N|-1)} + v(\{i\}).$$

The consensus value can be reformulated by means of a recursive formula, adopting the same technique as in the paper by O'Neill (1982) for the so-called Run to the Bank rule. The basic idea may be described as follows. As we aim to find the final gains of individual players in the grand coalition of a TU game, we focus on individual players' collaborating behavior as well. One can see that there are mainly two types of collaborating behavior possible for a player in a game: either joining in sub-coalitions or cooperating with all the others based on herself (*stand-alone*). If a player chooses stand-alone cooperation, her payoff will be determined by the 2-person game standard solution as we take all the others as one integrated/cooperating party. By assigning the rest of the wealth, one defines a reduced game for the group of players $N \backslash \{i\}$. Then, a solution concept f is said to satisfy the *stand-alone recursion* if, when applying to all reduced games, the average of the sum of the corresponding payoffs and the stand-alone payoff is the same as the payoff in the original game.

Formally, let $f : TU^N \longrightarrow \mathbb{R}^N$ be a solution concept. Let (N, v) be a TU game. For any player $i \in N$, we introduce the game $(N \backslash \{i\}, v^{-i})$ defined by for all $S \subset N \backslash \{i\}$

$$v^{-i}(S) = \begin{cases} v(S) & \text{if } S \subsetneqq N \backslash \{i\} \\ v(N \backslash \{i\}) + \frac{v(N) - v(N \backslash \{i\}) - v(\{i\})}{2} & \text{if } S = N \backslash \{i\} \end{cases}$$

and call v^{-i} the *stand-alone reduced game* of (N, v) with respect to player i.

We say that f satisfies *stand-alone recursion* if for every TU game (N, v) with $|N| \geq 3$ we have

$$f_i(N, v) = \frac{\sum_{j \in N\backslash\{i\}} f_i\left(N\backslash\{j\}, v^{-j}\right) + \left(v(\{i\}) + \frac{v(N) - v(N\backslash\{i\}) - v(\{i\})}{2}\right)}{|N|}$$

for all $i \in N$.

One can readily check that the consensus value is the unique one-point solution concept on the class of all n-person TU games with $n \geq 2$ which is standard for 2-person games and satisfies stand-alone recursion.

2.3 Characterizations

Let $f : TU^N \longrightarrow \mathbb{R}^N$ be a one-point solution concept. We consider the following properties.

- *Efficiency:* $\sum_{i \in N} f_i(v) = v(N)$ for all $v \in TU^N$;
- *Symmetry:* for two players $i, j \in N$, if $v(S \cup \{i\}) = v(S \cup \{j\})$ for any $S \subset N\backslash\{i, j\}$, we have $f_i(v) = f_j(v)$ for all $v \in TU^N$;
- *The quasi dummy property:* if for some player $i \in N$, $v(S \cup \{i\}) = v(S) + v(\{i\})$ for any $S \subset N\backslash\{i\}$, we have

  $$f_i(v) = \frac{1}{2} v(\{i\}) + \frac{1}{2}\left(v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}\right)$$

  for all $v \in TU^N$;
- *Additivity:* $f(v_i + v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in TU^N$.

The properties of efficiency, symmetry, and additivity are clear by themselves. The quasi dummy property is a modification of the classical dummy property.

As is argued in the introduction, the classical dummy property is utilitarianism oriented, or, put differently, individualism oriented. However, from the egalitarian point of view or from the collectivistic perspective, one can argue that all members of a society including dummies should share the joint surplus equally among them. Thus, assigning a dummy player either $v(\{i\})$ or $v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}$ can be viewed as consequences of two contrastive viewpoints. Concerning the tradeoff between these two extreme cases[7], we make a fair compromise and take the average as the gain of a dummy player, which results in the so-called quasi dummy property.

We show that the consensus value is the unique one-point solution concept that satisfies these four properties.

Theorem 2.3.1 *Let $f : TU^N \longrightarrow \mathbb{R}^N$. Then, f equals the consensus value if and only if it satisfies efficiency, symmetry, the quasi dummy property and additivity.*

Proof.

We first show that the consensus value satisfies those four properties.
(i) Efficiency is obvious since, by construction, $s^\sigma(v)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Now, let us check symmetry. Let i, j be two symmetric players in game $v \in TU^N$. Consider $\sigma \in \Pi(N)$, and set, without loss of generality, $\sigma(k) = i$, $\sigma(l) = j$, where $k, l \in \{1, ..., |N|\}$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from σ by interchanging the positions of i and j, i.e.,

$$\bar{\sigma}(m) = \begin{cases} \sigma(m) & \text{if } m \neq k, l \\ i & \text{if } m = l \\ j & \text{if } m = k. \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $s_i^\sigma(v) = s_j^{\bar{\sigma}}(v)$.
Case 1: $1 < k < l$.
By definition, we know

$$\begin{aligned} s_i^\sigma(v) = s_{\sigma(k)}^\sigma(v) &= v(\{\sigma(k)\}) + \frac{1}{2}\left(r(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})\right) \\ s_j^{\bar{\sigma}}(v) = s_{\bar{\sigma}(k)}^{\bar{\sigma}}(v) &= v(\{\bar{\sigma}(k)\}) + \frac{1}{2}\left(r(S_k^{\bar{\sigma}}) - v(S_{k-1}^{\bar{\sigma}}) - v(\{\bar{\sigma}(k)\})\right) \end{aligned}$$

Note that, $v(\{\sigma(k)\}) = v(\{i\}) = v(\{j\}) = v(\{\bar{\sigma}(k)\})$, $S_{k-1}^\sigma = S_{k-1}^{\bar{\sigma}}$, and thus $v(S_{k-1}^\sigma) = v(S_{k-1}^{\bar{\sigma}})$. It remains to show that $r(S_k^\sigma) = r(S_k^{\bar{\sigma}})$.
Clearly, $r(S_m^\sigma) = r(S_m^{\bar{\sigma}})$ for $m \geq l$. Recursively, we can show that $r(S_{l-t}^\sigma) = r(S_{l-t}^{\bar{\sigma}})$ for $t \in \{1, ..., l-k-1\}$ as

$$r(S_{l-t}^\sigma) = v(S_{l-t}^\sigma) + \frac{1}{2}\left(r(S_{l-t+1}^\sigma) - v(S_{l-t}^\sigma) - v(\{\sigma(l-t+1)\})\right)$$

and

$$r(S_{l-t}^{\bar{\sigma}}) = v(S_{l-t}^{\bar{\sigma}}) + \frac{1}{2}\left(r(S_{l-t+1}^{\bar{\sigma}}) - v(S_{l-t}^{\bar{\sigma}}) - v(\{\bar{\sigma}(l-t+1)\})\right).$$

[7]Cultural and philosophical factors may affect the propensity or choice between the two polar opinions.

Here, we also use the fact that $\sigma(l-t)=\bar{\sigma}(l-t)$. Moreover, since $S^{\sigma}_{l-t}\backslash\{i\}=S^{\bar{\sigma}}_{l-t}\backslash\{j\}$, and we know that $v(S^{\sigma}_{l-t})=v(S^{\bar{\sigma}}_{l-t})$.
Then, using symmetry of the players (twice),

$$\begin{aligned}
r(S^{\sigma}_{k}) &= v(S^{\sigma}_{k})+\frac{1}{2}\left(r(S^{\sigma}_{k+1})-v(S^{\sigma}_{k})-v(\{\sigma(k+1)\})\right)\\
&= v(S^{\bar{\sigma}}_{k})+\frac{1}{2}\left(r(S^{\bar{\sigma}}_{k+1})-v(S^{\bar{\sigma}}_{k})-v(\{\bar{\sigma}(k+1)\})\right)\\
&= r(S^{\bar{\sigma}}_{k}).
\end{aligned}$$

Case 2: $1<l<k$. The proof is analogous to Case 1.
Case 3: $1=k<l$.
In this case,

$$\begin{aligned}
s^{\sigma}_{i}(v)=s^{\sigma}_{\sigma(1)}(v) &= r(S^{\sigma}_{1})\\
s^{\bar{\sigma}}_{j}(v)=s^{\bar{\sigma}}_{\bar{\sigma}(1)}(v) &= r(S^{\bar{\sigma}}_{1}).
\end{aligned}$$

What remains is identical to Case 1.
Case 4: $1=l<k$. Analogously to Case 3.
(iii) As for additivity, it is immediate to see that $s^{\sigma}_{\sigma(k)}(v_1+v_2)=s^{\sigma}_{\sigma(k)}(v_1)+s^{\sigma}_{\sigma(k)}(v_2)$ for all $v_1,v_2\in TU^N$ and for all $k\in\{1,...,|N|\}$.
(iv) By relative invariance with respect to strategic equivalence (Lemma 2.2.4), it suffices to prove that the consensus value γ satisfies the quasi dummy property for zero-normalized games. Let $v\in TU^N$ be zero-normalized and $i\in N$ a dummy in v. It suffices to show that $\gamma_i(v)=\frac{v(N)}{2|N|}$.
For $\sigma\in\Pi(N)$, we have

$$\begin{aligned}
r(S^{\sigma}_{|N|}) &= v(N)\\
r(S^{\sigma}_{|N|-1}) &= \frac{1}{2}v(N)+\frac{1}{2}v(S^{\sigma}_{|N|-1})\\
r(S^{\sigma}_{|N|-2}) &= \frac{1}{4}v(N)+\frac{1}{4}v(S^{\sigma}_{|N|-1})+\frac{1}{2}v(S^{\sigma}_{|N|-2})\\
r(S^{\sigma}_{|N|-3}) &= \frac{1}{8}v(N)+\frac{1}{8}v(S^{\sigma}_{|N|-1})+\frac{1}{4}v(S^{\sigma}_{|N|-2})+\frac{1}{2}v(S^{\sigma}_{|N|-3})\\
&\cdots\\
r(S^{\sigma}_{2}) &= \frac{1}{2}(r(S^{\sigma}_{3})+v(S^{\sigma}_{2}))\\
r(S^{\sigma}_{1}) &= \frac{1}{2}(r(S^{\sigma}_{2}))
\end{aligned}$$

A general expression is provided below.

$$r(S_k^\sigma) = \begin{cases} v(N) & \text{if } k = |N| \\ (\frac{1}{2})^{|N|-k} v(N) + \sum_{l=k}^{|N|-1} (\frac{1}{2})^{l-k+1} v(S_l^\sigma) & \text{if } 2 \leq k \leq |N|-1 \\ (\frac{1}{2})^{|N|-1} v(N) + \sum_{l=2}^{|N|-1} (\frac{1}{2})^{l} v(S_l^\sigma) & \text{if } k = 1. \end{cases}$$

Let $i \in N$ be a dummy player in v. Let $\sigma(k) = i$. Then,

$$s_i^\sigma(v) = s_{\sigma(k)}^\sigma(v) = \begin{cases} \frac{1}{2}\left(r(S_k^\sigma) - v(S_{k-1}^\sigma)\right) & \text{if } k \geq 2 \\ r(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Hence,

$$\begin{aligned}
s_{\sigma(|N|)}^\sigma(v) = & \quad 0 \\
s_{\sigma(|N|-1)}^\sigma(v) = & \quad \frac{1}{4}v(N) + \frac{1}{4}v(S_{|N|-1}^\sigma) - \frac{1}{2}v(S_{|N|-2}^\sigma) \\
s_{\sigma(|N|-2)}^\sigma(v) = & \quad \frac{1}{8}v(N) + \frac{1}{8}v(S_{|N|-1}^\sigma) + \frac{1}{4}v(S_{|N|-2}^\sigma) - \frac{1}{2}v(S_{|N|-3}^\sigma) \\
s_{\sigma(|N|-3)}^\sigma(v) = & \quad \frac{1}{16}v(N) + \frac{1}{16}v(S_{|N|-1}^\sigma) + \frac{1}{8}v(S_{|N|-2}^\sigma) \\
+ & \quad \frac{1}{4}v(S_{|N|-3}^\sigma) - \frac{1}{2}v(S_{|N|-4}^\sigma) \\
... & \\
s_{\sigma(3)}^\sigma(v) = & \quad \frac{1}{2^{|N|-2}}v(N) + \frac{1}{2^{|N|-2}}v(S_{|N|-1}^\sigma) + \frac{1}{2^{|N|-3}}v(S_{|N|-2}^\sigma) \\
+ & \quad ... + \frac{1}{4}v(S_3^\sigma) - \frac{1}{2}v(S_2^\sigma) \\
s_{\sigma(2)}^\sigma(v) = & \quad \frac{1}{2^{|N|-1}}v(N) + \frac{1}{2^{|N|-1}}v(S_{|N|-1}^\sigma) + \frac{1}{2^{|N|-2}}v(S_{|N|-2}^\sigma) \\
+ & \quad ... + \frac{1}{8}v(S_3^\sigma) + \frac{1}{4}v(S_2^\sigma) \\
s_{\sigma(1)}^\sigma(v) = & \quad \frac{1}{2^{|N|-1}}v(N) + \frac{1}{2^{|N|-1}}v(S_{|N|-1}^\sigma) + \frac{1}{2^{|N|-2}}v(S_{|N|-2}^\sigma) \\
+ & \quad ... + \frac{1}{8}v(S_3^\sigma) + \frac{1}{4}v(S_2^\sigma).
\end{aligned}$$

A general expression is

$$s_i^\sigma(v) = s_{\sigma(k)}^\sigma(v) = \begin{cases} 0 & \text{if } k = |N| \\ (\frac{1}{2})^{|N|-k+1} v(N) + \sum_{l=k}^{|N|-1} (\frac{1}{2})^{l-k+2} v(S_l^\sigma) \\ -\frac{1}{2}v(S_{k-1}^\sigma) & \text{if } 2 \leq k \leq |N|-1 \\ (\frac{1}{2})^{|N|-1} v(N) + \sum_{l=2}^{|N|-1} (\frac{1}{2})^{l} v(S_l^\sigma) & \text{if } k = 1 \end{cases}$$

Consider a class P of $|N|$ permutations in $\Pi(N)$ such that for $\sigma, \tau \in P$ it holds that for all $j_1, j_2 \in N\backslash\{i\}$

$$\sigma^{-1}(j_1) < \sigma^{-1}(j_2) \Leftrightarrow \tau^{-1}(j_1) < \tau^{-1}(j_2).$$

That is, given an ordering of the players $N\backslash\{i\}$, let the dummy player i move from the end to the beginning without changing the other players' relative positions. Summing over the above equations, one readily checks that all the terms except $v(N)$ will cancel out. That is,

$$\sum_{\sigma \in P} s_i^{\sigma}(v) = \left((\frac{1}{2})^2 + (\frac{1}{2})^3 + ... + (\frac{1}{2})^{|N|-1} + (\frac{1}{2})^{|N|-1} \right) v(N) = \frac{1}{2} v(N).$$

Since $\Pi(N)$ can be partitioned in $(|N|-1)!$ of these classes, it follows that

$$\gamma_i(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s_i^{\sigma}(v) = \frac{(|N|-1)!}{|N|!} \cdot \frac{v(N)}{2} = \frac{v(N)}{2|N|}.$$

Conversely, let $f : TU^N \longrightarrow \mathbb{R}^N$ satisfy efficiency, symmetry, and the quasi dummy property. It easily follows that f is uniquely determined for (multiples of) unanimity games. Hence requiring a solution f to be additive too, it follows that f is uniquely determined for any game in TU^N, since the class of unanimity games constitutes a basis of TU^N. ■

Note that the quasi dummy property can be reformulated as $f_i(v) = \frac{1}{2}\Phi_i(v) + \frac{1}{2}E_i(v)$ for all $v \in TU^N$ and every dummy player i in v. Here, $\Phi(v)$ is the Shapley value of game v and $E(v)$ denotes the equal surplus solution of game v, i.e., $E_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}$ for all $i \in N$.

In fact, this property carries over to all players as is seen in Theorem 2.3.2.

Theorem 2.3.2 *For every v in TU^N it holds that*

$$\gamma(v) = \frac{1}{2}\Phi(v) + \frac{1}{2}E(v).$$

Proof. It is readily shown that $f(v) := \frac{1}{2}\Phi(v) + \frac{1}{2}E(v)$ satisfies the four characterizing properties: efficiency, symmetry, quasi dummy property and additivity. ■

We now provide an alternative characterization for the consensus value by means of the transfer property.

The transfer property (Dubey (1975)) in some sense substitutes for additivity. It is defined as follows. For any two games $v_1, v_2 \in TU^N$, we first define the games $(v_1 \vee v_2)$ and $(v_1 \wedge v_2)$ by $(v_1 \vee v_2)(S) = \max\{v_1(S), v_2(S)\}$ and $(v_1 \wedge v_2)(S) = \min\{v_1(S), v_2(S)\}$ for all $S \subset N$. Let $f : TU^N \longrightarrow \mathbb{R}^N$ be a solution concept on the class of TU games. Then, f satisfies the transfer property if $f(v_1 \vee v_2) + f(v_1 \wedge v_2) = f(v_1) + f(v_2)$ for all $v_1, v_2 \in TU^N$. Dubey (1975) characterized the Shapley value as the unique value on the class of monotonic simple games satisfying efficiency, symmetry, the dummy property, and transfer property. Feltkamp (1995) generalized this result to the class of all TU games. More specifically, the Shapley value is the unique value on the class of TU games satisfying efficiency, symmetry, the dummy property and the transfer property (cf. Feltkamp (1995, p.134, Theorem 9.1.5)).

We now have an alternative characterization of the consensus value for TU games.

Theorem 2.3.3 *The consensus value is the only one-point solution on the class of TU games that satisfies efficiency, symmetry, the quasi dummy property and the transfer property.*

Proof. A solution concept $f : TU^N \longrightarrow \mathbb{R}^N$ satisfying additivity on TU^N also satisfies the transfer property on TU^N. To prove this, take $v_1, v_2 \in TU^N$. Then, using additivity,

$$\begin{aligned} f(v_1 \vee v_2) + f(v_1 \wedge v_2) &= f(v_1 \vee v_2 + v_1 \wedge v_2) \\ &= f(v_1 + v_2) \\ &= f(v_1) + f(v_2). \end{aligned}$$

Therefore, the consensus value satisfies the transfer property. In addition, requiring a solution concept $f : TU^N \longrightarrow \mathbb{R}^N$ to satisfy efficiency, symmetry, and the quasi dummy property, it easily follows that f is uniquely determined for (multiples of) unanimity games. Moreover, by Feltkamp (1995, Lemma 9.1.4), it follows that if the solution concept f satisfies the transfer property too, it is uniquely determined for any game in TU^N. ■

We want to notice that despite the fact that the quasi dummy property balances those two extreme opinions, namely, utilitarianism (individualism) and egalitarianism (collectivism), it is still in favor of non-dummy players in terms of aggregate share of joint surplus in superadditive games. For instance, even if in the extreme cases where an n-person game may have $n-2$ dummies while only 2 non-dummies, non-dummy players will get $(n+2)/2n$ of the joint surplus $v(N)-\sum_{i\in N} v(\{i\})$, which is still greater than half of it.

As we know the consensus value is the average of the Shapley value and the equal surplus solution, it may be interesting to see under what condition those solution concepts are equivalent.

We call a game $v \in TU^N$ *zero-symmetric* if in the zero-normalization of v all players are symmetric. Now, we have the following theorem.

Theorem 2.3.4 *Let (N, v) be a TU game. If v is zero-symmetric then $\gamma(v) = \Phi(v)$. Moreover, if $|N| \leq 3$ then $\gamma(v) = \Phi(v)$ implies that v is zero-symmetric.*

Proof. The first part follows straightforward from γ and Φ satisfying symmetry and being relative invariant with respect to strategic equivalence. To prove the second part, consider a three-player game with player set $N = \{1, 2, 3\}$, and let v be zero-normalized. Since $\gamma(v) = \Phi(v) \Leftrightarrow E(v) = \Phi(v)$, it easily follows that $\Phi_1(v) = \Phi_2(v) = \Phi_3(v)$. From the expression of the Shapley value and the game being zero-normalized, we know that the Shapley value and the equal surplus solution assign to player 1 the payoffs

$$\Phi_1(v) = \frac{1}{6}v(\{1,2\}) + \frac{1}{6}v(\{1,3\}) - \frac{1}{3}v(\{2,3\}) + \frac{1}{3}v(N)$$

and

$$E_1(v) = \frac{1}{3}v(N),$$

respectively. Similar expressions can be found for players 2 and 3. These expressions yield the following equalities: $\Phi_1(v) = \Phi_2(v) \Leftrightarrow v(\{1,3\}) = v(\{2,3\})$ and $\Phi_1(v) = \Phi_3(v) \Leftrightarrow v(\{2,3\}) = v(\{1,2\})$. So, all worths of the two-player coalitions are equal and thus all players are symmetric in the zero-normalized game v. The result for three player games then follows from γ and Φ being relative invariant with respect to strategic equivalence. For one- and two-player games the result is obvious. ■

However, we could not get the similar result as the second part in Theorem 2.3.4 for games with more than three players. Consider the four-player game (N, v) with

$N = \{1, 2, 3, 4\}$ and characteristic function v is given by

$$v(S) = \begin{cases} 0 & \text{if } |S| = 1 \\ 1 & \text{if } S \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\} \\ 2 & \text{if } S = \{1,2\} \\ 5 & \text{if } S \in \{\{1,2,3\},\{1,2,4\}\} \\ 5\frac{1}{2} & \text{if } S \in \{\{1,3,4\},\{2,3,4\}\} \\ 14 & \text{if } S = N. \end{cases}$$

The Shapley value and the consensus value of this game are equal to each other, and are both equal to $\left(3\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}, 3\frac{1}{2}\right)$, but the game is not zero-symmetric.

2.4 A generalization of the consensus value

By relaxing the way of sharing remainders, we get a generalization of the consensus value: the *generalized consensus value*.

Let $v \in TU^N$. For given $\theta \in [0,1]$, we define the generalized remainder, with respect to an order $\sigma \in \Pi(N)$, recursively by

$$r_\theta(S_k^\sigma) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S_k^\sigma) \\ +(1-\theta)\left(r_\theta(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\})\right) & \text{if } k \in \{1, ..., |N|-1\}. \end{cases}$$

Correspondingly, the individual generalized remainder vector $s_\theta^\sigma(v)$ is the vector in $\mathbb{R}^N$ defined by

$$\left(s_\theta^\sigma\right)_{\sigma(k)}(v) = \begin{cases} v(\{\sigma(k)\}) \\ +\theta\left(r_\theta(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})\right) & \text{if } k \in \{2, ..., |N|\} \\ r_\theta(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 2.4.1 *For every $v \in TU^N$ and $\theta \in [0,1]$, the generalized consensus value $\gamma_\theta(v)$ is defined as the average of the individual generalized remainder vectors, i.e.,*

$$\gamma_\theta(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s_\theta^\sigma(v).$$

Note that the consensus value corresponds to the case $\theta = \frac{1}{2}$.

Similar to the consensus value, the generalized consensus value can also be reformulated by means of a recursive formula.

As mentioned in section 3, dependent on the degree to which individualism or collectivism is preferred by society, the dummy property can be generalized. Defining the *θ-dummy property* of a one-point solution concept $f : TU^N \longrightarrow \mathbb{R}^N$ by $f_i(v) = \theta v(\{i\}) + (1-\theta)\left(v(\{i\}) + \frac{v(N)-\sum_{j\in N} v(\{j\})}{|N|}\right)$ for all $v \in TU^N$ and every dummy player $i \in N$ with respect to v, we obtain the following theorem.

Theorem 2.4.2 *(a) The generalized consensus value γ_θ is the unique one-point solution concept on TU^N that satisfies efficiency, symmetry, the θ-dummy property and additivity.*

(b) For any $v \in TU^N$, it holds that

$$\gamma_\theta(v) = \theta\Phi(v) + (1-\theta)E(v).$$

(c) The generalized consensus value γ_θ is the unique function that satisfies efficiency, symmetry, the θ-dummy property and the transfer property over the class of TU games.

The expression of the generalized consensus value as provided in part (b) of Theorem 2.4.2 is in the same spirit as the so-called *compound measures* in the context of digraph competitions (cf. Borm, van den Brink and Slikker (2002)) and the *α-egalitarian Shapley value* (cf. Joosten (1996)).

Finally, we want to note that, in particular, for $\theta = 1$, the generalized consensus value is actually the Shapley value: the average serial remainder value turns to be the average serial marginal contribution value. When $\theta = 0$, the generalized consensus value equals to the equal surplus solution.

Consequently, we have the following characterizations for the equal surplus solution.

Corollary 2.4.3 *(a) The equal surplus solution E is the unique one-point solution concept that satisfies efficiency, symmetry, the 0-dummy property and additivity.*

(b) The equal surplus solution E is the unique one-point solution concept that satisfies efficiency, symmetry, the 0-dummy property and transfer property.

The above corollary suggests an interesting result: the equal surplus solution can be indirectly obtained through an average serial method, which implies that even if taking the partial cooperation into consideration we can still get the equal surplus solution. Meanwhile, it is also an intuitive result as Corollary 2.4.3 can be understood as a mathematical annotation for the well known adage "*One for all, all for one.*"

That is, if altruism is accepted as a common principle by all players in a game so that all that they do is just helping the others, then finally every one equally benefits from the joint cooperation.

Another insight we get from this corollary, or more precisely, from Theorem 2.4.2 is that how much is allocated to a dummy player plays a prominent role for determining the gains of all players and thereby is crucial for characterizing a solution concept. The classical dummy property is actually an individualist dummy property while the 0-dummy property is a collectivistic or egalitarian dummy property, and as a hybrid case or taking a mean course, the quasi dummy property is a compromise dummy property. Then, we may say that in some sense the quasi dummy property well balances the tradeoff between efficiency and equity, which would make the consensus value socially and morally acceptable.

2.5 Discussion: the coalition-size-based consensus value

If one takes the size of the incumbent party S into consideration, one can argue on the basis of a proportional principle that given an ordering of players the entrant should get $\frac{1}{|S|+1}$ of the joint surplus while the incumbents as a whole get a share of $\frac{|S|}{|S|+1}$. This results in another solution concept, namely, the *coalition-size-based consensus value.*

Let $v \in TU^N$. We define the coalition-size-based remainder for coalition S_k^σ, with respect to an order $\sigma \in \Pi(N)$, recursively by

$$\bar{r}(S_k^\sigma) = \begin{cases} v(N) & \text{if } k = |N| \\ v(S_k^\sigma) + \frac{|S_k^\sigma|}{|S_k^\sigma|+1}\left(\bar{r}(S_{k+1}^\sigma) - v(S_k^\sigma) - v(\{\sigma(k+1)\})\right) & \text{if } k \in \{1, ..., |N|-1\}. \end{cases}$$

Correspondingly, the coalition-size-based individual remainder vector $\bar{s}^\sigma(v)$ is the vector in $\mathbb{R}^N$ defined by

$$\bar{s}^\sigma_{\sigma(k)}(v) = \begin{cases} v(\{\sigma(k)\}) + \frac{1}{|S_k^\sigma|}\left(\bar{r}(S_k^\sigma) - v(S_{k-1}^\sigma) - v(\{\sigma(k)\})\right) & \text{if } k \in \{2, ..., |N|\} \\ \bar{r}(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 2.5.1 *For every $v \in TU^N$, the coalition-size-based consensus value $\bar{\gamma}(v)$ is defined as the average of the coalition-size-based individual remainder vectors, i.e.,*

$$\bar{\gamma}(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \bar{s}^\sigma(v).$$

The coalition-size-based consensus value can be reformulated by means of a recursive formula. Let $f : TU^N \longrightarrow \mathbb{R}^N$ be a solution concept. For a TU game (N, v) and $i \in N$, we introduce the game $(N\backslash\{i\}, \bar{v}^{-i})$ defined by

$$\bar{v}^{-i}(S) = \begin{cases} v(S) & \text{if } S \subsetneqq N\backslash\{i\} \\ v(N\backslash\{i\}) + \frac{|N|-1}{|N|}\left(v(N) - v(N\backslash\{i\}) - v(\{i\})\right) & \text{if } S = N\backslash\{i\} \end{cases}$$

and call $\bar{v}^{-i}$ the *coalition-size-based stand-alone reduced game* of (N, v) with respect to player i.

We say that a solution concept f satisfies *coalition-size-based stand-alone recursion* if for every game (N, v) with $|N| \geq 3$ we have

$$f_i(N, v) = \frac{\sum_{j \in N\backslash\{i\}} f_i\left(N\backslash\{j\}, \bar{v}^{-j}\right) + \left(v(\{i\}) + \frac{v(N) - v(N\backslash\{i\}) - v(\{i\})}{|N|}\right)}{|N|}$$

for all $i \in N$.

One can readily check that the coalition-size-based consensus value is the unique one-point solution concept on the class of all n-person TU games with $n \geq 2$ which is standard for 2-person games and satisfies coalition-size-based stand-alone recursion.

To further illustrate the value, we consider the game specified in Example 2.2.2 and the glove game in Example 2.2.3.

Example 2.5.2 *(a) Consider the 3-person TU game described in Example 2.2.2. For $\sigma = (2\ 1\ 3)$, we get*

$$\begin{aligned} \bar{s}_3^\sigma(v)(= \bar{s}_{\sigma(3)}^\sigma(v)) &= v(\{3\}) + \frac{1}{3}(v(N) - v(\{1,2\}) - v(\{3\})) = 4, \\ \bar{s}_1^\sigma(v)(= \bar{s}_{\sigma(2)}^\sigma(v)) &= v(\{1\}) + \frac{1}{2}(\bar{r}(\{2,1\}) - v(\{2\}) - v(\{1\})) = 18, \\ \bar{s}_2^\sigma(v)(= \bar{s}_{\sigma(1)}^\sigma(v)) &= \bar{r}(\{2\}) = v(\{2\}) + \frac{1}{2}(\bar{r}(\{2,1\}) - v(\{2\}) - v(\{1\})) = 8. \end{aligned}$$

All coalition-size-based individual remainder vectors are given by

σ	$\bar{s}_1^\sigma$	$\bar{s}_2^\sigma$	$\bar{s}_3^\sigma$
(123)	18	8	4
(132)	$18\frac{5}{6}$	$2\frac{1}{3}$	$8\frac{5}{6}$
(213)	18	8	4
(231)	$16\frac{2}{3}$	$6\frac{2}{3}$	$6\frac{2}{3}$
(312)	$18\frac{5}{6}$	$2\frac{1}{3}$	$8\frac{5}{6}$
(321)	$16\frac{2}{3}$	$6\frac{2}{3}$	$6\frac{2}{3}$

Hence, $\bar{\gamma}(v) = \left(17\frac{5}{6}, 5\frac{2}{3}, 6\frac{1}{2}\right)$.

(b) For the glove game described in Example 2.2.3, one can readily check that $\bar{\gamma}(v) = \left(\frac{4}{9}, \frac{5}{18}, \frac{5}{18}\right)$. *In a more general case where* $|N| > 3$ *but there still exists only one left hand glove player while all the others have one right hand glove each, according to the coalition-size-based consensus value, the left hand glove player gets* $\frac{1}{|N|}\left(\frac{1}{2} + \sum_{k=2}^{|N|}\frac{1}{k}\right)$ *and each right hand glove player gets* $\frac{1}{|N|-1}\left(1 - \frac{1}{|N|}\left(\frac{1}{2} + \sum_{k=2}^{|N|}\frac{1}{k}\right)\right)$.

2.6 An application: the consensus value and merger incentives

In this section, we apply the consensus value to analyse the issue of merger incentives in the network industries that are characterized by essential facilities. However, one should not view this section as a motivation for developing the consensus value, but should take it as an application only or somehow a separate study.

Jeon (2003) constructs a cooperative game theoretical framework to analyse the issue of merger incentives in the network industries with essential facilities. More specifically, those essential facilities are provided by one upstream operator for a number of downstream operators who supply final services for end-users. The examples of such industries are the Internet, cable, telecommunications, gas, and electricity industries. Essential bottleneck facilities in the relevant industries are backbones for Internet Service Providers, channel providers for local cable system operators, Local Exchange Carriers for long-distance telecom operators, distribution pipes for local gas suppliers, transmission grids for local electricity firms.

Assuming Shapley bargaining over access charges, Jeon finds that the main condition for mergers, either vertical or horizontal, to be beneficial to the merging parties is that the aggregate profit function of a coalition is concave in the size of the network that the coalition covers, i.e., that the network industry exhibits decreasing returns to network size. This is an interesting result. However, as Jeon himself remarks, the Shapley value is one of many bargaining solution concepts despite that it has been given many justifications, so it is worthwhile to check whether the results obtained in his paper are robust to other bargaining solutions. In this section, we use the same model (but more elaborate) and examine his results under the consensus value.

From the above sections, we know that the consensus value is featured by its constructive process for sharing the joint gains and is characterized as the unique one-point solution concept for TU games that satisfies efficiency, symmetry, the quasi dummy property and additivity. By the quasi dummy property, the consensus value well balances the tradeoff between utilitarianism/individualism and egalitarianism/collectivism. Therefore, this solution concept might be appropriate for analysing, among other, intra-firm issues. Consider the example/application of intra-firm wage bargaining (cf. Stole and Zwiebel (1996)) mentioned in Jeon (2003). It seems reasonable that a dummy player gets a share in the joint profit.

Consider a network industry that is composed of a set of operators $N = \{0, 1, 2, ..., n-1\}$ where 0 is an upstream operator and all the others form the set of downstream operators $K = \{1, 2, ..., n-1\}$. The upstream operator, i.e., operator 0, provides the essential facilities or inputs for downstream operator $i \in K$ which supplies final services for end-users. Thus, there are $|N|$ operators in such an industry. Operator $i \in N$ covers a network with size m_i. The size of the network may represent the number of end-users or service areas. Here, we assume that the upstream operator has no direct end-users, i.e. $m_0 = 0$. Then, the size of the whole network in the industry is $M \equiv \sum_{i=1}^{n-1} m_i$. Let $u(M)$ be the net benefit per end-user. It is assumed that $u'(M) > 0$ due to network externalities and $u(0) = 0$ by convention. Generally, let $u(m)$ be the net benefit per end-user with respect to network size m where $0 \leq m \leq M$. We assume that $u'(m) = 0$ when $m = 0$ and $u'(m) > 0$ otherwise. Each downstream operator $i \in K$ pays access charge a_i to operator 0 for the use of the essential facilities. Hence, if all operators pay for access, operator i obtains net profit after paying the charge, $m_i u(M) - a_i$. By assuming that the costs of facilities investment are sunk, we see that operator 0's profit is $\sum_{i=1}^{n-1} a_i$.

Defining the characteristic function $v(S)$ as

$$v(S) = \begin{cases} 0 & \forall S \subset N \text{ such that } 0 \notin S \\ \sum_{i \in S} m_i \cdot u(\sum_{i \in S} m_i) & \text{otherwise,} \end{cases}$$

i.e. any coalition's value is the aggregate of its members' profits in case only all members in S have access to the essential facilities, we get a cooperative game with transferable utility (N, v).

Let $\pi(\sum_{i \in S} m_i) = \sum_{i \in S} m_i \cdot u(\sum_{i \in S} m_i)$ for all $S \subset N$ such that $0 \in S$. Then, since $\pi(m) \equiv m \cdot u(m)$, $\pi(0) = 0$ and $\pi'(m) > 0$.

Along the same line as Jeon (2003), we do not explicitly model the process of bargaining over access charges but directly focus on the specific solution concepts. By assuming that each operator gets its share according to the consensus value γ, the access charges are determined by

$$\gamma_0(v) = \sum_{i=1}^{n-1} a_i \text{ and } \gamma_i(v) = m_i u(M) - a_i \text{ for all } i \in K.$$

By Theorem 2.3.2, we know

$$\gamma_i(v) = \frac{1}{2}\Phi_i(v) + \frac{1}{2}E_i(v) \text{ for all } i \in N.$$

Here,

$$\Phi_0(v) = \sum_{S \subset K} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \pi(\sum_{i \in S} m_i)$$

and

$$\Phi_j(v) = \sum_{S \subset N \backslash \{j\}, 0 \in S} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [\pi(m_j + \sum_{i \in S} m_i) - \pi(\sum_{i \in S} m_i)]$$

for all $j \in K$; and the equal surplus solution E is given by

$$E_i(v) = \frac{v(N)}{|N|} = \frac{\pi(M)}{|N|} \text{ for all } i \in N. \tag{2.1}$$

For any two operators $i, j \in N$, if they merge into one operator, we then denote the merged operator by $i + j$. Correspondingly, we have a well defined new game v_{i+j} derived from the original game v. Then, we use $\gamma_{i+j}(v_{i+j})$ (similar for the Shapley value $\Phi_{i+j}(v_{i+j})$ and the equal surplus solution $E_{i+j}(v_{i+j})$) to denote the merged operator's consensus value in the new game v_{i+j} while $\gamma_i(v)$ and $\gamma_j(v)$ denote i and j's consensus values in the original game v. The merger is said to be beneficial for the merging parties if $\gamma_{i+j}(v_{i+j}) > \gamma_i(v) + \gamma_j(v)$. Below we will check the merger incentives both in the case of vertical merger and horizontal merger. Considering the representative case of a vertical merger between the upstream operator 0 and a downstream operator, say 1, we can derive the share for the merged operator $0 + 1$ as

$$\begin{aligned} \gamma_{0+1}(v_{0+1}) = & \quad \frac{1}{2}\Phi_{0+1}(v_{0+1}) + \frac{1}{2}E_{0+1}(v_{0+1}) \\ = & \quad \frac{1}{2} \sum_{S \subset N \backslash \{0,1\}} \frac{|S|!(|N| - |S| - 2)!}{(|N| - 1)!} \pi(m_1 + \sum_{j \in S} m_j) \\ + & \quad \frac{1}{2} \left(\pi(m_1) + \frac{\pi(M) - \pi(m_1)}{|N| - 1} \right). \end{aligned}$$

Similarly, for the representative case of a horizontal merger between two downstream operators, say 1 and 2, we can derive the share for the merged operator $1+2$ as

$$\begin{aligned}
& \gamma_{1+2}(v_{1+2}) \\
= \quad & \frac{1}{2}\Phi_{1+2}(v_{1+2}) + \frac{1}{2}E_{1+2}(v_{1+2}) \\
= \quad & \frac{1}{2}\sum_{S\subset N\setminus\{1,2\},0\in S}\frac{|S|!(|N|-|S|-2)!}{(|N|-1)!}\left(\pi(m_1+m_2+\sum_{j\in S}m_j)-\pi(\sum_{j\in S}m_j)\right) \\
+ \quad & \frac{1}{2}\cdot\frac{\pi(M)}{|N|-1}.
\end{aligned}$$

We make the following assumption.

Assumption 2.6.1 *If* $|S|>|S'|$*, then* $\sum_{i\in S}m_i \geq \sum_{i\in S'}m_i$, $\forall S,S'\subset K$.

This assumption implies that there do not exist dominant downstream operators whose network sizes are considerably larger than others.

Theorem 2.6.2 *Here and below, let* v *be given. Then*

(a) with Assumption 2.6.1, we have

$$\text{for all } m,\ \pi''(m)<0 \ \Rightarrow\ \Phi_{0+1}(v_{0+1}) > \Phi_0(v)+\Phi_1(v);$$
$$\text{for all } m,\ \pi''(m)>0 \ \Rightarrow\ \Phi_{0+1}(v_{0+1}) < \Phi_0(v)+\Phi_1(v).$$

(b)

$$\left.\begin{array}{l}\text{for all } m,\ \pi''(m)<0 \\ \frac{M}{|N|}<m_1\end{array}\right\}\Rightarrow E_{0+1}(v_{0+1}) > E_0(v)+E_1(v);$$

$$\left.\begin{array}{l}\text{for all } m,\ \pi''(m)>0 \\ \frac{M}{|N|}>m_1\end{array}\right\}\Rightarrow E_{0+1}(v_{0+1}) < E_0(v)+E_1(v).$$

Proof. Theorem 2.6.2 (a) is due to Jeon (2003). Now we prove part (b). From equation (2.1), we know

$$\begin{aligned}
& E_{0+1}(v_{0+1}) - (E_0(v)+E_1(v)) \\
= \quad & \pi(m_1) + \frac{\pi(M)-\pi(m_1)}{|N|-1} - 2\frac{\pi(M)}{|N|} \\
= \quad & \frac{|N|-2}{|N|(|N|-1)}\left(|N|\pi(m_1)-\pi(M)\right).
\end{aligned}$$

Since $\pi'(m) > 0$, if $M < |N|m_1$, then $\pi(M) < \pi(|N|m_1)$. Moreover $\pi''(m) < 0 \Rightarrow \pi(|N|m_1) < |N|\pi(m_1)$. Therefore, $\pi(M) < |N|\pi(m_1)$. So, $E_{0+1}(v_{0+1}) > E_0(v)+E_1(v)$. Similarly, we can prove the case of $E_{0+1}(v_{0+1}) < E_0(v) + E_1(v)$. ■

With the above theorem and since $\gamma_{0+1}(v_{0+1}) - (\gamma_0(v) + \gamma_1(v)) = \frac{1}{2}(\Phi_{0+1}(v_{0+1}) - (\Phi_0(v) + \Phi_1(v))) + \frac{1}{2}(E_{0+1}(v_{0+1}) - (E_0(v) + E_1(v)))$, we then readily get the following corollary.

Corollary 2.6.3 *With Assumption 2.6.1, for all $m \geq 0$, we have*

(a)

$$\left.\begin{array}{l} \text{for all } m,\ \pi''(m) < 0 \\ \frac{M}{|N|} < m_1 \end{array}\right\} \Rightarrow \gamma_{0+1}(v_{0+1}) > \gamma_0(v) + \gamma_1(v);$$

(b)

$$\left.\begin{array}{l} \text{for all } m,\ \pi''(m) > 0 \\ \frac{M}{|N|} > m_1 \end{array}\right\} \Rightarrow \gamma_{0+1}(v_{0+1}) < \gamma_0(v) + \gamma_1(v).$$

Corollary 2.6.3 (a) shows that under the consensus value the main condition for vertical mergers to be beneficial (costly) to the merging parties is that the aggregate profit function of a coalition is concave (convex) in the size of the network that the coalition covers, i.e., that the network industry exhibits decreasing (increasing) returns to network size. Moreover, the size of the involved downstream merging operator should be sufficiently large, exactly speaking, larger than the average size $\frac{M}{|N|}$. Basically, this result is consistent with the result obtained by Jeon (2003) for the case of the Shapley value. In addition, if adopting the generalized consensus value γ_θ, one can find that we will get the same result as this corollary.

Under the consensus value, we have the following result on costly horizontal merger.

Theorem 2.6.4 *For all m, $\pi''(m) > 0$ and $\pi'''(m) \geq 0 \Rightarrow \gamma_{1+2}(v_{1+2}) < \gamma_1(v) + \gamma_2(v)$.*

Proof. One can readily check that

$$E_{1+2}(v_{1+2}) - (E_1(v) + E_2(v)) = \frac{\pi(M)}{|N|-1} - 2\frac{\pi(M)}{|N|} = \frac{2-|N|}{|N|(|N|-1)}\pi(M) < 0.$$

That is, if applying the equal surplus solution, downstream operators will always prefer to stay apart. Moreover, Jeon (2003) shows that under the conditions of the theorem,

$\Phi_{1+2}(v_{1+2}) < \Phi_1(v) + \Phi_2(v)$, so clearly, $\gamma_{1+2}(v_{1+2}) < \gamma_1(v) + \gamma_2(v)$ as well. ■

From the above analysis, we have the following results. In the case of vertical mergers, if adopting the consensus value, the main conditions for the mergers to be beneficial (costly) to the merging parties are that the network industry exhibits decreasing (increasing) returns to network size and the size of the involved downstream merging operator should be sufficiently large. Although another (weaker) condition is introduced, the results here are basically consistent with the results obtained by Jeon (2003) for the case of the Shapley value. In the case of horizontal mergers, we find that if adopting the consensus value the main conditions for the mergers to be costly to the merging parties are the same as in Jeon (2003).

Therefore, we can conclude that despite the minor modifications, the results obtained in Jeon (2003) are also the main conditions for the merger incentives under the consensus value, which implies that those results are robust.

Chapter 3

The consensus value for games in partition function form

3.1 Introduction

The problem of sharing the joint gains of cooperation is well captured by cooperative game theory. The Shapley value (Shapley (1953)) has been proven to be the most studied and widely used single-valued solution concept for cooperative games with transferable utility in characteristic function form as it satisfies some desirable properties. In some sense, the value captures the expected outcome of a game, and represents a distinct approach to the problems of complex strategic interactions that game theory seeks to illuminate (Roth (1988)).

However, considering an economy with externalities one cannot easily recommend a division of the joint profits in the same way as the final profits depend on the coalition structure which has been formed. This feature was first captured by Thrall and Lucas (1963) by the concept of *partition function form games*: A partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. The advantage of this model is that it takes both internal factors (coalition itself) and external factors (coalition structure) that may affect cooperation outcomes into account and allows to go deeper into cooperation problems. Thus, it is closer to real life although more complex to analyse.

Values for such games can be found in Myerson (1977), Bolger (1989), Feldman (1994), Potter (2000), and Pham Do and Norde (2002). All of them are in some way extensions of the Shapley value (Shapley (1953)) for cooperative TU games in characteristic function form. Myerson (1977) introduced a value based on the extensions of the three axioms in the Shapley's original paper. Bolger's value assigns zero to dummies and assigns nonnegative values to players in monotone simple games. Potter (2000) added another axiom, coalitional symmetry, and reformed the regular definition of the dummy player such that the dummy player can get nonnegative worth. But note that a null player defined in this chapter still gets zero worth by Potter's value. When $|N| = 3$, Potter's value coincides with the values introduced by Bolger and Feldman. But they are different when $|N| > 3$. The difference is due to the fact that Potter defined the worth of each embedded coalition as the average worth. Pham Do and Norde (2002) studied another extension of the Shapley value for the class of partition function form games, which is the average of a collection of marginal vectors. Fujinaka (2004) provided alternative characterizations for the Shapley value defined by Pham Do and Norde (2002) based on a marginality axiom and a monotonicity axiom. Moreover, he found an error in the proof of the axiomatization which is based on the axiom of additivity in Pham Do and Norde (2002) and amended it in his paper.

This chapter[1] takes a different perspective and aims to derive a solution concept which not only satisfies "reasonable" properties but also has a constructive sharing procedure. Following a simple and natural way of generalizing the standard solution for 2-person partition function form games into n-person cases, a new solution concept for partition function form games is obtained: *the consensus value*. It is, in fact, a natural extension of the consensus value for TU games in characteristic function form introduced in chapter 2. This value differs from all the previous values as it is characterized to be the unique function that satisfies efficiency, complete symmetry, additivity and the *quasi-null player property*. The first three requirements are relatively weak, especially the property of complete symmetry is a natural and obvious requirement. A quasi-null player is a player who has zero payoff in the complete breakdown situation (every player stands alone) and whose marginal contributions to all non-empty coalitions are also zero. Instead of the "regular" marginal contribution perspective requiring zero payoff to a quasi-null player (we may call it the *marginal* quasi-null player property) which is implicitly specified by the Shapley value in Pham Do and Norde (2002), this chapter introduces the so-called quasi-null player property based on the positive or negative externalities that the quasi-null player might benefit or suffer from.

One may argue that the efficiency[2] postulate and the marginal quasi-null player property seem to be contradictory to each other when considering solution concepts

[1]This chapter is based on Ju (2004).

[2]A more general criticism on the efficiency postulate can be traced back to Luce and Raiffa (1957); and, more recently, is seen in Maskin (2003).

for partition function form games because a quasi-null player, given the positive externalities she might enjoy, can hardly participate in coalitions where she contributes nothing and will get zero payoff. More generally, we have no reason to ignore the externality effect in partition function form games while the marginal quasi-null player property rules out the considerations on externalities and completely favors coalitions. That is, from the positive externality point of view, any quasi-null player could obtain nonnegative worth when standing alone, and analogously, she might get nonpositive worth in the presence of negative externality, which opens up the possibility to relax the marginal quasi-null player postulate. In this spirit, the quasi-null player property is introduced and discussed.

By defining *the expected stand-alone value*, we can determine, in some sense, the maximum and minimum that a quasi-null player might get in a game due to the positive and negative externalities[3], respectively. In order to balance the tradeoff between those two contrastive opinions, i.e. emphasizing coalitions or focusing on externalities, we make a fair compromise and take the average as the value for a quasi-null player, resulting in the quasi-null player property. At the same time, introducing the quasi-null player property actually affects all other players in the same way such that any player's value is determined by her contributions to coalitions and the externalities imposed on her if stand-alone, which is further confirmed by the general formula of the consensus value: It is the average of the Shapley value introduced by Pham Do and Norde (2002) and the expected stand-alone value.

A novel feature of the consensus value for TU games in characteristic function form is its underlying sharing process. It is shown in this chapter that such a process is particularly suitable for the settings of games in partition function form because, given a coalition structure, the standard solution for 2-person partition function form games can be well implemented. Here, given an ordering of players, we also take a bilateral perspective and consider to allocate the joint surplus of an existing coalition of players (i.e., the incumbents) and an entrant, which means that the unilateral viewpoint like the marginal contribution approach focusing on entrants is abandoned. By taking the incumbents as one party and the entrant as a second party, the standard solution for 2-person games can be applied all the way with consensus. That is, all the joint surpluses are always equally split between the corresponding two parties. Since no specific ordering is pre-determined, we average over all possible permutations. Hence,

[3]More strictly, since the externalities from different coalitions imposed on a player could be both positive and negative in a game, the expected stand-alone value is just a value focusing on externalities, in contrast with the value derived from the contributions to coalitional values.

by this rule, not only the concern of all the possible orderings but also what happens in each ordering are mutually accepted: Consensus is obtained.

By means of the transfer property, a second characterization for the consensus value is provided. Based on a modification of the stand-alone reduced game introduced in Ju, Borm and Ruys (2004) and a related recursive formula, the consensus value for partition function form games is reformulated. Furthermore, by introducing a share parameter on the splitting of joint surpluses, a generalization of the consensus value is obtained. In particular, the Shapley value and the expected stand-alone value are the two polar cases of the generalized consensus value. Accordingly, characterizations for the expected stand-alone value are obtained. A special case of the partition function form games is that any player's stand-alone values are the same as that in the complete breakdown situation. Then, the consensus value is equivalent to that in TU games in characteristic function form, which equals to the average of the Shapley value and the equal surplus solution.

In addition to this section introducing the chapter and reviewing the seminal works briefly, the remaining part proceeds as follows. In the next section, we briefly recall the basic features of partition function form games. In section 3, we address 2-person partition function form games and take the corresponding solutions as a standardization and define the consensus value for partition function form games. The consensus value is characterized in an axiomatic way in section 4. It is shown that the consensus value is the average of the Shapley value for partition function form games and the expected stand-alone value. Section 5 discusses a generalization of this solution concept. The final section shows the applications of the consensus value by providing two illustrative examples: one is about oligopoly games in partition function form and the other is about the participation incentives in free-rider situations.

3.2 Preliminaries

This section, based on Pham Do and Norde (2002), recalls some basic definitions and notations related to games in partition function form.

A *partition* κ of the player set N, a so-called *coalition structure*, is a set of mutually disjoint coalitions, $\kappa = \{S_1, ..., S_m\}$, so that their union is N. Let $\mathbb{P}(N)$ be the set of all partitions of N. For any coalition $S \subset N$, the set of all partitions of S is denoted by $\mathbb{P}(S)$. A typical element of $\mathbb{P}(S)$ is denoted by κ_S. Note that two partitions will be considered equal if they differ only by the insertion or deletion of $\emptyset$. That is, $\{\{1,2\},\{3\}\} = \{\{1,2\},\{3\},\emptyset\}$.

A pair (S,κ) consisting of a coalition S and a partition κ of N to which S belongs is called an *embedded coalition*, and is nontrivial if $S \neq \emptyset$. Let $\mathbb{E}(N)$ denote the set of embedded coalitions, i.e.

$$\mathbb{E}(N) = \left\{(S,\kappa) \in 2^N \times \mathbb{P}(N) | S \in \kappa\right\}.$$

Definition 3.2.1 *A mapping*

$$w : \mathbb{E}(N) \longrightarrow \mathbb{R}$$

that assigns a real value, $w(S,\kappa)$, to each embedded coalition (S,κ) is called a partition function. *By convention, $w(\emptyset,\kappa) = 0$ for all $\kappa \in \mathbb{P}(N)$. The ordered pair (N,w) is a* partition function form game. *The set of partition function form games with player set N is denoted by PG^N.*

The value $w(S,\kappa)$ represents the payoff of coalition S, given the coalition structure κ forms. For a given partition $\kappa = \{S_1,...,S_m\}$ and a partition function w, let $\bar{w}(S_1,...,S_m)$ denote the m-vector $(w(S_i,\kappa))_{i=1}^m$. It will be convenient to economize brackets and suppress the commas between elements of the same coalition. Thus, where no confusion can arise, we will write, for example, $w(\{i,j,k\},\{\{i,j,k\},\{l,h\}\})$ as $w(ijk,\{ijk,lh\})$, and $\bar{w}(\{i,j,k\},\{l,h\})$ as $\bar{w}(ijk,lh)$. For a partition $\kappa \in \mathbb{P}(N)$ and $i \in N$, we denote the coalition in κ to which player i belongs by $S(\kappa,i)$.

The typical partition which consists of *singleton coalitions* only, $\kappa = \{\{1\},...,\{n\}\}$, is denoted by $[N]$, whereas the partition, which consists of the grand coalition only is denoted by $\{N\}$. For any subset $S \subset N$, let $[S]$ denote the typical partition which consists of the singleton elements of S, i.e., $[S] = \{\{j\}|j \in S\}$

Definition 3.2.2 *A solution concept on PG^N is a function f, which associates with each game (N,w) in PG^N a vector $f(N,w)$ of individual payoffs in $\mathbb{R}^N$, i.e.,*

$$f(N,w) = (f_i(N,w))_{i \in N} \in \mathbb{R}^N.$$

Since the consensus value for partition function form games is related to the Shapley value defined by Pham Do and Norde (2002), it is necessary to recall that definition.

For a given $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$, we define the partition κ_k^σ associated with σ and k, by $\kappa_k^\sigma = \{S_k^\sigma\} \cup [N \backslash S_k^\sigma]$ where $S_k^\sigma := \{\sigma(1), ..., \sigma(k)\}$, and $\kappa_0^\sigma = [N]$. So, in κ_k^σ the coalition S_k^σ has already formed, whereas all other players still form singleton coalitions.

For a game $w \in PG^N$, define the marginal vector $m^\sigma(w)$ as the vector in $\mathbb{R}^N$ by

$$m_{\sigma(k)}^\sigma(w) = w(S_k^\sigma, \kappa_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma)$$

for all $\sigma \in \Pi(N)$ and $k \in \{1, ..., |N|\}$.

Definition 3.2.3 *(Pham Do and Norde (2002)) The* Shapley value $\Phi(w)$ *of the partition function form game* (N, w) *is the average of the marginal vectors, i.e.*

$$\Phi(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(w).$$

3.3 The consensus value

One may notice that the Shapley value for partition function form games defined by Pham Do and Norde (2002) actually ignores all the new information provided in a partition function form game compared to a TU game in characteristic function form. For instance, this value is in dependent of $w(\{i\}, (\kappa_{N\backslash\{i\}}) \cup \{\{i\}\})$ for all $i \in N$ and for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ such that $\kappa_{N\backslash\{i\}} \neq [N\backslash\{i\}]$. Put differently, partition functions play no role here and the Shapley value defined above does not take externality into account.

The idea to define the consensus value for TU games in characteristic function form (cf. chapter 2) well fits the settings of partition function form games: It takes a bilateral perspective and allocates payoffs based on a fair compromise between coalition effect and externality effect. Hence, it reflects the role of coalition structures in determining players' final payoffs.

To illustrate the idea, we first consider an arbitrary 2-person partition function form game with player set $N = \{1, 2\}$ and partition function w determined by the values: $w(1, \{1, 2\})$, $w(2, \{1, 2\})$ and $w(12, \{12\})$. Note that, as mentioned in section 2, here we use shortcut notations, for example, $w(1, \{1, 2\})$ is for $w(\{1\}, \{\{1\}, \{2\}\})$, and $w(12, \{12\})$ is for $w(\{1, 2\}, \{\{1, 2\}\})$. A reasonable solution is that player 1 gets

$$w(1, \{1, 2\}) + \frac{w(12, \{12\}) - w(1, \{1, 2\}) - w(2, \{1, 2\})}{2}$$

and player 2 gets

$$w(2, \{1, 2\}) + \frac{w(12, \{12\}) - w(2, \{1, 2\}) - w(1, \{1, 2\})}{2}.$$

That is, the (net) surplus generated by the cooperation between player 1 and 2,

$$w(12, \{12\}) - w(2, \{1, 2\}) - w(1, \{1, 2\}),$$

is equally shared between the two players. This solution is called the *standard* solution for 2-person partition function form games.

Then, we consider a generalization of the standard solution for 2-person games into n-person cases. It follows the following line of reasoning.

Consider a 3-person game (N, w) with player set $N = \{1, 2, 3\}$. Suppose we have the ordering $(1, 2, 3)$: player 1 shows up first, then player 2, and finally player 3. When player 2 joins 1, we in fact have a 2-person situation where the surplus sharing problem is solved by the standard solution. Next, player 3 enters the scene, who would like to cooperate with player 1 and 2. Because coalition $\{12\}$ has already been formed before she joins, player 3 will actually cooperate with the existing coalition $\{12\}$ instead of simply cooperating with 1 and 2 individually. If $\{12\}$ agrees to cooperate with player 3 as well, the value of the grand coalition, $w(123, \{123\})$ will be generated. Now, the question is how to share it?

Again, following the standard solution for 2-person games, one can argue that both parties should get half of the joint surplus

$$w(123, \{123\}) - w(12, \{12, 3\}) - w(3, \{12, 3\})$$

in addition to their stand-alone payoffs. The reason is simple: coalition $\{12\}$ should be regarded as one player instead of two players because they have already formed a cooperating coalition. Internally, 1 and 2 will receive equal shares of the surplus because this part is obtained extra by the coalition $\{12\}$ cooperating with coalition $\{3\}$.

One can also tell the story in a reverse way, which yields the same outcome in terms of surplus sharing. Initially, three players cooperate with each other and $w(123, \{123\})$ is obtained. We now consider players leaving the existing coalitions one by one in the opposite order $(3, 2, 1)$. So player 3 leaves first. By the standard solution for 2-person games, player 3 should get half of the joint surplus plus her stand-alone payoff, i.e.

$$w(3, \{12, 3\}) + \frac{w(123, \{123\}) - w(3, \{12, 3\}) - w(12, \{12, 3\})}{2},$$

as 1 and 2 remain as one cooperating coalition $\{12\}$. Thus, the value left for coalition $\{12\}$, which we call the *standardized remainder* (the value left for the corresponding remaining coalition) for $\{12\}$, is

$$w(12, \{12, 3\}) + \frac{w(123, \{123\}) - w(12, \{12, 3\}) - w(3, \{12, 3\})}{2}.$$

In the same fashion, the standardized remainder for $\{1\}$ will be

$$w(1, \{1, 2, 3\}) + \frac{\frac{w(123,\{123\})+w(12,\{12,3\})-w(3,\{12,3\})}{2} - w(1, \{1, 2, 3\}) - w(2, \{1, 2, 3\}))}{2}.$$

Extending this argument to an n-person case, we then have a general method, which can be understood as a *standardized remainder rule* since we take the later entrant (or earlier leaver) and all her pre-entrants (or post-leavers) as two parties and apply the standard solution for 2-person games all the way. Furthermore, since no ordering is pre-determined for a partition function form game, we will average all possible orderings.

Formally, for a game in partition function form we shall define the standardized remainder as follows.

$$r(S_k^\sigma) = \begin{cases} w(N, \{N\}) & \text{if } k = |N| \\ w(S_k^\sigma, \kappa_k^\sigma) + \frac{r(S_{k+1}^\sigma) - w(S_k^\sigma, \kappa_k^\sigma) - w(\{\sigma(k+1)\}, \kappa_k^\sigma)}{2} & \text{if } k \in \{1, ..., |N| - 1\}, \end{cases}$$

where $r(S_k^\sigma)$ is the *standardized remainder* for coalition S_k^σ: the value left for S_k^σ after allocating surplus to later entrants (earlier leavers) $N \backslash S_k^\sigma$. Note that for notational simplicity we still use the same notation, i.e. $r(S_k^\sigma)$, as that for TU games in characteristic function form.

We construct the *individual standardized remainder vector* $s^\sigma(w)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ (or leave the game one by one in the order $\sigma(|N|), \sigma(|N| - 1), ..., \sigma(1))$ and assign each player $\sigma(k)$, besides her stand-alone payoff $w(\{\sigma(k)\}, \kappa_{k-1}^\sigma)$, half of the net surplus from the standardized remainder $r(S_k^\sigma)$. Formally, it is the vector in $\mathbb{R}^N$ recursively (start with $|N|$) defined by

$$s_{\sigma(k)}^\sigma(w) = \begin{cases} w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) + \frac{r(S_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma) - w(\{\sigma(k)\}, \kappa_{k-1}^\sigma)}{2} & \text{if } k \in \{2, ..., |N|\} \\ r(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 3.3.1 *The* consensus value $\gamma(w)$ *of the partition function form game* (N,w) *is the average of the individual standardized remainder vectors, i.e.*

$$\gamma(w)=\frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} s^{\sigma}(w).$$

Hence, the consensus value can be interpreted as the expected individual standardized remainder that a player can get by participating in coalitions.

Example 3.3.2 *This game is from Pham Do and Norde (2002). Consider the partition function form game* (N,w) *defined by*

$$\begin{aligned}\bar{w}(1,2,3) &= (0,0,0),\\ \bar{w}(12,3) &= (2,0),\ \bar{w}(13,2)=(2,1),\ \bar{w}(23,1)=(3,2),\\ \bar{w}(123) &= (10).\end{aligned}$$

With $\sigma:\{1,2,3\}\longrightarrow N$ *defined by* $\sigma(1)=2$*,* $\sigma(2)=1$ *and* $\sigma(3)=3$*, which is shortly denoted by* $\sigma=(2\ 1\ 3)$*, we get*

$$s_3^{\sigma}(w)(=s_{\sigma(3)}^{\sigma}(w))=w(3,\{12,3\})+\frac{w(123,\{123\})-w(12,\{12,3\})-w(3,\{12,3\})}{2}=4.$$

And one can readily calculate

$$s_1^{\sigma}(w)(=s_{\sigma(2)}^{\sigma}(w))=w(1,\{1,2,3\})+\frac{r(\{2,1\})-w(2,\{1,2,3\})-w(1,\{1,2,3\})}{2}=3,$$

and $s_2^{\sigma}(w)(=s_{\sigma(1)}^{\sigma}(w))=r(\{2\})=3$*. All individual standardized remainder vectors are given by*

σ	$s_1^{\sigma}(w)$	$s_2^{\sigma}(w)$	$s_3^{\sigma}(w)$
(123)	3	3	4
(132)	$2\frac{3}{4}$	$4\frac{1}{2}$	$2\frac{3}{4}$
(213)	3	3	4
(231)	$4\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{3}{4}$
(312)	$2\frac{3}{4}$	$4\frac{1}{2}$	$2\frac{3}{4}$
(321)	$4\frac{1}{2}$	$2\frac{3}{4}$	$2\frac{3}{4}$

Then, we get $\gamma(w)=\left(3\frac{5}{12},3\frac{5}{12},3\frac{1}{6}\right)$ *whereas the Shapley value of this game (Pham Do and Norde (2002)) is* $\Phi(w)=\left(3,3\frac{1}{2},3\frac{1}{2}\right)$*. One can verify that the value introduced by Potter (2000) as well as the value introduced by Bolger (1989) yield the same vector* $(3\frac{1}{4},3\frac{1}{2},3\frac{1}{4})$ *for this game as when* $|N|=3$ *the value introduced by Potter coincides* *with Bolger's value (Potter (2000)). The difference between the consensus value and the others stems from the way to share joint surpluses and the fact that the externalities of players are taken into account. The Shapley value still focuses on marginal vectors and rules out externality effects. As for Bolger's value, it considers a different collection of marginal vectors. Potter's value is obtained by considering the sum of an "average worth" of coalitions.*

Similar to the stand-alone recursion of the consensus value for TU games in characteristic function form, we can reformulate the consensus value for partition function form games by modifying the stand-alone reduced game and defining a corresponding recursive formula.

Formally, let $f : PG^N \longrightarrow \mathbb{R}^N$ be a solution concept. For any partition function form game (N, w) and $i \in N$, we introduce the game $(N\backslash\{i\}, w^{-i})$ defined by for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and for all $S \in \kappa_{N\backslash\{i\}}$

$$w^{-i}(S, \kappa_{N\backslash\{i\}}) = \begin{cases} w(S, \kappa_{N\backslash\{i\}} \cup \{i\}) & \text{if } S \subsetneq N\backslash\{i\} \\ w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) \\ +\frac{w(N,\{N\}) - w(N\backslash\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\}) - w(\{i\},\{N\backslash\{i\}\}\cup\{\{i\}\})}{2} & \text{if } S = N\backslash\{i\} \end{cases}$$

and call w^{-i} the *stand-alone reduced game* of w with respect to player i.

We say that a solution concept f satisfies the *stand-alone recursion* if and only if for any game (N, w) with $|N| \geq 3$ we have

$$\begin{aligned} & f_i(N, w) \\ = \quad & \frac{1}{|N|} \sum_{j \in N\backslash\{i\}} f_i\left(N\backslash\{j\}, w^{-j}\right) \\ + \quad & \frac{1}{|N|} \cdot \frac{w(N, \{N\}) - w(N\backslash\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\}) + w(\{i\}, \{N\backslash\{i\}\} \cup \{\{i\}\})}{2} \end{aligned}$$

for all $i \in N$.

One can readily check that the consensus value is the unique one-point solution concept on the class of all n-person partition function form games with $n \geq 2$ which is standard for 2-person partition function form games and satisfies stand-alone recursion.

3.4 Characterizations

This section characterizes the consensus value for partition function form games in an axiomatic way.

Definition 3.4.1 *In a partition function form game $w \in PG^N$, two players i and j are* completely symmetric *if for all $\kappa_{N\backslash\{i,j\}} \in \mathbb{P}(N\backslash\{i,j\})$ and $S \in \kappa_{N\backslash\{i,j\}}$,*

$$w(S\cup\{i\}, (\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{j\}\}\cup\{S\cup\{i\}\}) = w(S\cup\{j\}, (\kappa_{N\backslash\{i,j\}}\backslash S)\cup\{\{i\}\}\cup\{S\cup\{j\}\})$$

and

$$w(\{i\}, (\kappa_{N\backslash\{i,j\}}\backslash S) \cup \{\{i\}\} \cup \{S \cup \{j\}\}) = w(\{j\}, (\kappa_{N\backslash\{i,j\}}\backslash S) \cup \{\{j\}\} \cup \{S \cup \{i\}\}).$$

Definition 3.4.2 *In a partition function form game $w \in PG^N$, player i is a* null player *if for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and $S \in \kappa_{N\backslash\{i\}}$,*

$$w(S, \kappa_{N\backslash\{i\}} \cup \{\{i\}\}) = w(S \cup \{i\}, (\kappa_{N\backslash\{i\}}\backslash S) \cup \{S \cup \{i\}\}).$$

So, a null player always makes zero marginal contributions to any coalition and obtains zero payoff when standing alone. Moreover, we define a quasi-null player as follows.

Definition 3.4.3 *In a game $w \in PG^N$, player i is a* quasi-null player *if for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$ and $S \in \kappa_{N\backslash\{i\}}$ such that $S \neq \emptyset$,*

$$w(S, \kappa_{N\backslash\{i\}} \cup \{\{i\}\}) = w(S \cup \{i\}, (\kappa_{N\backslash\{i\}}\backslash S) \cup \{S \cup \{i\}\})$$

and

$$w(\{i\}, [N]) = 0.$$

Thus, a quasi-null player i will be a null player if $w(\{i\}, \kappa_{N\backslash\{i\}} \cup \{\{i\}\}) = 0$ for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$.

By Definition 3.2.3, one can find that $\Phi_i(w) = 0$ for all $w \in PG^N$ and for any quasi-null player i in (N, w), which implies that the Shapley value is not so convincing: If a quasi-null player can get positive payoffs due to positive externalities, i.e., $w(\{i\}, \kappa_{N\backslash\{i\}} \cup \{\{i\}\}) > 0$ for all $\kappa_{N\backslash\{i\}} \in \mathbb{P}(N\backslash\{i\})$, why would she join the others to form the grand coalition and obtain zero payoff finally?

In order to find how much a quasi-null player should obtain, we first introduce the concept of expected stand-alone value. For a partition function form game $w \in PG^N$ and a player $i \in N$, we define player i's *expected stand-alone value* as

$$\begin{aligned} e_i(w) = \quad & \frac{w(N, \{N\})}{|N|} \\ + & \sum_{S \subset N \backslash \{i\}: S \neq \emptyset} \frac{|S|!(|N| - |S| - 1)!}{|N|!} w(\{i\}, \{S\} \cup [N \backslash (S \cup \{i\})] \cup \{\{i\}\}) \\ - & \sum_{j \in N \backslash \{i\}} \sum_{S \subset N \backslash \{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{|N|!} w(\{j\}, [N \backslash (S \cup \{i\})] \cup \{S \cup \{i\}\}). \end{aligned}$$

The expected stand-alone value tells us how much a player may obtain in a partition function form game (N, w) when we focus on the stand-alone side of the game[4]. Since we rule out the consideration on coalition values, immediately, a reference point could be that the value of the grand coalition is equally shared among players, i.e. $\frac{w(N,\{N\})}{|N|}$. Focusing on stand-alone situations implies that we take externality as the only determinant. Given a player $i \in N$, she has two choices concerning externalities, either choosing stand-alone and enjoying the externalities from coalitions consisting of other players or joining some coalitions generating externalities to the players standing alone. Thus, the second term in the above expression corresponds to the first choice and can be understood as player i's *expected gain* from the externalities of all possible coalitions without containing i, where the distribution of coalitions is such that any ordering of the players is equally likely. The last term, corresponding to the second choice, is player i's *expected loss* due to joining coalitions, which is expressed as the other players' gain from the externalities of coalitions containing i.

One can find that in the case that any player has identical stand-alone payoffs in a partition function form game, the expected stand-alone value is comparable to the equal surplus solution for TU games in characteristic function form. Let TU^N denote the set of all TU games in characteristic function form with player set N. The equal solution surplus E is defined as $E_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{|N|}$ for all $v \in TU^N$ and for all $i \in N$.

[4]Or directly in some special situations that people have no information about the values of coalitions but only know players' stand-alone values and the value of the grand coalition, we then could get such a sharing rule, which is actually an equal-surplus-solution style value in partition function form games.

Proposition 3.4.4 *For a game $w \in PG^N$, if $w(\{i\}, \{S\} \cup [N\backslash(S \cup \{i\})] \cup \{\{i\}\}) = w(\{i\}, [N])$ for all $i \in N$ and for all $S \subset N\backslash\{i\}$, then*

$$e_i(w) = \frac{w(N, \{N\}) - \sum_{j \in N} w(\{j\}, [N])}{|N|} + w(\{i\}, [N])$$

for all $i \in N$.

Proof. By the definition of the expected stand-alone value and since $w(\{i\}, \{S\} \cup [N\backslash(S \cup \{i\})] \cup \{\{i\}\}) = w(\{i\}, [N])$ for all $S \subset N\backslash\{i\}$ and for all $i \in N$, it follows that

$$\begin{aligned} & \sum_{S \subset N\backslash\{i\}: S \neq \emptyset} \frac{|S|!(|N| - |S| - 1)!}{|N|!} w(\{i\}, \{S\} \cup [N\backslash(S \cup \{i\})] \cup \{\{i\}\}) \\ = & \sum_{|S|=1}^{|N|-1} \frac{|S|!(|N| - |S| - 1)!}{|N|!} \binom{|N| - 1}{|S|} w(\{i\}, [N]) \\ = & \frac{|N| - 1}{|N|} w(\{i\}, [N]). \end{aligned}$$

Similarly, we can show that

$$\begin{aligned} & \sum_{j \in N\backslash\{i\}} \sum_{S \subset N\backslash\{i,j\}} \frac{|S|!(|N| - |S| - 2)!}{|N|!} w(\{j\}, [N\backslash(S \cup \{i\})] \cup \{S \cup \{i\}\}) \\ = & \frac{\sum_{j \in N\backslash\{i\}} w(\{j\}, [N])}{|N|}. \end{aligned}$$

Hence, what remains is obvious. ■

Therefore, the equal surplus solution E for TU games in characteristic function form is actually a special case of the expected stand-alone value. It can be expressed as

$$E_i(v) = \frac{v(N)}{|N|} + \frac{|N| - 1}{|N|} v(\{i\}) - \frac{1}{|N|} \sum_{j \in N\backslash\{i\}} v(\{j\})$$

for all $v \in TU^N$ and for all $i \in N$. Here, the second term is the expected gain as being a stand-alone player while the last term is the expected loss due to joining coalitions.

Let $f : PG^N \longrightarrow \mathbb{R}^N$ be a one-point solution concept. We consider the following properties.

- *Efficiency:* $\sum_{i \in N} f_i(w) = w(N, \{N\})$ for all $w \in PG^N$;
- *Complete symmetry:* $f_i(w) = f_j(w)$ for all $w \in PG^N$, and for all completely symmetric players i, j in (N, w);
- *The quasi-null player property:*

$$f_i(w) = \frac{1}{2} e_i(w)$$

  for all $w \in PG^N$ and for any quasi-null player i in (N, w);
- *Additivity:* $f(w_1 + w_2) = f(w_1) + f(w_2)$ for all $w_1, w_2 \in PG^N$, where $w_1 + w_2$ is defined by $(w_1 + w_2)(S, \kappa) = w_1(S, \kappa) + w_2(S, \kappa)$ for every $(S, \kappa) \in \mathbb{E}(N)$.

The properties of efficiency, complete symmetry, and additivity are clear by themselves. Here, it is necessary to stress the new property: the quasi-null player property.

Let us first discuss the marginal quasi-null player property that assigns zero payoff to a quasi-null player, which is implicitly specified by the Shapley value introduced by Pham Do and Norde (2002). Requiring a solution concept for partition function form games satisfying both efficiency and this marginal quasi-null player property seems inappropriate. For instance, a quasi-null player i who may obtain positive payoff due to the positive externality from coalition $N \backslash \{i\}$ has to accept zero payoff in the game according to this marginal quasi-null player property. Then, it is hard to imagine that player i could have any incentive to join the grand coalition. As a consequence, it is difficult to justify the efficiency axiom. More generally, the players who may enjoy extremely high positive externalities from other coalitions will choose stand-alone as those effects are not well reflected by the solution concepts that adopt a marginal contribution approach. So, the externality has to be taken into consideration.

As we know, the marginal quasi-null player property favors coalitions while it biases against the outside individuals. In order to give a fair treatment to both sides, we have to balance the coalition effect and the externality effect. More specifically, to assign a quasi-null player 0 or $e_i(w)$ can be viewed as consequences of two contrastive viewpoints. Concerning the tradeoff between these two extreme opinions[5], an impartial decision could be choosing the average as the gain of a quasi-null player, which results in the so-called quasi-null player property.

[5]Cultural and philosophical factors may affect the propensity or choice between the two extreme opinions.

In addition, one can see that a null player, as a special quasi-null player, could still get positive worth as long as her expected loss from externalities is less than the average value $\frac{w(N,\{N\})}{|N|}$. This observation implies that the quasi-null player property also has the flavor of egalitarianism or collectivism. The justification is similar to that for the consensus value for TU games in characteristic function form in Ju, Borm and Ruys (2004).

It is shown that the consensus value is the unique function that satisfies these four properties.

Theorem 3.4.5 *The consensus value satisfies efficiency, complete symmetry, the quasi-null player property and additivity.*

Proof.
(i) Efficiency: Clearly, by construction, $s^\sigma(w)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Complete symmetry: Let i, j be two completely symmetric players in a partition function form game $w \in PG^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i$, $\sigma(l) = j$, where $i, j \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained by interchanging in σ the positions of i and j, i.e.

$$\bar{\sigma}(m) = \begin{cases} \sigma(m) & \text{if } m \neq k, l \\ i & \text{if } m = l \\ j & \text{if } m = k \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $s_i^\sigma(w) = s_j^{\bar{\sigma}}(w)$.
Case 1: $1 < k < l$.
By definition, we know

$$s_i^\sigma(w) = s_{\sigma(k)}^\sigma(w) = w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) + \frac{1}{2}\left(r(S_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma) - w(\{\sigma(k)\}, \kappa_{k-1}^\sigma)\right),$$

$$s_j^{\bar{\sigma}}(w) = s_{\bar{\sigma}(k)}^{\bar{\sigma}}(w) = w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}}) + \frac{1}{2}\left(r(S_k^{\bar{\sigma}}) - w(S_{k-1}^{\bar{\sigma}}, \kappa_{k-1}^{\bar{\sigma}}) - w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}})\right).$$

Note that, by complete symmetry,

$$w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) = w(\{i\}, \kappa_{k-1}^\sigma) = w(\{j\}, \kappa_{k-1}^{\bar{\sigma}}) = w(\{\bar{\sigma}(k)\}, \kappa_{k-1}^{\bar{\sigma}}),$$

$S_{k-1}^\sigma = S_{k-1}^{\bar{\sigma}}$, and apparently $w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma) = w(S_{k-1}^{\bar{\sigma}}, \kappa_{k-1}^{\bar{\sigma}})$. It remains to show that $r(S_k^\sigma) = r(S_k^{\bar{\sigma}})$.
Clearly, $r(S_m^\sigma) = r(S_m^{\bar{\sigma}})$ for $m \geq l$. Recursively, we can show that $r(S_{l-t}^\sigma) = r(S_{l-t}^{\bar{\sigma}})$ for $t \in \{1, ..., l-k-1\}$ as

$$r(S_{l-t}^\sigma) = w(S_{l-t}^\sigma, \kappa_{l-t}^\sigma) + \frac{1}{2}\left(r(S_{l-t+1}^\sigma) - w(S_{l-t}^\sigma, \kappa_{l-t}^\sigma) - w(\{\sigma(l-t+1)\}, \kappa_{l-t}^\sigma)\right)$$

and

$$r(S^{\bar{\sigma}}_{l-t}) = w(S^{\bar{\sigma}}_{l-t}, \kappa^{\bar{\sigma}}_{l-t}) + \frac{1}{2}\left(r(S^{\bar{\sigma}}_{l-t+1}) - w(S^{\bar{\sigma}}_{l-t}, \kappa^{\bar{\sigma}}_{l-t}) - w(\{\bar{\sigma}(l-t+1)\}, \kappa^{\bar{\sigma}}_{l-t})\right).$$

Here, since $\sigma(l-t) = \bar{\sigma}(l-t)$ and $S^{\sigma}_{l-t}\backslash\{i\} = S^{\bar{\sigma}}_{l-t}\backslash\{j\}$, by complete symmetry, we know $w(S^{\sigma}_{l-t}, \kappa^{\sigma}_{l-t}) = w(S^{\bar{\sigma}}_{l-t}, \kappa^{\bar{\sigma}}_{l-t})$.
Then, it immediately follows that $r(S^{\sigma}_k) = r(S^{\bar{\sigma}}_k)$ as

$$\begin{aligned} r(S^{\sigma}_k) &= w(S^{\sigma}_k, \kappa^{\sigma}_k) + \frac{1}{2}\left(r(S^{\sigma}_{k+1}) - w(S^{\sigma}_k, \kappa^{\sigma}_k) - w(\{\sigma(k+1)\}, \kappa^{\sigma}_k)\right) \\ &= w(S^{\bar{\sigma}}_k, \kappa^{\bar{\sigma}}_k) + \frac{1}{2}\left(r(S^{\bar{\sigma}}_{k+1}) - w(S^{\bar{\sigma}}_k, \kappa^{\bar{\sigma}}_k) - w(\{\bar{\sigma}(k+1)\}, \kappa^{\bar{\sigma}}_k)\right) \\ &= r(S^{\bar{\sigma}}_k). \end{aligned}$$

Case 2: $1 < l < k$. The proof is analogous to Case 1.
Case 3: $1 = k < l$.
In this case,

$$\begin{aligned} s^{\sigma}_i(w) = s^{\sigma}_{\sigma(1)}(w) &= r(S^{\sigma}_1), \\ s^{\bar{\sigma}}_j(w) = s^{\bar{\sigma}}_{\bar{\sigma}(1)}(w) &= r(S^{\bar{\sigma}}_1). \end{aligned}$$

What remains is identical to Case 1.
Case 4: $1 = l < k$. The proof is analogous to Case 3.
(iii) Additivity: It is immediate, by definition, to see that $s^{\sigma}_{\sigma(k)}(w_1 + w_2) = s^{\sigma}_{\sigma(k)}(w_1) + s^{\sigma}_{\sigma(k)}(w_2)$ for all $w_1, w_2 \in PG^N$ and for all $k \in \{1, 2, ..., |N|\}$.
(iv) The quasi-null player property: By definition, we know for a partition function form game $w \in PG^N$ and a given ordering $\sigma \in \Pi(N)$,

$$\begin{aligned} r(S^{\sigma}_{|N|}) = \;& w(N, \{N\}) \\ r(S^{\sigma}_{|N|-1}) = \;& \frac{1}{2}w(N, \{N\}) + \frac{1}{2}w(S^{\sigma}_{|N|-1}, \kappa^{\sigma}_{|N|-1}) - \frac{1}{2}w(\{\sigma(|N|)\}, \kappa^{\sigma}_{|N|-1}) \\ r(S^{\sigma}_{|N|-2}) = \;& \frac{1}{4}w(N, \{N\}) + \frac{1}{4}w(S^{\sigma}_{|N|-1}, \kappa^{\sigma}_{|N|-1}) - \frac{1}{4}w(\{\sigma(|N|)\}, \kappa^{\sigma}_{|N|-1}) \\ +\;& \frac{1}{2}w(S^{\sigma}_{|N|-2}, \kappa^{\sigma}_{|N|-2}) - \frac{1}{2}w(\{\sigma(|N|-1)\}, \kappa^{\sigma}_{|N|-2}) \\ ...& \\ r(S^{\sigma}_2) = \;& \frac{1}{2}r(S^{\sigma}_3) + \frac{1}{2}w(S^{\sigma}_2, \kappa^{\sigma}_2) - \frac{1}{2}w(\{\sigma(3)\}, \kappa^{\sigma}_2) \\ r(S^{\sigma}_1) = \;& \frac{1}{2}r(S^{\sigma}_2) + \frac{1}{2}w(S^{\sigma}_1, \kappa^{\sigma}_1) - \frac{1}{2}w(\{\sigma(2)\}, \kappa^{\sigma}_1). \end{aligned}$$

Hence, a general expression is provided as follows.

$$r(S_k^\sigma) = \begin{cases} w(N,\{N\}) & \text{if } k = |N| \\ (\frac{1}{2})^{|N|-k} w(N,\{N\}) \\ + \sum_{l=k}^{|N|-1} (\frac{1}{2})^{l-k+1} \left(w(S_l^\sigma, \kappa_l^\sigma) - w(\{\sigma(l+1)\}, \kappa_l^\sigma) \right) & \text{if } 1 \leq k \leq |N|-1. \end{cases}$$

Let player $i \in N$ be a quasi-null player in game w. Let $\sigma(k) = i$. Then, by definition, this quasi-null player's individual standardized remainders in σ, $s_i^\sigma(w) = s_{\sigma(k)}^\sigma(w)$, are explicitly given as

$$\begin{aligned}
s_{\sigma(|N|)}^\sigma(w) = & \quad \frac{1}{2} w(\{i\}, \kappa_{|N|-1}^\sigma) \\
s_{\sigma(|N|-1)}^\sigma(w) = & \quad \frac{1}{4} w(N,\{N\}) + \frac{1}{4} \left(w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) - w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) \right) \\
- & \quad \frac{1}{2} w(S_{|N|-2}^\sigma, \kappa_{|N|-2}^\sigma) + \frac{1}{2} w(\{i\}, \kappa_{|N|-2}^\sigma) \\
s_{\sigma(|N|-2)}^\sigma(w) = & \quad \frac{1}{8} w(N,\{N\}) + \frac{1}{8} \left(w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) - w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) \right) \\
+ & \quad \frac{1}{4} \left(w(S_{|N|-2}^\sigma, \kappa_{|N|-2}^\sigma) - w(\{\sigma(|N|-1)\}, \kappa_{|N|-2}^\sigma) \right) \\
- & \quad \frac{1}{2} w(S_{|N|-3}^\sigma, \kappa_{|N|-3}^\sigma) + \frac{1}{2} w(\{i\}, \kappa_{|N|-3}^\sigma) \\
\ldots & \\
s_{\sigma(2)}^\sigma(w) = & \quad \frac{1}{2^{|N|-1}} w(N,\{N\}) + \frac{1}{2^{|N|-1}} w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) \\
- & \quad \frac{1}{2^{|N|-1}} w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) + \ldots + \frac{1}{4} w(S_2^\sigma, \kappa_2^\sigma) \\
- & \quad \frac{1}{4} w(\{\sigma(3)\}, \kappa_2^\sigma) - \frac{1}{2} w(S_1^\sigma, \kappa_1^\sigma) + \frac{1}{2} w(\{i\}, \kappa_1^\sigma) \\
s_{\sigma(1)}^\sigma(w) = & \quad \frac{1}{2^{|N|-1}} w(N,\{N\}) + \frac{1}{2^{|N|-1}} w(S_{|N|-1}^\sigma, \kappa_{|N|-1}^\sigma) \\
- & \quad \frac{1}{2^{|N|-1}} w(\{\sigma(|N|)\}, \kappa_{|N|-1}^\sigma) + \ldots + \frac{1}{4} w(S_2^\sigma, \kappa_2^\sigma) \\
- & \quad \frac{1}{4} w(\{\sigma(3)\}, \kappa_2^\sigma) - \frac{1}{2} w(\{\sigma(2)\}, \kappa_1^\sigma) + \frac{1}{2} w(\{i\}, \kappa_1^\sigma).
\end{aligned}$$

A general expression is

$$s^{\sigma}_{\sigma(k)} = \begin{cases} \frac{1}{2}w(\{i\},\kappa^{\sigma}_{|N|-1}) & \text{if } k=|N| \\ (\frac{1}{2})^{|N|-k+1}w(N,\{N\}) \\ +\sum_{l=k}^{|N|-1}(\frac{1}{2})^{l-k+2}\left(w(S^{\sigma}_l,\kappa^{\sigma}_l)-w(\{\sigma(l+1)\},\kappa^{\sigma}_l)\right) \\ -\frac{1}{2}w(S^{\sigma}_{k-1},\kappa^{\sigma}_{k-1})+\frac{1}{2}w(\{i\},\kappa^{\sigma}_{k-1}) & \text{if } 2\leq k\leq |N|-1 \\ r(S^{\sigma}_1) & \text{if } k=1. \end{cases}$$

Consider a class P of $|N|$ permutations $\sigma \in \Pi(N)$ such that for $\sigma, \tau \in P$ it holds that for all $j_1, j_2 \in N\backslash\{i\}$

$$\sigma^{-1}(j_1) < \sigma^{-1}(j_2) \Leftrightarrow \tau^{-1}(j_1) < \tau^{-1}(j_2).$$

That is, given an ordering of the players $N\backslash\{i\}$, let quasi-null player i move from the end to the beginning without changing the other players' relative positions. Summing over the above equations, we get

$$\begin{aligned} \sum_{\sigma\in P} s^{\sigma}_i(w) = & \sum_{\sigma\in P} s^{\sigma}_{\sigma(k)}(w) \\ = & \frac{1}{2}w(N,\{N\})+\frac{1}{2}\sum_{k=1}^{|N|}\left(w(S^{\sigma}_k,\kappa^{\sigma}_k)-w(S^{\sigma}_{k-1},\kappa^{\sigma}_{k-1})\right) \\ + & \frac{1}{2}\sum_{k=2}^{|N|}w(\{\sigma(k)\},\kappa^{\sigma}_{k-1})-\frac{1}{2}\sum_{k=1}^{|N|-1}w(\{\sigma(k+1)\},\kappa^{\sigma}_k). \end{aligned}$$

Since i is a quasi-null player, $w(S^{\sigma}_k,\kappa^{\sigma}_k)-w(S^{\sigma}_{k-1},\kappa^{\sigma}_{k-1})=0$ for all $k\in\{1,2,...,|N|\}$. Then,

$$\sum_{\sigma\in P} s^{\sigma}_i(w) = \frac{1}{2}\left(w(N,\{N\})+\sum_{k=2}^{|N|}w(\{\sigma(k)\},\kappa^{\sigma}_{k-1})-\sum_{k=1}^{|N|-1}w(\{\sigma(k+1)\},\kappa^{\sigma}_k)\right).$$

Taking all orderings of players into account, we then get

$$\begin{aligned}
\gamma_i(w) = & \quad \frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} s_i^\sigma(w) \\
= & \quad \frac{1}{2}\frac{w(N,\{N\})}{|N|} \\
& + \quad \frac{1}{2}\sum_{S\subset N\backslash\{i\}:S\neq\emptyset} \frac{|S|!(|N|-|S|-1)!}{|N|!} w(\{i\},\{S\}\cup[N\backslash(S\cup\{i\})]\cup\{\{i\}\}) \\
& - \quad \frac{1}{2}\sum_{j\in N\backslash\{i\}}\sum_{S\subset N\backslash\{i,j\}} \frac{|S|!(|N|-|S|-2)!}{|N|!} w(\{j\},[N\backslash(S\cup\{i\})]\cup\{S\cup\{i\}\}) \\
= & \quad \frac{1}{2}e_i(w).
\end{aligned}$$

■

Before proving the uniqueness of the consensus value, we first check the relationship between the consensus value and the Shapley value for partition function form games defined by Pham Do and Norde (2002). Since the Shapley value assigns zero worth to a quasi-null player, one can see that the quasi-null player property can be reformulated as $f_i(w) = \frac{1}{2}\Phi_i(w) + \frac{1}{2}e_i(w)$ for all $w \in PG^N$ and quasi-null player i in (N, w). In fact, interestingly, introducing this property influences all the players in the same way: Each player finally gets an average of her Shapley value and the expected stand-alone value. Formally, we have the following theorem.

Theorem 3.4.6 *The consensus value is the average of the Shapley value and the expected stand-alone value. That is, for every w in PG^N it holds that*

$$\gamma(w) = \frac{1}{2}\Phi(w) + \frac{1}{2}e(w).$$

Proof. Similar to part (iv) in the proof for Theorem 3.4.5, one can show that for every $w \in PG^N$,

$$\gamma_i(w) = \frac{1}{2}\left(\frac{1}{|N|!}\sum_{\sigma\in\Pi(N)} m_i^\sigma(w) + e_i(w)\right)$$

for all $i \in N$. ■

In order to prove[6] that the consensus value is the unique solution that satisfies efficiency, complete symmetry, the quasi-null player property and additivity, one needs to consider the "standard" basis of partition function form games. Let $\mathbb{E}'(N)$ be the set of all $(S,\kappa) \in \mathbb{E}(N)$ such that $S \neq \emptyset$. That is, $\mathbb{E}'(N) = \{(S,\kappa) \in \mathbb{E}(N) : S \neq \emptyset\}$. For any $(S,\kappa) \in \mathbb{E}'(N)$, define the partition function

$$\delta_{(S,\kappa)}(S',\kappa') = \begin{cases} 1 & \text{if } (S',\kappa') = (S,\kappa) \\ 0 & \text{otherwise.} \end{cases}$$

We call $\delta_{(S,\kappa)}$ the Dirac game with respect to (S,κ). One can see that the set of all Dirac games, $\left\{\delta_{(S,\kappa)} : (S,\kappa) \in \mathbb{E}'(N)\right\}$, forms a basis of partition function form games. Each $w \in PG^N$ can be uniquely written as

$$w = \sum_{(S,\kappa)\in\mathbb{E}'(N)} w(S,\kappa)\delta_{(S,\kappa)}.$$

If f is a solution on PG^N satisfying additivity, then for all $w \in PG^N$,

$$f(w) = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}).$$

Lemma 3.4.7 *Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$, and f be a solution on PG^N satisfying additivity and the quasi-null player property. We have*

$$f_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } S(\kappa,i) \neq \{i\} \\ \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) & \text{if } S(\kappa,i) = \{i\}, \end{cases}$$

where w is the partition function such that

$$w(S',\kappa') = \begin{cases} c & \text{if } (S',\kappa') = (S,\kappa) \\ c & \text{if } (S',\kappa') = (S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\}) \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$.
Case 1: $S(\kappa,i) \neq \{i\}$. Here, one can readily verify that i is a quasi-null player of game $c\delta_{(S,\kappa)}$. Hence, by the quasi-null player property, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$.
Case 2: $S(\kappa,i) = \{i\}$. Since we can write $w = c\delta_{(S,\kappa)} + c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}$, and i is a quasi-null player in w, by additivity and the quasi-null player property, we have $f_i(w) = \frac{1}{2}e_i(w)$ and $f_i(w) = f_i\left(c\delta_{(S,\kappa)}\right) + f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. ■

[6] We want to note that the proof for the uniqueness is in the same line as Fujinaka (2004).

Theorem 3.4.8 *There is a unique solution on PG^N satisfying efficiency, complete symmetry, the quasi-null player property and additivity. This solution is the consensus value.*

Proof. From Theorem 3.4.5, it follows that the consensus value γ satisfies efficiency, complete symmetry, the quasi-null player property and additivity.

Conversely, suppose a solution concept f satisfies these four properties. We have to show that $f = \gamma$. By additivity, it suffices to show that for any $c \in \mathbb{R}$ and any $(S,\kappa) \in \mathbb{E}'(N)$, $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

For any $c \in \mathbb{R}$ and $(S,\kappa) \in \mathbb{E}'(N)$, $\gamma\left(c\delta_{(S,\kappa)}\right)$ is defined as follows. If $S \neq N$ and $\kappa \neq \{S\} \cup [N\backslash S]$, then

$$\gamma_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } |S| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$ because all players are quasi-null players; if $\kappa = \{S\} \cup [N\backslash S]$, by Theorem 3.4.6, we have

$$\gamma_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) + \frac{1}{2}\cdot c\cdot\frac{(|S|-1)!(|N|-|S|)!}{|N|!} & \text{for all } i \in S \\ \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) + \frac{1}{2}\cdot c\cdot\left(-\frac{|S|!(|N|-|S|-1)!}{|N|!}\right) & \text{otherwise.} \end{cases}$$

For any $c \in \mathbb{R}$ and $(S,\kappa) \in \mathbb{E}'(N)$, let $I\left(c\delta_{(S,\kappa)}\right) = |S|$. In order to prove that $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$, we use a (converse-)induction argument on the number $I\left(c\delta_{(S,\kappa)}\right)$.

If $I\left(c\delta_{(S,\kappa)}\right) = |N|$, then $c\delta_{(S,\kappa)} = c\delta_{(N,\{N,\{N\}\})}$. One can readily check that for all $i \in N$, $\gamma_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$ because $e_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$. Efficiency and complete symmetry imply that for all $i \in N$, $f_i\left(c\delta_{(N,\{N,\{N\}\})}\right) = \frac{c}{|N|}$. Thus, $f\left(c\delta_{(N,\{N,\{N\}\})}\right) = \gamma\left(c\delta_{(N,\{N,\{N\}\})}\right)$. We then complete the first step for the induction argument.

Next, as an induction hypothesis, suppose that for each $k' \geq k+1$, if $I\left(c\delta_{(S,\kappa)}\right) = k'$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$. We need to show that if $I\left(c\delta_{(S,\kappa)}\right) = k$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$. Assume that $I\left(c\delta_{(S,\kappa)}\right) = k$.

Claim 1: If $\kappa \neq \{S\} \cup [N\backslash S]$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

First, we shall show that for each $i \notin S$, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. Let $i \notin S$, there are two cases:

Case 1: $S(\kappa, i) \neq \{i\}$. By Lemma 3.4.7, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$. Moreover, if $|S| \neq 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right)$$

and if $|S| = 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Case 2: $S(\kappa, i) = \{i\}$. By Lemma 3.4.7, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. Since $I\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = k+1$ and $(\kappa\setminus S)\cup\{S\cup\{i\}\} \neq \kappa_{S\cup\{i\}}$, by the induction hypothesis,

$$f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = \gamma_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = 0.$$

Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w)$. Hence, if $|S| \neq 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right);$$

if $|S| = 1$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) = 0 = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Now we need to show that for each $i \in S$, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. Let $i \in S$. If $S \neq \{i\}$, since for all $j \notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = 0$ and all players in S are completely symmetric in $c\delta_{(S,\kappa)}$, by efficiency and complete symmetry, we have for all $i \in S$,

$$f_i\left(c\delta_{(S,\kappa)}\right) = 0 = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

If $S = \{i\}$, since for all $j \notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = \gamma_j\left(c\delta_{(S,\kappa)}\right)$, by efficiency, obviously, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$.

Claim 2: If $\kappa = \{S\} \cup [N\setminus S]$, then $f\left(c\delta_{(S,\kappa)}\right) = \gamma\left(c\delta_{(S,\kappa)}\right)$.

Let $i \notin S$. Since $S(\kappa, i) = \{i\}$, by Lemma 3.4.7,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right).$$

Since $I\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = k+1$ and $(\kappa\setminus S)\cup\{S\cup\{i\}\} = \kappa_{S\cup\{i\}}$, by the induction hypothesis,

$$\begin{aligned} f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) &= \gamma_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) \\ &= \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) + \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!}. \end{aligned}$$

Therefore,

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!}.$$

Thus, if $|S| > 1$, $\frac{1}{2}e_i(w) - \frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = 0 = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$, so

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

If $|S| = 1$, $\frac{1}{2}e_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) = 0$ and $\frac{1}{2}e_i(w) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right)$, so we also have

$$f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) - \frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} = \gamma_i\left(c\delta_{(S,\kappa)}\right).$$

Let $i \in S$. If $S \neq \{i\}$, since for all $j \notin S$, $f_j\left(c\delta_{(S,\kappa)}\right) = \gamma_j\left(c\delta_{(S,\kappa)}\right) = -\frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!}$ and all players in S are completely symmetric in $c\delta_{(S,\kappa)}$, by efficiency and complete symmetry, we have for all $i \in S$,

$$\begin{aligned} f_i\left(c\delta_{(S,\kappa)}\right) &= \frac{|N|-k}{k}\cdot\frac{1}{2}\cdot c\cdot\frac{k!(|N|-k-1)!}{|N|!} \\ &= \frac{1}{2}\cdot c\cdot\frac{(k-1)!(|N|-k)!}{|N|!} \\ &= \gamma_i\left(c\delta_{(S,\kappa)}\right). \end{aligned}$$

If $S = \{i\}$, by efficiency, obviously, $f_i\left(c\delta_{(S,\kappa)}\right) = \gamma_i\left(c\delta_{(S,\kappa)}\right)$. ■

We now provide an alternative characterization for the consensus value by means of the transfer property, which is in the same spirit as that for the Shapley value for the case of TU games in characteristic function form (cf. Feltkamp (1995)).

The transfer property, introduced by Dubey (1975), that in some sense substitutes for additivity, is defined as follows. For any two partition function form games $w_1, w_2 \in PG^N$, we first define the games $(w_1 \vee w_2)$ and $(w_1 \wedge w_2)$ by $(w_1 \vee w_2)(S,\kappa) = \max\{w_1(S,\kappa), w_2(S,\kappa)\}$ and $(w_1 \wedge w_2)(S,\kappa) = \min\{w_1(S,\kappa), w_2(S,\kappa)\}$ for all $S \in \kappa$ and $\kappa \in \mathbb{P}(N)$. Let $f : PG^N \longrightarrow \mathbb{R}^N$ be a solution concept on the class of partition function form games. Then, f satisfies the transfer property if $f(w_1 \vee w_2) + f(w_1 \wedge w_2) = f(w_1) + f(w_2)$ for all $w_1, w_2 \in PG^N$.

In order to characterize the consensus value on the class of all partition function form games by the transfer property, we need the following lemma. Here, the zero game in PG^N that is defined by $w(S,\kappa) = 0$ for all $(S,\kappa) \in \mathbb{E}(N)$ is denoted by $\underline{0}$.

Lemma 3.4.9 *Let f be a solution on PG^N satisfying the transfer property, with*[7] *$f(\underline{0}) = 0$. Then, for all games $w \in PG^N$,*

$$f(w) = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}). \tag{3.1}$$

Proof. We prove in three steps that equation (3.1) holds.

Step 1: For the class of all non-negative games w the proof is by induction on

$$k(w) := |\{S|(S,\kappa) \in \mathbb{E}(N) \text{ and } w(S,\kappa) > 0\}|.$$

Here, a game w is non-negative if $w(S,\kappa) \geq 0$ for all $(S,\kappa) \in \mathbb{E}(N)$.
If $k(w) = 0$, then $w = \underline{0}$, so $f(w) = 0 = \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)})$.
Take $k > 0$ and suppose equation (3.1) holds for all non-negative games w with $k(w) < k$. For a non-negative game w with $k(w) = k$, choose an embedded coalition $(S',\kappa') \in \mathbb{E}(N)$ such that $w(S',\kappa') > 0$. Then $k(w - w(S',\kappa')\delta_{(S',\kappa')}) = k - 1$, $(w - w(S',\kappa')\delta_{(S',\kappa')}) \vee (w(S',\kappa')\delta_{(S',\kappa')}) = w$ and $(w - w(S',\kappa')\delta_{(S',\kappa')}) \wedge (w(S',\kappa')\delta_{(S',\kappa')}) = 0$. Hence, using the induction hypothesis and the transfer property, we obtain

$$\begin{aligned}
f(w) = & \quad f(w - w(S',\kappa')\delta_{(S',\kappa')}) + f(w(S',\kappa')\delta_{(S',\kappa')}) \\
- & \quad f((w - w(S',\kappa')\delta_{(S',\kappa')}) \wedge (w(S',\kappa')\delta_{(S',\kappa')})) \\
= & \sum_{(S,\kappa)\in\mathbb{E}'(N)} f[(w - w(S',\kappa')\delta_{(S',\kappa')})(S,\kappa)\delta_{(S,\kappa)}] + f(w(S',\kappa')\delta_{(S',\kappa')}) - f(\underline{0}) \\
= & \sum_{(S,\kappa)\in\mathbb{E}'(N);\ (S,\kappa)\neq(S',\kappa')} f(w(S,\kappa)\delta_{(S,\kappa)}) + f(w(S',\kappa')\delta_{(S',\kappa')}) - f(\underline{0}) \\
= & \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}).
\end{aligned}$$

Step 2: For non-positive games one proves analogously (interchanging the operations $\wedge$ and $\vee$) that equation (3.1) holds.

Step 3: For an arbitrary game w, split the game into its non-negative part $v \vee \underline{0}$ and its non-positive part $v \wedge \underline{0}$. The transfer property and steps 1 and 2 imply

$$\begin{aligned}
f(w) &= f(w) + f(\underline{0}) \\
&= f(w \vee \underline{0}) + f(w \wedge \underline{0}) \\
&= \sum_{(S,\kappa)\in\mathbb{E}'(N)} [f((w \vee \underline{0})(S,\kappa)\delta_{(S,\kappa)}) + f((w \wedge \underline{0})(S,\kappa)\delta_{(S,\kappa)})] \\
&= \sum_{(S,\kappa)\in\mathbb{E}'(N)} f(w(S,\kappa)\delta_{(S,\kappa)}).
\end{aligned}$$

[7] *Note that $f(\underline{0}) = 0$ is a weak requirement because any solution concept satisfying efficiency and (complete) symmetry yields this outcome.*

Hence equation (3.1) holds for all partition function form games. ■

Note that the converse is also true: If a solution concept f on PG^N satisfies equation (3.1) for all games $w \in PG^N$, then f satisfies the transfer property and $f(\underline{0}) = 0$.

Below we introduce a lemma which is similar to Lemma 3.4.7.

Lemma 3.4.10 *Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$, and f be a solution on PG^N satisfying the transfer property with $f(\underline{0}) = 0$ and the quasi-null player property. We have*

$$f_i\left(c\delta_{(S,\kappa)}\right) = \begin{cases} \frac{1}{2}e_i\left(c\delta_{(S,\kappa)}\right) & \text{if } S(\kappa,i) \neq \{i\} \\ \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right) & \text{if } S(\kappa,i) = \{i\}, \end{cases}$$

where w is defined in Lemma 3.4.7.

Proof. Let $c \in \mathbb{R}$, $(S,\kappa) \in \mathbb{E}'(N)$ and $i \notin S$. The proof is the same as that for Lemma 3.4.7 except for the case when $S(\kappa,i) = \{i\}$. Since here we can write $w = c\delta_{(S,\kappa)} \vee c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}$, by the transfer property, we have

$$\begin{aligned} & f(c\delta_{(S,\kappa)} \vee c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}) + f(c\delta_{(S,\kappa)} \wedge c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}) \\ = \; & f(w) + f(\underline{0}) \\ = \; & f(c\delta_{(S,\kappa)}) + f(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}). \end{aligned}$$

Thus, $f_i(w) = f_i\left(c\delta_{(S,\kappa)}\right) + f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. Moreover, since i is a quasi-null player in w, by the quasi-null player property, we know $f_i(w) = \frac{1}{2}e_i(w)$. Therefore, $f_i\left(c\delta_{(S,\kappa)}\right) = \frac{1}{2}e_i(w) - f_i\left(c\delta_{(S\cup\{i\},(\kappa\setminus S)\cup\{S\cup\{i\}\})}\right)$. ■

Using Lemma 3.4.9 and Lemma 3.4.10, we now prove the following.

Theorem 3.4.11 *The consensus value is the only one-point solution on the class of partition function form games that satisfies efficiency, symmetry, the quasi-null player property and the transfer property.*

Proof. First of all, we claim that a solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ satisfying additivity on PG^N also satisfies the transfer property on PG^N. To prove this, take $w_1, w_2 \in PG^N$. Then, using additivity,

$$\begin{aligned} f(w_1 \vee w_2) + f(w_1 \wedge w_2) &= f(w_1 \vee w_2 + w_1 \wedge w_2) \\ &= f(w_1 + w_2) \\ &= f(w_1) + f(w_2). \end{aligned}$$

Therefore, the consensus value satisfies the transfer property.

By Lemma 3.4.10 and using the same technique in the proof for Theorem 3.4.8, one can readily see that requiring a solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ to satisfy efficiency, complete symmetry, and the quasi-null player property, it easily follows that f is uniquely determined for (multiples of) Dirac games. Moreover, based on Lemma 3.4.9, we know that a solution f satisfying the transfer property is uniquely determined for any game in PG^N, since the class of Dirac games forms a basis of PG^N. ■

Similarly, we can characterize the Shapley value for partition function form games introduced by Pham Do and Norde (2002) by means of this transfer property.

Theorem 3.4.12 *The Shapley value is the only one-point solution on the class of partition function form games that satisfies efficiency, symmetry, the null player property (or the marginal quasi-null player property) and the transfer property.*

One may notice that a nice feature of the consensus value for partition function form games lies in the individual rationality.

First, we define superadditivity for partition function form games. A partition function form game $w \in PG^N$ is called *superadditive* if it satisfies

$$\begin{aligned} w(S \cup T, \{S \cup T\} \cup \kappa_{N\setminus(S\cup T)}) \geq \quad & w(S, \{S\} \cup \{T\} \cup \kappa_{N\setminus(S\cup T)}) \\ + \quad & w(T, \{S\} \cup \{T\} \cup \kappa_{N\setminus(S\cup T)}) \end{aligned}$$

for all $S, T \subset N$ and $\kappa_{N\setminus(S\cup T)} \in \mathbb{P}(N\setminus(S \cup T))$ with $S \cap T = \emptyset$.

Theorem 3.4.13 *If a partition function form game $w \in PG^N$ is superadditive and with nonnegative externalities on individual players, i.e. $w(\{i\}, \kappa_{N\setminus\{i\}}) \geq w(\{i\}, [N])$ for all $i \in N$ and $\kappa_{N\setminus\{i\}} \in \mathbb{P}(N\setminus\{i\})$, then the consensus value satisfies* individual rationality, *that is, $\gamma_i(w) \geq w(\{i\}, [N])$ for all $i \in N$.*

Proof. By Definition 3.3.1, it is easy to see that in any superadditive game with nonnegative externalities on individual players, the individual standardized remainder $s_i^\sigma(w)$ is greater than or equal to the stand-alone value $w(\{i\}, [N])$ for all $i \in N$ and $\sigma \in \Pi(N)$. ■

This is a very reasonable property. However, not all solution concepts satisfy it. See the following example where the game is taken from Cornet (1998).

Example 3.4.14 *Let the game* (N, w) *be given by* $N = \{1, 2, 3\}$ *and*

$$\begin{aligned} \bar{w}(1,2,3) &= (0,0,0), \\ \bar{w}(12,3) &= (0,3),\ \bar{w}(13,2) = (0,3),\ \bar{w}(23,1) = (3,0), \\ \bar{w}(123) &= (4). \end{aligned}$$

As the game is superadditive and with nonnegative externalities on individual players, the consensus value satisfies individual rationality. Indeed, one can check that the consensus value of this game $\gamma(w) = (\frac{1}{3}, \frac{11}{6}, \frac{11}{6})$*, which coincides with the Shapley value defined by Pham Do and Norde (2002) in this game, is greater than* $(0, 0, 0)$*. However, the Myerson value is* $(-\frac{5}{3}, \frac{17}{6}, \frac{17}{6})$*; the Feldman value as well as the Bolger's and Potter's value are* $(-\frac{1}{6}, \frac{25}{12}, \frac{25}{12})$*; the Shapley value defined by Feldman (1994) is* $(-\frac{2}{3}, \frac{7}{3}, \frac{7}{3})$*.*

3.5 A generalization of the consensus value

By relaxing the way of sharing remainders, we get a generalization of the consensus value: the *generalized consensus value*, which is in the same spirit as section 4 of chapter 2.

We define the generalized remainder, with respect to an order $\sigma \in \Pi(N)$, for given $\theta \in [0, 1]$, recursively by

$$r_\theta(S_k^\sigma) = \begin{cases} w(N, \{N\}) & \text{if } k = |N| \\ \\ w(S_k^\sigma, \kappa_k^\sigma) + (1-\theta)(r_\theta(S_{k+1}^\sigma) \\ -w(S_k^\sigma, \kappa_k^\sigma) - w(\{\sigma(k+1)\}, \kappa_k^\sigma)) & \text{if } k \in \{1, ..., |N| - 1\}. \end{cases}$$

The generalized remainder is the value left for S_k^σ after allocating surplus to later entrants $N \backslash S_k^\sigma$ according to share parameter θ. Correspondingly, the individual generalized remainder vector $s_\theta^\sigma(w)$ is the vector in $\mathbb{R}^N$ defined by

$$(s_\theta^\sigma)_{\sigma(k)}(w) = \begin{cases} w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) \\ +\theta \left(r_\theta(S_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma) - w(\{\sigma(k)\}, \kappa_{k-1}^\sigma) \right) & \text{if } k \in \{2, ..., |N|\} \\ \\ r_\theta(S_1^\sigma) & \text{if } k = 1. \end{cases}$$

Definition 3.5.1 *For any* $w \in PG^N$*, the generalized consensus value,* $\gamma_\theta(w)$*,* $\theta \in [0, 1]$*, is the average of the individual generalized remainder vectors, i.e.*

$$\gamma_\theta(w) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} s_\theta^\sigma(w).$$

Note that the consensus value corresponds to the case $\theta = \frac{1}{2}$.

As mentioned in section 4, dependent on the degree to which that the coalition effect or externality effect is preferred by a society, the quasi-null player property can be generalized. Defining the *θ-quasi-null player property* of a one-point solution concept $f : PG^N \longrightarrow \mathbb{R}^N$ by $f_i(w) = (1-\theta)e_i(w)$ for all $w \in PG^N$ and any quasi-null player $i \in N$ for w, we obtain the following theorem.

Theorem 3.5.2 *For $\theta \in [0,1]$:*

(a) The generalized consensus value γ_θ is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the θ-quasi-null player property and additivity.

(b) For any $w \in PG^N$, it holds that

$$\gamma_\theta(w) = \theta\Phi(w) + (1-\theta)e(w)$$

(c) The generalized consensus value γ_θ is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the θ-quasi-null player property and the transfer property.

Proof. Following the same way to prove Theorem 3.4.5, Theorem 3.4.6, Theorem 3.4.8, and Theorem 3.4.11, it is easily established. ∎

In particular, for $\theta = 1$, the generalized consensus value is the Shapley value defined by Pham Do and Norde (2002); for $\theta = 0$, the generalized consensus value equals the expected stand-alone value.

Corollary 3.5.3 *(a) The expected stand-alone value is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the 0-quasi-null player property and additivity.*

(b) The expected stand-alone value is the unique one-point solution concept on PG^N that satisfies efficiency, complete symmetry, the 0-quasi-null player property and the transfer property.

The proof is omitted as it is obvious.

The idea of defining the consensus value can also be adapted. If taking the size of the incumbent party S into consideration, we can argue on a basis of a proportional principle that given an ordering of players the entrant should get $\frac{1}{|S|+1}$ of the joint surplus while the incumbents get a share of $\frac{|S|}{|S|+1}$, which results in another solution concept, namely, *the coalition-size-based consensus value* for partition function form games.

3.6 Some applications of the consensus value

3.6.1 Application to oligopoly games

Along the same line as Pham Do and Norde (2002), this section first applies the consensus value to oligopoly games in partition function form.

Let us focus on a linear oligopoly market of a homogeneous good with asymmetric costs, no fixed costs and no capacity constraints. Such an oligopoly is defined by the vector $(b; c) \in \mathbb{R}_+^{n+1}$, where $b > 0$ is the intercept of the inverse demand function, $c = (c_1, c_2, ..., c_n) \geq 0$ is the marginal cost vector. Without loss of generality, assume $c_1 \leq c_2 \leq ... \leq c_n$. We also assume that an equilibrium price always exceeds the largest marginal cost, i.e. $\frac{b+\sum_{j=1}^n c_j}{n+1} > c_n$. Note that this assumption is equivalent to the requirement of positive market shares at the equilibrium for all players (Zhao (2001)). For each supply (input) vector $x = (x_1, x_2, ..., x_n)$, the price is $p(x) = b - \sum_{i=1}^n x_i$, whereas player i's cost and profit (payoff) are $C_i(x_i) = c_i x_i$ and

$$\pi_i(x) = p(x)x_i - C_i(x_i) = \left(b - \sum_{i=1}^n x_i\right) x_i - c_i x_i,$$

respectively. Player i's reaction curve is implicitly defined by the first order condition:

$$\frac{\partial \pi_i(x)}{\partial x_i} = p(x) - x_i - c_i = 0, \text{ or } x_i = \frac{b - c_i - \sum_{j \neq i} x_j}{2}. \tag{3.2}$$

A Cournot-Nash equilibrium is a vector such that each player's action x_i is a best response to the complementary choice $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. This equilibrium is graphically the intersection point of all reaction curves and algebraically the solution of the system of the equations (3.2). The unique equilibrium, $x^* = (x_1^*, ..., x_n^*)$, is determined by

$$x_i^* = \frac{b - n \cdot c_i + \sum_{j \neq i} c_j}{n+1}$$

and the payoff of player i at this equilibrium is

$$\pi_i(x^*) = (x_i^*)^2 = \frac{(b - n \cdot c_i + \sum_{j \neq i} c_j)^2}{(n+1)^2}.$$

Now suppose that after sufficient communication, some players may agree to cooperate (for example, players intend to adjust negative externalities which are caused by decreasing returns to inputs). In such a situation a coalition structure might form, in which, however, the payoff of coalition S depends on the behavior of the players outside S, and visa versa. Notice that the payoff for coalition S under one coalition structure is different from that under another coalition structure if the number of coalitions is different. Assume that the marginal cost of coalition S is $c_S = min_{i\in S}c_i$, that is, the most efficient technology in a coalition can be costlessly adopted by all players in that coalition. Moreover, if a coalition structure $\kappa = \{S_1, ..., S_k\}$ is formed, then, in equilibrium each coalition S in κ will choose the total (input) quantity levels to maximize the sum of its members' profits, given the total inputs of the other coalitions in κ.

Let $x_{S_j} = \sum_{i\in S_j} x_i$ denote the total input level for a coalition S_j and $\pi_{S_j}(x)$ denote the profit of coalition S_j under coalition structure κ,

$$\pi_{S_j}(x) = p(x)x_{S_j} - C_S(x_{S_j}) = \left(b - \sum_{i=1}^{k} x_{S_i}\right) x_{s_j} - c_{S_j}x_{S_j}.$$

Coalition S_j's reaction curve under coalition structure κ is also implicitly defined by the first order condition:

$$\frac{\partial \pi_{S_j}(x)}{\partial x_{S_j}} = p(x) - x_{S_j} - c_{S_j} = 0, \text{ or } x_{S_j} = \frac{b - c_{S_j} - \sum_{i\neq j} x_{S_i}}{2}.$$

The unique equilibrium under coalition structure κ with quantities $x^* = (x^*_{S_1}, ..., x^*_{S_k})$, and profit $\pi_{S_j}(x^*)$ of coalition S_j, is given by

$$x^*_{S_j} = \frac{b - k\cdot c_{S_j} + \sum_{i\neq j} c_{S_i}}{k+1}$$

and

$$\pi_{S_j}(x^*) = \frac{(b - k\cdot c_{S_j} + \sum_{i\neq j} c_{S_i})^2}{(k+1)^2}.$$

The oligopoly game in partition function form (N, w) is determined for every (S_j, κ) by $w(S_j, \kappa) = \pi_{S_j}(x^*)$, where x^* is the equilibrium vector under coalition structure κ.

To get further illustration of how the consensus value can be used we specify the 3-person oligopoly game in partition function form (N, w). The partition function form game is given by $\bar{w}(1, 2, 3) = (a_1, a_2, a_3)$, $\bar{w}(12, 3) = (a_{12}, b_3)$, $\bar{w}(13, 2) = (a_{13}, b_2)$,

$\bar{w}(23,1) = (a_{23}, b_1)$, $\bar{w}(123) = (a_{123})$, where

$$\begin{aligned}
a_1 &= \frac{1}{16}(b - 3c_1 + c_2 + c_3)^2, \\
a_2 &= \frac{1}{16}(b - 3c_2 + c_1 + c_3)^2, \\
a_3 &= \frac{1}{16}(b - 3c_3 + c_1 + c_2)^2, \\
a_{12} &= \frac{1}{9}(b - 2c_1 + c_3)^2, \; b_3 = \frac{1}{9}(b - 2c_3 + c_1)^2 \\
a_{13} &= \frac{1}{9}(b - 2c_1 + c_2)^2, \; b_2 = \frac{1}{9}(b - 2c_2 + c_1)^2 \\
a_{23} &= \frac{1}{9}(b - 2c_2 + c_1)^2, \; b_1 = \frac{1}{9}(b - 2c_1 + c_2)^2 \\
a_{123} &= \frac{1}{9}(b - c_1)^2.
\end{aligned}$$

Given the ordering of marginal costs, one can easily see that $a_1 \geq a_2 \geq a_3$, and $a_{12} \geq a_{13} = b_1 \geq a_{23} = b_2 \geq b_3$.

The consensus value of this game, $\gamma(w) = (\gamma_i(w))_{i=1,2,3}$, can be computed as follows:

$$\begin{aligned}
\gamma_1(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_1 - a_2 - a_3 + \frac{a_{12} + 3a_{13} - 3a_{23} - b_3}{2}\right) \\
\gamma_2(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_2 - a_1 - a_3 + \frac{a_{12} + 3a_{23} - 3a_{13} - b_3}{2}\right) \\
\gamma_3(w) &= \frac{a_{123}}{3} + \frac{1}{6}\left(2a_3 - a_1 - a_2 + b_3 - a_{12}\right).
\end{aligned}$$

Note that if players have identical costs, then $a_1 = a_2 = a_3$ and $a_{12} = a_{13} = a_{23} = b_1 = b_2 = b_3$, and obviously, the consensus value yields an equal payoff to all players, i.e. $\gamma_i(w) = \frac{a_{123}}{3}$.

Consider the following example for further illustration.

Example 3.6.1 *The game (cf. Pham Do and Norde (2002)) in partition function form* (N, w) *associated with a linear oligopoly market* $(b; c)$, *where* $b = 20$, $c = (1, 3, 4)$, *is given by*

$$\begin{aligned}
\bar{w}(1,2,3) &= (36, 16, 9), \\
\bar{w}(12,3) &= (53.78, 18.78), \; \bar{w}(13,2) = (49, 25), \; \bar{w}(23,1) = (25, 49), \\
\bar{w}(123) &= (90.25).
\end{aligned}$$

The consensus value for this game is $\gamma(w) = (46.833, 24.833, 18.583)$, *whereas the Shapley value is* $\Phi(w) = (46.70, 24.71, 18.83)$.

3.6.2 Free-rider, sharing rule and participation incentive

Since the partition function form games can well capture externalities, they provide a suitable framework to analyse the associated issues such as the free-rider problem. Below we will investigate the effects of different solution concepts on the participation incentives of the players who may free-ride in a game.

Consider the following partition function form game (N, w) (we may call it a free-rider game) defined by $\bar{w}(1,2,3) = (0,0,0)$, $\bar{w}(12,3) = (1,1)$, $\bar{w}(13,2) = (1,1)$, $\bar{w}(23,1) = (0,0)$, $\bar{w}(123) = 1$. This game can be interpreted as follows: Three players are considering to set up a joint project. Each player has two choices: *participate* or *stand by*. The success of the project depends on the players' participation. Here, obviously, player 2 and 3 are possible free-riders.

Since both player 2 and 3 are prone to standing by, it is very likely that the project will fail in the end. Thus, everybody becomes a loser due to their "selfish rationality". Given the different sharing rules, which one is better for increasing the possible free-riders' incentive to contribute instead of standing idle? We now check the following solution concepts and compare their influences on players' choices.

	1	2	3
the Shapley value $\Phi(w)$	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
Bolger, Feldman or Potter's value	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
the consensus value $\gamma(w)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Now we first discuss the effect of the Shapley value on the free-riders's participation incentive. Given the Shapley value as the solution concept for the above game, we know the three players will play the following strategic game.

If player 1 chooses participating, the payoff matrix is provided as below. (Here, the payoffs in each cell are listed in the order of player 1, 2 and 3.)

	3 participate	3 stand by
2 participate	$\frac{2}{3}, \frac{1}{6}, \frac{1}{6}$	$\frac{1}{2}, \frac{1}{2}, 1$
2 stand by	$\frac{1}{2}, 1, \frac{1}{2}$	$0, 0, 0$

While if player 1 chooses standing by, all of them will get zero payoff no matter what strategies player 2 and 3 will choose.

	3 participate	3 stand by
2 participate	$0, 0, 0$	$0, 0, 0$
2 stand by	$0, 0, 0$	$0, 0, 0$

So, obviously, choosing participating is the weakly dominant strategy for player 1. One can easily check this game has three pure-strategy Nash equilibria: (1 stands by, 2 stands by, 3 stands by), (1 participates, 2 participates, 3 stands by) and (1 participates, 2 stands by, 3 participates); and another equilibrium which involves mixed strategies of players 2 and 3: (1 participates, 2 participates with probability $\frac{3}{8}$ and stands by with probability $\frac{5}{8}$, 3 participates with probability $\frac{3}{8}$ and stands by with probability $\frac{5}{8}$). We may call such an equilibrium semi mixed-strategy equilibrium as player 1 still plays a pure strategy.

Similarly, one can check the results due to the implementation of the values by Bolger, Feldman or Potter. The corresponding three pure-strategy equilibria are the same as above, while the third equilibrium is different: (1 participates, 2 participates with probability $\frac{2}{5}$ and stands by with probability $\frac{3}{5}$, 3 participates with probability $\frac{2}{5}$ and stands by with probability $\frac{3}{5}$).

Now we check the consensus value in this game. Despite the fact that the three pure-strategy equilibria are the same as above, the semi mixed-strategy equilibrium is different: (1 participates, 2 participates with probability $\frac{3}{7}$ and stands by with probability $\frac{4}{7}$, 3 participates with probability $\frac{3}{7}$ and stands by with probability $\frac{4}{7}$).

Apparently, $\frac{3}{7} > \frac{2}{5} > \frac{3}{8}$; and the corresponding expected payoff due to the consensus value is also greater than the others: $1\frac{8}{49} > 1\frac{3}{25} > 1\frac{5}{64}$.

Therefore, from the semi mixed-strategy equilibrium, we can see that the consensus value helps to increase the participation incentives of the possible free-riders.

Chapter 4

Compensating losses and sharing surpluses in project-allocation situations

4.1 Introduction

This chapter[1] has two aims. Firstly, it develops a general framework for studying a class of economic environments in which coalitions of agents are optimally reassigned to some bundles of projects: project-allocation situations. Secondly, since this reassignment causes some agents losing jobs or positions, we analyse the problems associated with valuating such reshuffling, such as compensation for losses and sharing surpluses arising from the enhanced efficiency.

In an economy characterized by changing capabilities and preferences of agents and changing technology embodied in projects, people need to continuously adapt their positions to obtain efficiency. That is how our societies have evolved into prosperity. Every change in the production structure requires a reshuffling of responsibilities, which is hard or impossible to implement if possible "losers" are not sufficiently compensated to cooperate. Only when all parties gain from the reassignment, is it a win-win situation. Examples are abundant. Similarly, when extra profit is generated simply by cooperation after reshuffling, a surplus sharing problem occurs.

Obviously, solving the problems of compensation and surplus sharing is essential for creating and maintaining flexibility and creating efficiency in an economy. However, generally speaking, the two concepts are not well distinguished in theoretical research so that the corresponding practical problems cannot be treated adequately. In a strict

[1] This chapter is based on Ju, Ruys and Borm (2004).

sense, compensation refers to a financial remuneration to an agent for the loss caused by her being removed from some project. On the other hand, surplus sharing deals with the extra benefits created by cooperation among agents assigned to some combination of projects, which benefits are in excess of the sum of individual payoffs. Hence, if the compensation is not properly distinguished from surplus sharing, some individuals may lose on the whole after a reshuffling. Those individuals will strongly oppose and obstruct such a reshuffling. If there exists an authority who can impose reassignments from above, without minding too much about individual sacrifices, the overall approach is sufficient. But even then there are several value concepts available that have a characteristic influence on the outcome. That will be our point of departure.

One may observe that trade unions have forced firms to adopt generic rules for laborers that include some compensation for lay-offs in a firm, as well as labor laws and other safety nets on the macro-level. This chapter focuses on the micro-level. We assume that gains and losses for every particular situation can be endogenously specified and may serve as a basis for the issues of compensation of losses and surplus sharing.

Consider, for example, a restaurant and a boat company, working independently, both situated on the shore of the same lake. The restaurant, project A, is operated by agent 1 who can be understood as a group of managers, waiters and kitchen staff. Agent 2, a group of people as well, manages project B, the boat company. They are considering collaboration and have two proposals. The first one is simply setting up a joint lunch-sightseeing program, $\{1_A, 2_B\}$, that will benefit both parties. The second proposal is more involved and induces a reshuffling of the two projects, i.e. the restaurant and the boat company. Since agent 2 has excellent expertise in both travelling and restaurant management, much more profit will be generated if the restaurant is also managed by her. The technical possibilities in this situation are represented in the following diagram.

$\{1_A\}$	$\{2_B\}$	$\{1_A, 2_B\}$	$\{1, 2_{A,B}\}$
5	10	18	26

Note that the two types of cooperation, $\{1_A, 2_B\}$ and $\{1, 2_{A,B}\}$ are different in nature: the former corresponds to the first proposal where those two agents have their own projects and coordinate with each other; the latter can be understood as that agent 1 renders A to agent 2 and then works with 2 (just on his human capital). Whereas the first proposal only entails a surplus sharing problem, the second one is further complicated by the problem of compensating agent 1 for giving up his access or user rights of the restaurant project.

This chapter analyses both the loss compensation and surplus sharing problem as illustrated by the second proposal in the above example from a cooperative game theoretic point of view. In our framework, the value of some coalition of agents crucially depends on the involvement of the agents in this coalition in a well-defined set of projects. The involvement is measured by the notion of an agent's shares in the projects. That defines a so-called *project-allocation situation* (in short, P-A situation) and an associated *project-allocation game* (in short, P-A game). The value function of the project-allocation game is derived from the underlying profit functions for every coalition given a specific share profile of the projects. So, in particular, the value function of this game can be viewed as a generalization of the neoclassical profit function, with labor and capital as inputs and with prices given.

Naturally, any specific solution concept for a cooperative TU game may of course be applied to solve project-allocation games, and implicitly solve the *combined* loss compensation and surplus sharing problems. (The combined compensation and surplus sharing problem corresponding to the second proposal in the above example can be modelled as a TU game in which $v(\{1,2\})$ equals 26, the joint value generated by the cooperation between agent 1 and agent 2 after agent 1 transferred the restaurant project to agent 2.) We restrict our attention to two additive one-point solution concepts: the Shapley value (Shapley (1953)) and the *consensus value* (cf. chapter 2). Arguments for the suitability of these rules in this specific context are provided.

However, since P-A games are just a partial abstraction of P-A situations, this traditional approach is incapable of disentangling all necessary details to adequately model the basic mechanisms concerning the physical reallocation of projects (loss compensation) and cooperation in joint projects (surplus sharing). In fact, the process to realize the maximal gain of a coalition is a blackbox. Therefore, in order to make the framework operational for solving practical problems, one further step has to be made. By explicitly incorporating an underlying cooperative structure in terms of project reallocation and cooperation afterwards, we devise two different stages in such a way that the loss compensation due to project reallocation and the sharing of extra surplus from cooperation can be clearly and logically distinguished. Hence, this two stage approach makes voluntary acceptance of a reshuffling and a bottom up approach possible, and is even more compelling if reassignment means that agents are laid off and have no chance to participate in the benefit sharing. For each of the two stages, a game is constructed.[2] Consistently, the same solution concept is applied to each of the stage games. Thus, following a general stage approach, also a solution concept for the combined problem is obtained.

Although there exists some fundamental work that is helpful for our research, it seems that the problem of compensating losses has largely been ignored in economic research. The analysis of cost sharing situations (cf. Moulin (1987), Tijs and Branzei (2002)) and linear production situations (Owen (1975)) is in the same spirit but does not explicitly discriminate between the problems of surplus sharing and loss compensation. An exception in a somewhat different context is the work on sequencing games (Curiel et al (1989), Hamers (1995), Klijn (2000)). In this framework, time slots could be considered as projects. Agents change the initial order (shares or rights on time slots) into an optimal one so that the individual payoffs are changed and compensation is needed. Moreover, since joint total costs decrease as well, also the issue of surplus sharing becomes prominent.

In addition to this section introducing the problem and reviewing the literature briefly, the remaining part of the chapter is structured as follows. In the next section, we present the main analytical framework by formally introducing project-allocation situations and define project-allocation games. Section 3 addresses the possible solution concepts. Section 4 distinguishes stages to explicitly solve the problems of compensation and surplus sharing in project-allocation situations separately. Section 5 provides an example of public-private partnerships, which indicates an interesting application of the framework into real economic issues. The final section concludes the chapter by discussing the cases that the set of feasible share profiles is restricted.

4.2 Project-allocation situations and games

Consider a situation in which there exists a finite set N of agents/players who can operate a finite set M of *projects*. We use the word "project" in this chapter in a very general sense. A project is a specific entity that can be exploited or operated for some purpose(s) (and mostly for value-creation). It can be a machine, a research

[2] The combined problem in the above example can be decomposed into two stages. The first stage considers project reallocation such that agent 1 only renders his A project to agent 2 but does not make further cooperation. The second stage may correspond to a new situation: imagine that agent 1 does not have any project while agent 2 has both A and B; now agent 2 would like agent 1 to work for her.

project, a firm, or a public utility, etc. Generally, a project can either be *divisible* or *indivisible*. A project is divisible if it is capable of being separated into parts and can be partially operated or owned by some party, without loss of its original function. For instance, a tree farm as a project can be perfectly divided among agents. Indivisibility means that for the purpose of value-creation, a project can only be completely owned or exploited as a whole. A truck is then an indivisible project because it will lose the basic function if divided into parts. Since divisibility is a context-dependent concept which may imply physical divisibility, operational divisibility or ownership divisibility, we have to point out that this chapter focuses on the operational divisibility.

The basic idea of a project-allocation situation is that individual agents from the set N have received user rights to operate individual projects from the set M. Each such agent-project combination results in an outcome, called a payoff. Agents may also cooperate and form a coalition that operates a bundle of projects. Since both the agents and the projects are specialized, some agent-project combinations may generate a higher payoff than other combinations. So, for a given assignment of user rights over agents, each coalition of agents operates a specific bundle of projects, which generates a payoff. When, however, some coalition would perform better when it is assigned another bundle of projects, then a feasible reshuffling of coalitions of agents may increase the efficiency of the situation.

This problem of project-allocation is formalized as follows. Each agent receives a *share* in each project, which is a real number $\rho_{i,k}$ between 0 and 1, indicating[3] the fraction of project k that agent i may use or operate. If the agent has an exclusive right the share equals 1, and if he has no right on the project his share equals 0. If the agent has to share rights with other agents in N, and the project is divisible, the fraction corresponds with the distribution of the project over the agents, satisfying the feasibility restriction $\sum_{i \in N} \rho_{i,k} \leq 1$. For instance, one agent may own half of a project, the share is then 0.5. If the project is not divisible and assigned to a coalition S of agents, then the project is - fictitiously - equally distributed among the agents in S. So in that case the individual share $\rho_{i,k}$ equals $1/|S|$, for all $i \in S$. This allows for describing each agent's share in any indivisible project and for solving the problems of loss compensation and surplus sharing in such cases mathematically. That is, despite

[3]However, we do not restrict the implications of *shares*. Put differently, we do not give a definite economic interpretation but only care about how the shares (in a general sense) in projects that agents have will affect cooperation or even determine compensation and surplus sharing. It may have different meanings in different contexts. For example, it can also represent the ownership/property rights or managerial rights.

that an indivisible project can not be divided in itself, the value generated from it can be shared among the agents who jointly own it in some way.

The assignment of individual user rights for operating individual projects to the agents in N is thus specified by an $N \times M$ matrix[4] ρ, called a *share profile*. The set of share profiles, R, is defined by

$$R := \left\{ \rho \in [0,1]^{N \times M} | \sum_{i=1}^{n} \rho_{i,k} \leq 1, \text{ for all } k \in M \right\} \tag{4.1}$$

A share profile determines *feasible* agent-project combinations. Feasibility is integrated with technical performance by the following function. The map $f^\rho : 2^N \longrightarrow \mathbb{R}$, assigning for any share profile ρ in R, to any coalition S in N, a real number - the payoff - is called the *payoff function under share profile* ρ. So $f^\rho(S)$ is the payoff of coalition S under share profile ρ. An empty coalition has a zero payoff. Specialization implies that an arbitrary bundle of agent-project combinations may neither be optimal: other combinations may perform better; nor feasible: according to the given share profile it may not have access to projects required for a performance, in which case the payoff equals 0.

Thus, feasible reshuffling of a share profile is required to obtain optimality or efficiency. For that purpose we define the concept of a feasible allocation. Let an initial share profile ρ^0 in R be given. For any coalition $S \subset N$, a reallocation of shares within coalition S is called *feasible* for S, if the sum of initially allocated shares in each project to the agents in S equals the sum of the reallocated shares in each corresponding project to the agents in S, while the other agents in N keep their initial shares. So, the set of feasible allocations or feasible share profiles for coalition S with respect to ρ^0, $F(S, \rho^0)$, is defined by

$$F(S, \rho^0) := \left\{ \rho \in R | \sum_{i \in S} \rho_{i,k} = \sum_{i \in S} \rho^0_{i,k}, \text{ for all } k \in M \text{ and } \rho_{N \setminus S} = \rho^0_{N \setminus S} \right\} \tag{4.2}$$

For notational simplicity, we use $F(S)$ if there is no confusion about ρ^0. Thus, feasibility means that agents can re-arrange their shares in projects subject to the capacity determined by the initial share profile within the coalition they participate in, without affecting the allocations outside the coalition.

Based on the above description, we are able to define a project-allocation situation.

[4]Given agent set N of size n and project set M of size m, ρ is in fact an $n \times m$ matrix. We use $N \times M$ to emphasize that ρ is a matrix associated with an agent set and a project set. The same explanation applies in cases of other matrices, for instance, when we say that ρ_S is an $S \times M$ matrix.

Definition 4.2.1 *A* project-allocation situation $P(\rho^0)$ *is a tuple* $(N, M, \rho^0, \{f^\rho\}_{\rho\in R})$, *where* N *is the set of agents,* M *is the set of projects,* R *is the set of share profiles,* ρ^0 *is the initial share profile, and* $f^\rho : 2^N \longrightarrow \mathbb{R}$ *is the payoff function under a share profile* $\rho \in R$.

For analytical convenience, we use the following assumptions at different stages.

Assumption 4.2.2 (continuity)
for any $S \in 2^N$, $f^\rho(S)$ *is continuous with respect to the share profile* $\rho \in R$.

So a small change in the share profile has only a small effect on the value distribution.

Assumption 4.2.3 (no externality among coalitions)
$f^{\rho^1}(S) = f^{\rho^2}(S)$,
for all $S \in 2^N$, *whenever* $\rho^1_S = \rho^2_S$.

Here ρ_S is the $S \times M$ submatrix of ρ. It follows that the distribution of values within a coalition is independent from the share profile outside that coalition.

Assumption 4.2.4 (gains from cooperation)
$f^\rho(S \cup T) \geq f^\rho(S) + f^\rho(T)$,
for all $\rho \in R$ *and for all* $S, T \in 2^N$ *with* $S \cap T = \emptyset$.

Assumption 4.2.5 (ordinary cooperation)
$f^{\rho^1}(S) \geq f^{\rho^2}(S) \Rightarrow \sum_{i\in S} f^{\rho^1}(\{i\}) \geq \sum_{i\in S} f^{\rho^2}(\{i\})$,
for all $\rho^1, \rho^2 \in R$ *and for all* $S \in 2^N$.

The last assumption means that if a share profile is preferable for a coalition of players in cooperation, then the corresponding stand-alone situation is also preferable in terms of the sum of their individual payoffs. It can be understood as a type of consistency between cooperation and its stand-alone basis.

The class of all project-allocation situations with player set N and project set M and the payoff functions satisfying the above assumptions is denoted by $PAS^{N,M}$.

The project-allocation situation $P(\rho^0)$ provides room for reshuffling and optimizing the initial share profile ρ^0. This reallocation situation can be described as a TU game, in which the value of a coalition is defined as the maximal payoff that this coalition can achieve by means of feasible share profile.

Given a project-allocation situation $P(\rho^0) = (N, M, \rho^0, \{f^\rho\}_{\rho \in R}) \in PAS^{N,M}$, the cooperative game with transferable utility (N, v) defined by

$$v(S) := \max_{\rho \in F(S)} f^\rho(S) \tag{4.3}$$

for all coalitions S in N with $v(\emptyset) = 0$, is called a **project-allocation (P-A) game**.

A share profile $\rho \in F(S)$ with $f^\rho(S) = v(S)$ is called an *optimal share profile* for coalition S, denoted by $\rho^*(S)$.

The P-A games in this chapter satisfy the following property.

Proposition 4.2.6 *Under the Assumptions* 4.2.2 − 4.2.4*, project-allocation games are superadditive.*

Proof. Let $P(\rho^0) = (N, M, \rho^0, \{f^\rho\}_{\rho \in R})$ be a project-allocation situation and let the corresponding P-A game be given by (N, v). We need to show $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.

Consider optimal share profiles $\rho^*(S \cup T)$, $\rho^*(S)$ and $\rho^*(T)$ for coalitions $S \cup T$, S, T, respectively. Since $S \cap T = \emptyset$, we can construct a new share profile $\tilde{\rho} \in F(S \cup T)$ such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, by definition and Assumptions 4.2.3 and 4.2.4, we have

$$\begin{aligned} v(S \cup T) &= f^{\rho^*(S \cup T)}(S \cup T) \\ &\geq f^{\tilde{\rho}}(S \cup T) \\ &\geq f^{\tilde{\rho}}(S) + f^{\tilde{\rho}}(T) \\ &= f^{\rho^*(S)}(S) + f^{\rho^*(T)}(T) \\ &= v(S) + v(T). \end{aligned}$$

■

4.3 Solution concepts for project-allocation games

In this section, we consider two related solution concepts: the well known Shapley value Φ and the consensus value γ (see chapter 2).

When we go over the definition and the properties of the Shapley value, we can find that it may not be entirely adequate to analyse the project-allocation situations mainly by the following two reasons.

Firstly, the Shapley value relies on the basic notion of marginal vectors. Here, given some ordering of players entering a game, the payoffs are determined by the marginal contributions, which is not satisfying in a constructive or bargaining type of physical setting since a later entrant gets the whole surplus. In a superadditive game, the incumbents will not accept such an arrangement as their contributions are not reflected. While if a game is subadditive, the entrant will not accept such a contract. Apparently, in the practice of a project-allocation situation, a marginal vector is even harder to implement as it may involve reshuffling of projects by current incumbents.

Secondly, the dummy property does not seem too imperative in P-A situations. Rather, this purely utilitarian requirement assigning nothing more than the individual value to a dummy player may hinder the possible collaborations in a P-A situation. Payoffs can only be verified after actual project reallocations. Each agent can be a dummy player, while no one would like to pay effort for nothing. Furthermore, the balance in tradeoff between utilitarianism and egalitarianism is also critical in real life situations.

We propose an alternative solution concept for TU games: the consensus value. This rule follows from a natural and simple idea to share coalition values. For more information including an axiomatic characterization of this solution concept, we refer to chapter 2.

One can find the difference between those two values from the following P-A situation example.

Example 4.3.1 *Consider a P-A situation* $P(\rho^0) = (N, M, \rho^0, \{f^\rho\}_{\rho\in R}) \in PAS^{N,M}$ *where* $N := \{1,2,3\}$*,* $M := \{A, B\}$*, and*

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Furthermore,

$$\begin{aligned} f^\rho(\{1\}) &= 10\rho_{1,A} + 1\rho_{1,B} + \rho_{1,A}\rho_{1,B} \\ f^\rho(\{2\}) &= 8\rho_{2,A} + 3\rho_{2,B} + \rho_{2,A}\rho_{2,B} \\ f^\rho(\{3\}) &= 6\rho_{3,A} + 5\rho_{3,B} + \rho_{3,A}\rho_{3,B} \end{aligned}$$

$$
\begin{aligned}
f^{\rho}(\{1,2\}) &= \sum_{i\in\{1,2\}} f^{\rho}(\{i\}) + 3\left(\sum_{i\in\{1,2\}}\sum_{k\in M}\rho_{i,k}\right)\\
f^{\rho}(\{1,3\}) &= \sum_{i\in\{1,3\}} f^{\rho}(\{i\}) + 4\left(\sum_{i\in\{1,3\}}\sum_{k\in M}\rho_{i,k}\right)\\
f^{\rho}(\{2,3\}) &= \sum_{i\in\{2,3\}} f^{\rho}(\{i\}) + 2\left(\sum_{i\in\{2,3\}}\sum_{k\in M}\rho_{i,k}\right)\\
f^{\rho}(\{1,2,3\}) &= \sum_{i\in\{1,2,3\}} f^{\rho}(\{i\}) + 6\left(\sum_{i\in\{1,2,3\}}\sum_{k\in M}\rho_{i,k}\right).
\end{aligned}
$$

It is easy to see that these payoff functions satisfy Assumptions 4.2.2 − 4.2.5. *The corresponding P-A game is given by*

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	12	0	19	0	17	27

with $\gamma(v) = \left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$ *and* $\Phi(v) = \left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$.

It is our opinion that the consensus value fits quite well in the reshuffling process and the admission structure of the P-A situations. Consider an existing coalition S and a new entrant i. Ex ante, S is a well formed coalition: they had reallocated projects with each other and now cooperate well; they also share the joint surplus in some way. Now, player i would join this coalition. What happens? Obviously, i could not work with any sub-coalition of S but only with S as a whole since S has already been formed over there, comparable to the case that two players cooperate. The immediate (standard) and also a practical solution is then to share the extra revenues from the cooperation equally between S and i.

However, the approach to model the whole P-A situation as one cooperative game is not completely satisfying: P-A games do not take all practical features of P-A situations into account. The underlying process of realizing and allocating the maximal gain of the grand coalition $f^{\rho^*(N)}(N)$ starting from the individual payoffs is still a blackbox.

4.4 The two stage approach: compensation and surplus sharing

We now focus on an underlying process of obtaining and redistributing the maximal payoff of the grand coalition $f^{\rho^*(N)}(N)$ in a project-allocation situation $P(\rho^0)$.

The initial share distribution ρ^0 yields a *stand-alone value distribution before reallocation*:

$$\beta_i^0 := f^{\rho^0}(\{i\}), \text{ for } i = 1, ..., n.$$

Reallocation of shares not only changes this individual value distribution based on stand-alone activities, but also the payoff generated by new combinations of coalitions: the surplus generated by cooperation. For determining the boundaries of individual compensation, we focus on the stand-alone situations and compare the initial stand-alone value with the stand-alone value generated by the optimal share distribution[5] $\rho^*(N)$. So the optimal share distribution $\rho^*(N)$ yields a *stand-alone value distribution after reallocation*:

$$\beta_i^* := f^{\rho^*(N)}(\{i\}), \text{ for } i = 1, ..., n.$$

If the difference $(\beta_i^* - \beta_i^0)$ is positive, it indicates the stand-alone gain from reallocation, which is also the maximal compensation agent i is willing to pay to other agents. If it is negative, it gives the stand-alone loss from reallocation, which is the minimal compensation agent i is asking from other agents for agreeing with the reallocation of shares.

Taking β^* as a watershed, we can distinguish two stages in the reshuffling process. The first stage considers the compensation issue while the second one focuses on surplus sharing. In order to further illustrate why we take β^* as the critical point and divide the whole P-A situation into two stages, let us consider the following time line which might happen in a P-A situation. At time 1, agents with initial share profile ρ^0 consider cooperation. Due to various constraints or simply because they are only interested in the exchanges of projects at this moment, those agents only reallocate shares of projects to get a better share profile, say, ρ^*, in terms of the total welfare of the whole group, but put other types of cooperation aside. At time 2, based on the new share profile ρ^*,

[5] As noted in section 2, there may exist multiple optimal share profiles for N. For simplicity, in this chapter, we focus the analysis on the cases with a unique optimal share profile. For coalitions we do not have to impose such a condition because by Assumption 4.2.5 possible multiplicity does not play a role in the procedure, which will be further explained in this section.

they are going to work together. Combining those two cases together yields exactly the same outcome as modelled by the corresponding P-A game. However, in a practical situation like the above, the two cases should be treated separately. Moreover, even if we do not explicitly distinguish the two cases based on the time line, one would still like to clarify the share based contribution from the specialization based contribution, while the former is related to project reallocation and the latter corresponds to the cooperation based on a share profile.

- **Stage 1: The compensation game** $(N, \bar{w})$

This stage consists of project reallocation towards the optimal share profile $\rho^*(N)$ for the grand coalition N and finally yields β^*. To determine agents' true values in this reallocation stage, we construct a *stand-alone game*, in which not only the stand-alone values for the grand coalition are taken into account, but also the stand-alone values generated by other coalitions. Given a P-A situation $P(\rho^0)$, the stand-alone game (N, w) is defined by $w(S) = \sum_{i\in S} f^{\rho^*(S)}(\{i\})$, where $\rho^*(S) = \arg\max_{\rho\in F(S,\rho^0)} f^\rho(S)$. So, in particular, $w(\{i\}) = \beta_i^0$ and $w(N) = \sum_{i\in N} \beta_i^*$.

Here, we want to note that $w(S)$ is well defined.

Let $P(\rho^0) = (N, M, \rho^0, \{f^\rho\}_{\rho\in R})$ be a project-allocation situation. For a coalition $S \subset N$, suppose there exist two optimal share profiles $\rho^{*1}(S)$ and $\rho^{*2}(S)$. Then,

$$f^{\rho^{*1}(S)}(S) = f^{\rho^{*2}(S)}(S) = \max_{\rho\in F(S)} f^\rho(S).$$

It suffices to show $\sum_{i\in S} f^{\rho^{*1}(S)}(\{i\}) = \sum_{i\in S} f^{\rho^{*2}(S)}(\{i\})$. By Assumption 4.2.5, it follows that

$$\left.\begin{array}{l} f^{\rho^{*1}(S)}(S) = f^{\rho^{*2}(S)}(S) \Rightarrow f^{\rho^{*1}(S)}(S) \geq f^{\rho^{*2}(S)}(S) \\ \Rightarrow \sum_{i\in S} f^{\rho^{*1}(S)}(\{i\}) \geq \sum_{i\in S} f^{\rho^{*2}(S)}(\{i\}) \\ \\ f^{\rho^{*2}(S)}(S) = f^{\rho^{*1}(S)}(S) \Rightarrow f^{\rho^{*2}(S)}(S) \geq f^{\rho^{*1}(S)}(S) \\ \Rightarrow \sum_{i\in S} f^{\rho^{*2}(S)}(\{i\}) \geq \sum_{i\in S} f^{\rho^{*1}(S)}(\{i\}) \end{array}\right\}$$

$$\Rightarrow \sum_{i\in S} f^{\rho^{*1}(S)}(\{i\}) = \sum_{i\in S} f^{\rho^{*2}(S)}(\{i\}).$$

Hence, $w(S)$ is well defined.

Proposition 4.4.1 *Stand-alone games are superadditive.*

Proof. Let $P(\rho^0) = (N, M, \rho^0, \{f^\rho\}_{\rho \in R})$ be a project-allocation situation and let the corresponding stand-alone game be given by (N, w). We need to show $w(S \cup T) \geq w(S) + w(T)$ for all $S, T \subset N$ with $S \cap T = \emptyset$.

Let $\rho^*(S \cup T)$, $\rho^*(S)$ and $\rho^*(T)$ be the optimal share profiles for coalitions $S \cup T$, S, T, respectively. Let $\tilde{\rho} \in F(S \cup T)$ be such that $\tilde{\rho_S} = \rho^*_S(S)$ and $\tilde{\rho_T} = \rho^*_T(T)$. Then, we have

$$\begin{aligned} w(S \cup T) &= \sum_{i \in S \cup T} f^{\rho^*(S \cup T)}(\{i\}) \\ &\geq \sum_{i \in S \cup T} f^{\tilde{\rho}}(\{i\}) \\ &= \sum_{i \in S} f^{\tilde{\rho}}(\{i\}) + \sum_{i \in T} f^{\tilde{\rho}}(\{i\}) \\ &= \sum_{i \in S} f^{\rho^*(S)}(\{i\}) + \sum_{i \in T} f^{\rho^*(T)}(\{i\}) \\ &= w(S) + w(T). \end{aligned}$$

Here, the inequality follows from the fact that $f^{\rho^*(S \cup T)}(S \cup T) \geq f^{\tilde{\rho}}(S \cup T)$ and Assumption 4.2.5. ■

In general, $\beta^* \neq \beta^0$; and apparently, the agents who incurred losses due to project reallocation need to be compensated. To explicitly determine compensations, we will consider solutions of the associated *compensation game* $(N, \bar{w})$ defined by

$$\bar{w}(S) = w(S) - \sum_{i \in S} \beta^*_i.$$

Note that $\bar{w}(N) = 0$. The specific values of compensation depend on the solution concept to be chosen, such as the Shapley value $\Phi(\bar{w})$ or the consensus value $\gamma(\bar{w})$.

- **Stage 2: The surplus sharing game** $(N, \bar{\omega})$

This stage considers cooperation after the reallocation in the first stage, i.e. co-working on projects based on the optimal share profile $\rho^*(N)$. This type of cooperation yields a *co-working* game (N, ω) defined as the project-allocation game corresponding to the project-allocation situation where its initial share profile is its optimal share profile, i.e., $(N, M, \rho^*(N), \{f^\rho\}_{\rho \in R})$.

Proposition 4.4.2 *Co-working games are superadditive.*

Proof. Co-working games are superadditive as they are project-allocation games. ∎

Indeed, this game still takes project reallocation into consideration so that agents are allowed to reallocate shares before joint production. However, the initial share profile itself in this situation is the optimal share profile and $\omega(N) = v(N) = f^{\rho^*(N)}(N)$, players do not reallocate projects in the grand coalition any more (although it may happen in theory within sub-coalitions) but directly work with each other with their current shares. So no compensation is needed. What entails is only surplus sharing. For this aspect, we consider solutions of the associated *surplus sharing* game $(N, \bar{\omega})$ given by $\bar{\omega}(S) = \omega(S) - \sum_{i \in S} \beta_i^*$.

It is obvious that the two stage approach decomposes the maximal payoff of the grand coalition into three elements: $v(N) = \sum_{i \in N} \beta_i^* + \bar{w}(N) + \bar{\omega}(N)$.

The above description on the two stages in a project-allocation situation implies a natural and reasonable way to share the maximal payoff $f^{\rho^*(N)}(N)$. Firstly, an agent i has her stand-alone value after reallocation, β_i^*, due to optimal project reallocation. In addition, to determine the compensations, one solves the compensation game $\bar{w}$; and to solve the surplus sharing problem, one solves the surplus sharing game $\bar{\omega}$. A player's final payoff is the sum of these three parts.

Solving both games with the same one-point solution concept yields a (stage-based) consistent one-point solution concept for P-A situations:

$$\psi^*(P(\rho^0)) := \beta^* + \psi(\bar{w}) + \psi(\bar{\omega})$$

where $\psi : TU^N \longrightarrow \mathbb{R}^N$ is a one-point solution concept for TU games.

Generally, it will be the case that the immediate application of ψ to the P-A game v will yield a different solution, i.e., $\psi^*(P(\rho^0)) \neq \psi(v)$. Moreover, we know for a P-A situation, the corresponding P-A game v is well defined. Therefore, the application of ψ to v will yield a unique solution no matter which optimal share profile is adopted and implemented. In contrast to the one stage approach based on the P-A game, for different optimal share profiles, the two stage approach based on the distinction between compensation for share changes and surplus sharing for joint work might generate different solutions because there are different surplus sharing games. Hence, this approach discovers the role of project reallocation in measuring agents' contributions in P-A situations, which seems consistent with our intuition: based on different reallocation profiles, agents could make different contributions to the value of the grand coalition, and thereby different rewards are required.

One may wonder under which conditions the equality holds and both the one stage and the two stage approach give the same result. We require two weak conditions on ψ, i.e., $\psi(0) = 0$ and translation invariance $\psi(v + b) = \psi(v) + b$ for all $v \in TU^N$ and $b \in \mathbb{R}^N$ (b is an additive game), and strengthen Assumption 4.2.4 in the following way.

Assumption 4.4.3 *f is additive with respect to disjoint coalitions, i.e. $f^\rho(S \cup T) = f^\rho(S) + f^\rho(T)$ for all $\rho \in R$ and for all $S, T \subset N$ with $S \cap T = \emptyset$.*

Now we can show the following.

Proposition 4.4.4 *With Assumption 4.4.3, if ψ satisfies translation invariance and $\psi(\underline{0}) = 0$, then $\psi_i^*(P(\rho^0)) = \psi_i(v)$, for all $i \in N$, where ψ, $P(\rho^0)$ and v are defined as above.*

Proof. Clearly, Assumption 4.4.3 implies that $f^{\rho^*(N)}(N) = \sum_{i \in N} f^{\rho^*(N)}(\{i\})$. Consequently, $w(S) = v(S)$ for all S in N and $\bar{\omega}(S) = 0$ for all $S \subset N$. What remains is obvious: $\psi^*(P(\rho^0)) = \psi(v)$. ■

In particular, the Shapley value $\Phi^*(P(\rho^0))$ of a project-allocation situation $P(\rho^0)$ is given by

$$\Phi_i^*(P(\rho^0)) = \beta_i^* + \Phi_i(\bar{w}) + \Phi_i(\bar{\omega}) \text{ for all } i \in N.$$

Similarly, the consensus value $\gamma^*(P(\rho^0))$ of a project-allocation situation $P(\rho^0)$ is given by

$$\gamma_i^*(P(\rho^0)) = \beta_i^* + \gamma_i(\bar{w}) + \gamma_i(\bar{\omega}) \text{ for all } i \in N.$$

One may also, in principle, choose a solution concept for the compensation game that is different from the solution concept for the surplus sharing game.

4.5 An example: disintegration in the water sector

Let us look at an example considering the reform of disintegration and reallocation in the water sector. In this setting, we have three players $N := \{1, 2, 3\}$: player 1 is a provincial government, 2 is a local government, and 3 is a company; two projects: water business (A) and related business (B) such as a golf club or recreation park built on the water source land. So, $M := \{A, B\}$. Initially, both projects are owned by the local government. Consequently, the initial share profile is

$$\rho^0 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Unlike the for-profit project B, water business is usually seen as a public utility. So, the payoff of running the water project can be interpreted as the social welfare instead of individual profit. Moreover, we assume that the company has speciality in operating a commercial business while the provincial government may create higher social value if she controls the water project. However, they do have some relative weaknesses. For example, the private company is not good at running public utilities. This type of situation is modelled by the corresponding payoff functions, which are provided in Example 4.3.1.

Without cooperation, players' individual payoffs come from two parts: the stand-alone payoffs generated from project A or B and the payoff due to the cross-subsidy effect between the two projects. With cooperation, in addition to players' individual payoffs, there are some extra gains from cooperation, which, for instance, is expressed by the term $2\left(\sum_{i\in\{2,3\}}\sum_{k\in M}\rho_{i,k}\right)$ in the payoff function of coalition $\{2,3\}$.

The optimal reform plans for the various coalitions will be: $\rho^*(\{i\}) = \rho^0$, $i = 1, 2, 3$;

$$\rho^*(\{1,2\}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \rho^*(\{1,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}; \quad \rho^*(\{2,3\}) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and for grand coalition $\{1, 2, 3\}$,

$$\rho^*(\{1,2,3\}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

One readily checks that $\beta^0 := (0, 12, 0)$, and $\beta^* = (10, 0, 5)$.

Moreover, beside the project-allocation game (N, v) for the whole situation in this example, we have a compensation game and a surplus sharing game:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	12	0	19	0	17	27
$\bar{w}(S)$	-10	12	-5	3	-15	8	0
$\bar{\omega}(S)$	0	0	0	3	8	2	12

The solutions based on the Shapley value or the consensus value can be found in the following two tables.

$\Phi(v)$	$\left(4\frac{1}{2}, 19, 3\frac{1}{2}\right)$
β^*	$(10, 0, 5)$
$\Phi(\bar{w})$	$\left(-9\frac{1}{6}, 13\frac{1}{3}, -4\frac{1}{6}\right)$
$\Phi(\bar{\omega})$	$\left(5\frac{1}{6}, 2\frac{1}{6}, 4\frac{2}{3}\right)$
$\Phi^*(P(\rho^0))$	$\left(6, 15\frac{1}{2}, 5\frac{1}{2}\right)$

$\gamma(v)$	$\left(4\frac{3}{4}, 18, 4\frac{1}{4}\right)$
β^*	$(10, 0, 5)$
$\gamma(\bar{w})$	$\left(-9\frac{1}{12}, 13\frac{1}{6}, -4\frac{1}{12}\right)$
$\gamma(\bar{\omega})$	$\left(4\frac{7}{12}, 3\frac{1}{12}, 4\frac{1}{3}\right)$
$\gamma^*(P(\rho^0))$	$\left(5\frac{1}{2}, 16\frac{1}{4}, 5\frac{1}{4}\right)$

Hence, according to the consensus value, the local government is compensated by the provincial government and the company with a total amount of $13\frac{1}{6}$ due to project reallocation, and obtains $3\frac{1}{12}$ from the joint surplus generated by joint production based on the new share profile.

4.6 Discussion: restricted set of feasible share profiles

The above sections deal with the problems of compensating losses and sharing surpluses in the general project-allocation situations where no restriction is imposed on the set of share profiles. More specifically, the set of feasible share profiles plays a crucial role in determining the games associated with the two stage approach and the combined approach. Here, according to equation (4.2) and the definitions of those games, given an initial share profile ρ^0, any reallocation scheme in $F(S)$ can be carried out.

However, in the real world, there are many cases that the optimal share profile (or more generally, some profiles) can not be implemented to attain efficiency due to a number of practical reasons and hence, for instance, a suboptimal reallocation scheme (or an arbitrary reallocation profile which is available) is popular. We now briefly discuss a possible way to model such a situation and apply the two stage approach to solve the associated loss compensation and surplus sharing problem.

We call $F'(S) \subset F(S)$ a subset of feasible share profiles for coalition S. Given a project-allocation situation $P(\rho^0)$, we can define a restricted P-A game (N, v') by $v'(S) := \max_{\rho \in F'(S)} f^\rho(S)$ for all $S \subset N$.

For such cases, one can find an easy way to reestablish the analysis as above by defining $\rho^*(S) = \arg\max_{\rho \in F'(S,\rho^0)} f^\rho(S)$.

Here, we want to note that there could exist a special class of restricted set of feasible share profiles as

$$F'(S,\rho^0) = \begin{cases} \{\rho'(N)\} & \text{if } S = N \\ F(S,\rho^0) & \text{otherwise,} \end{cases}$$

where $\rho'(N) \in F(N,\rho^0)\backslash\{\arg\max_{\rho \in F(N,\rho^0)} f^\rho(N)\}$. Or more directly, sometimes we are interested in the loss compensation and surplus sharing problem under an arbitrary reallocation profile $\rho'(N)$ $(\neq \rho^*(N))$ for the grand coalition which is not optimal. For such situations, one could simply take $\rho'(N)$ as the critical point to separate the two stages. The compensation game for the first stage solves the compensation problem arising from the project reallocation $\rho^0 \to \rho'$. In the same spirit, the surplus sharing game for the second stage can be readily constructed.

Finally, we propose a possible way to apply the two stage approach to the P-A situations where Assumption 4.2.5 is relaxed. That is, with the absence of the ordinary cooperation assumption, we need to modify the stand-alone game (N, w) as follows.

$$w(S) = \begin{cases} \sum_{i \in N} f^{\rho^*}(\{i\}) & \text{if } S = N \\ \max_{\rho \in F(S,\rho^0)} \sum_{i \in S} f^\rho(\{i\}) & \text{otherwise,} \end{cases}$$

where ρ^* is still the optimal share profile for N, i.e., $\rho^* = \arg\max_{\rho \in F(N)} f^\rho(N)$. One may observe that in such cases a stand-alone game could be subadditive. However, it provides a basis for measuring how much the agents should sacrifice for obtaining the efficiency in the next stage. The sum of the losses and gains from the two stages plus the stand-alone value after reallocation serve as the final payoff for an agent in such a P-A situation.

Chapter 5

Externalities and compensation

5.1 Introduction

This chapter focuses on the issue of externality and the associated compensation problem. Externalities arise whenever an (economic) agent undertakes an action that has an effect on another agent. When the effect turns out to be a cost imposed on the other agent(s), it is called a negative externality. When agents benefit from an activity in which they are not directly involved, the benefit is called a positive externality. An associated fundamental question in real life is how to compensate the losses incurred by the negative externalities.

Pigou (1920) suggests a solution that involves intervention by a regulator who imposes a Pigovian tax. An alternative solution, known as the Coase theorem (Coase (1960)), involves negotiation between the agents. Coase claims that if transactions costs are zero and property rights are well defined, agents should be able to negotiate their way to an efficient outcome. A third class of solutions, associated with Arrow (1970), involves setting up a market for the externality. If a firm produces pollution that harms another firm, then a competitive market for the right to pollute may allow for an efficient outcome. In a relatively recent study, Varian (1994) designs the so-called compensation mechanisms for internalizing externalities which encourage the firms to correctly reveal the costs they impose on the other.

In fact, all solutions and approaches above try and solve the inefficiency problems arising from externalities, whereas they can not be viewed as normative answers in terms of fairness. Put differently, one may say that those solutions do provide the normative answers but from a utilitarian perspective. However, this indicates that the study of the “general” externality problem is not adequate. In particular, the theories can not answer a basic question like how much a household should be compensated by a polluting firm. Therefore, we are still in search of basic normative solutions which might serve as benchmarks to determine adequate compensations in environments that are featured by externalities.

Solving an externality-incurred compensation problem boils down to recommending solutions for profit/cost sharing problems with externalities. A first model to solve this problem was developed by Thrall and Lucas (1963) by the concept of *partition function form games*: A partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition. Values for such games can be found in Myerson (1977), Bolger (1986), Feldman (1994), Potter (2000), Pham Do and Norde (2002), and Ju (2004). For more details, we refer to chapter 3.

However, one may observe that the framework of partition function form games does not model the externalities among individuals but restrict to specific coalitional effects. The reason is simple: The partition function form games as well as the cooperative games with transferable utility in characteristic function form always assume all the players in the player set N are present even if they do not form a coalition. Consider a partition function form game and a player i in this game. What we know about the values with respect to i has the following three cases only: complete breakdown, i.e., all the players in this game do not cooperate with each other; partial cooperation, i.e., i participates in some coalition or i stands alone while some other players cooperate; complete cooperation, i.e., all the players form a grand coalition. In fact, the externalities among individual players (inter-individual externalities) are "internalized" or "incorporated" from the very beginning because there is no distinction between the case when only one player is in the game and the case when all appear.

The task attempted in this chapter is essentially twofold. First, it takes players' initial situations (no other players, in an absolute stand-alone sense) into account and constructs a new class of games which model the externalities among individual players. Second, it discusses several solution concepts which can actually serve as different benchmarks to solve the compensation issue related to externality problems.

The model of so-called primeval games has a flavor of TU games but is more alike as the partition function form games in structure. The two basic differences from the classical cooperative games are that primeval games do not consider cooperation, and primeval games take into account the situations in which only one player exists. Meanwhile, they also consider all other possible cases where other players appear. In this way, all possible externalities among players are modelled.

We introduce several solution concepts for primeval games: a modification of the Shapley value for TU games (Shapley (1953)), the consensus value, and a more context-specific solution concept: the primeval value. The first two solution concepts are axiomatically characterized. Properties of the primeval value and the comparison between it and the other two values are discussed.

In addition to this section introducing the chapter briefly, the remaining part proceeds as follows. The next section presents an example that motivates the approach and the model. In section 3, we lay out the general model: primeval games. Section 4 defines three solution concepts for primeval games. Section 5 introduces unanimity games for the class of primeval games which facilitates the characterizations of the solution concepts satisfying the additivity property. The Shapley value and the consensus value are characterized in an axiomatic way in section 6. Properties of the primeval value are discussed although it is not characterized in this chapter. By comparison, it is shown that the primeval value is more appropriate for the class of primeval games and is therefore easier to be implemented in practice. A generalization of the consensus value is discussed in section 7. Taking cooperation into consideration, the final section briefly introduces a possible way to connect primeval games with TU games and suggests some solution concepts.

5.2 An example: a village with three households

Consider an example of three players, a, b, and c. One can assume that they are three households living in the same district; or more generally, some (economic) agents in a certain situation. To be more specific, consider the following figure.

<table>
<tr><td colspan="2">slot 1 (a)</td></tr>
<tr><td>slot 2 (b)</td><td>slot 3 (c)</td></tr>
</table>

Figure 5.1: A three-household village

Figure 5.1 can be thought of as a village in a geographic sense, which is divided into three parts or slots. Each household lives in one slot, as shown above. In the current structure, without any friendly cooperating behavior or any hostile competing behavior among them, the utilities of a, b, and c are given as 8, 2 and 2, respectively.

It is quite common that one household may generate positive or negative externalities to the others. For instance, a's utility is not only dependent on a himself, but may also depend on the activities of b and c. That is, the realization of 8 is the outcome of every household's activities in the current structure. It could be higher or lower if some other household were absent or would stop any possible activities that may generate externalities to a.

Therefore, it is necessary and interesting to go "back" to see the "primeval" situations of the current structure.

In the case that only household b lives in the village, or equivalently, in the case that b comes into this village first while a and c are not present the utility for b would be 3 instead of 2. This case is described by the second column in the following table. There are also some other cases, for instance, the fifth column implies that when both a and c live in the village while b is not present, a and c's utilities would be 5 and 1, respectively. All the other possible cases are provided as well.

(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
(5)	(3)	(2)	$(8,2)$	$(5,1)$	$(3,0)$	$(8,2,2)$

From the above table, one can easily see that a is in fact a beneficiary from the externality point of view while b is somehow a loser. Naturally, some associated questions arise: Should b be compensated? If so, how to compensate b? And how much?

5.3 The model: primeval games

To capture all the possibilities of inter-individual externalities and further discuss the associated compensation problem, we now construct a new model, which has a flavor of cooperative games, and more appropriately, looks like TU games in partition function form.

Let $N = \{1, 2, ..., n\}$ be the finite set of players. A subset S of N, in order to be distinguished from the usual concept of *coalition* in cooperative games, is directly called a *group S of individuals* (in short, group S). Here, the term of group should be understood as a neutral concept, which has nothing to do with cooperation or anything else, but simply means a set of individual players in N.

A pair (i, S) that consists of a player i and a group S of N to which i belongs is called an *embedded player* in S. Let $\mathcal{E}(N)$ denote the set of embedded players, i.e.

$$\mathcal{E}(N) = \left\{(i, S) \in N \times 2^N | i \in S\right\}.$$

Definition 5.3.1 *A mapping*

$$u : \mathcal{E}(N) \longrightarrow \mathbb{R}$$

that assigns a real value, $u(i, S)$, to each embedded player (i, S) is an individual-group function*. The ordered pair (N, u) is an* individual-group function form game*, or a* primeval game[1]*. The set of primeval games with player set N is denoted by PRI^N.*

The value $u(i, S)$ represents the payoff, or utility, of player i, given all the players in S come into view (but no cooperation at all) in the absence of any other players. For a given group S and an individual-group function u, let $\bar{u}(S)$ denote the vector $(u(i, S))_{i \in S}$. We call $\bar{u}(N)$ the *status quo* of a primeval game u; and call $u(i, \{i\})$ the *Rubinson Crusoe payoff* (in short, R-C payoff) of player i in game u.

Definition 5.3.2 *A solution concept on PRI^N is a function f, which associates with each game (N, u) in PRI^N a vector $f(N, u)$ of individual payoffs in $\mathbb{R}^N$, i.e.,*

$$f(N, u) = (f_i(N, u))_{i \in N} \in \mathbb{R}^N.$$

5.4 Solution concepts

In this section, we introduce several possible solution concepts for primeval games.

5.4.1 The Shapley value

Focusing on one-point solution concepts, one might immediately think of the Shapley value. We now introduce a modification of the Shapley value in primeval games.

The idea is based on a general principle that players should not do harm to the others. If a player's activities do not cause negative effect on the others, then he has no responsibility for the consequences of such activities; otherwise, he must pay for that as the compensation for the victims. On the other hand, equivalently, if a player's activities impose positive effect on the others, then he has the right to ask them to pay for that. Meanwhile we need to take the orders that players enter a game into account and might adopt a practical principle as *first come, first served*. That is, the player who comes into a game first should be well protected: Any later entrant must compensate him if she causes loss on him while he does not have to worry about any

[1] Since it models inter-individual externalities and aims to solve the associated compensation problem, an alternative name would be individual externality-compensation game.

possible negative effects he could impose on the later entrants, i.e., he has the right to assume no responsibility for his behavior, irrespective of what consequence it might cause on the others. Along the same line of reasoning, the second entrant only cares about the first player but does not have any responsibility for his successors whereas all his successors should take care of the first two entrants' payoffs. More specifically, given an ordering of players, the early entrants should be well protected such that the losses due to negative externalities are compensated. Correspondingly, the gains from positive externalities should be transferred to whom they are produced by. Those effects can be well captured by the so-called marginal values.

The formal definition is provided as follows. For a primeval game $u \in PRI^N$, we construct the marginal vector $m^\sigma(u)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ and where each player $\sigma(k)$ is given the marginal value he creates by entering. Formally, it is the vector in $\mathbb{R}^N$ defined by

$$m^\sigma_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) & \text{if } k = 1 \\ \\ u(\sigma(k), S^\sigma_k) + \sum_{j=1}^{k-1} \left(u(\sigma(j), S^\sigma_k) - u(\sigma(j), S^\sigma_{k-1}) \right) & \text{if } k \in \{2, ..., |N|\}. \end{cases}$$

Therefore, player $\sigma(k)$ might be involved in four kinds of compensating behavior or circumstances: compensating the incumbents if he produces negative externalities on them, being compensated from the incumbents if they benefit from his showing up (i.e., he produces positive externalities on the incumbents), being compensated by the later entrants if they impose negative externalities on him; paying compensation to the later entrants if they generate positive externalities on him.

Here, one can readily check that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{t} m^\sigma_{\sigma(k)}(u) = \sum_{k=1}^{t} u(\sigma(k), S^\sigma_t)$$

for all $t \in \{1, ..., |N|\}$.

Furthermore, since there is no predetermined ordering of players, we take all possible permutations into consideration. Thus, *the Shapley value* $\Phi(u)$ is defined as the average of the marginal vectors, i.e.,

$$\Phi(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(u).$$

Example 5.4.1 *Consider the three-household village example.*

With $\sigma : \{1,2,3\} \longrightarrow N$ *defined by* $\sigma(1) = b$, $\sigma(2) = a$ *and* $\sigma(3) = c$, *which is shortly denoted by* $\sigma = (b\ a\ c)$, *we get*

$$\begin{aligned}
m_b^\sigma(u)(= m_{\sigma(1)}^\sigma(u)) = &\quad u(b,\{b\}) = 3,\\
m_a^\sigma(u)(= m_{\sigma(2)}^\sigma(u)) = &\quad u(a,\{a,b\}) + u(b,\{a,b\}) - u(b,\{b\}) = 8+2-3 = 7,\\
m_c^\sigma(u)(= m_{\sigma(3)}^\sigma(u)) = &\quad u(c,\{a,b,c\}) + u(b,\{a,b,c\})\\
+ &\quad u(a,\{a,b,c\}) - u(b,\{a,b\}) - u(a,\{a,b\})\\
= &\quad 2+2+8-2-8 = 2.
\end{aligned}$$

Similarly, all marginal vectors are given by

σ	$m_a^\sigma(u)$	$m_b^\sigma(u)$	$m_c^\sigma(u)$
$(a\ b\ c)$	5	5	2
$(a\ c\ b)$	5	6	1
$(b\ a\ c)$	7	3	2
$(b\ c\ a)$	9	3	0
$(c\ a\ b)$	4	6	2
$(c\ b\ a)$	9	1	2

Then, we get $\Phi(u) = \left(6\frac{1}{2}, 4, 1\frac{1}{2}\right)$. *Thus, in respect of compensation for externalities,* a *needs to pay* $1\frac{1}{2}$ *to* b, *and* c *will pay* $\frac{1}{2}$ *to* b.

The Shapley value can be expressed by the following formula. Given a game $u \in PRI^N$,

$$\Phi_i(u) = \sum_{S\subset N\setminus\{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}\left(u(i,S\cup\{i\}) + \sum_{j\in S}(u(j,S\cup\{i\}) - u(j,S))\right)$$

for all $i \in N$.

5.4.2 The consensus value

One might oppose the "first come, first served" idea and rather prefer an equal responsibility based rule. The argument could be as follows: Every player has the right to enter the game although he could be later. From the bilateral point of view, both parties (the incumbents and the entrant) should be equally responsible for the externality due to the showing up of the new entrant. Take the village example. In the cases that player b comes first, he only takes up slot 2 but has no rights of the other slots. Suppose player a follows. One possible argument could be as follows. a may deserve the payoff of 8 as he has the right for slot 1 in any case. Meanwhile we also observe that b is negatively affected. However, not only a but also b should account for the loss of 1 because it is the outcome of the joint effect between a's activities and b's feelings. An alternative argument could be that the households have the property rights of their slots. Therefore they equally enjoy the rights to produce externalities on their slots, irrespective of the timing about entering the village. Then, a 50-50 rule seems more suitable. Put differently and directly, this rule takes a different perspective to define the property rights of the externalities, which means that two parties involved should equally assume the corresponding responsibility.

In order to define the consensus value for primeval games, we construct the concession vector $C^\sigma(u)$, which corresponds to the situation where players enter the game u one by one in an order $\sigma \in \Pi(N)$ and where every new entrant, say $\sigma(k)$, first obtains the payoff when entering, $u(\sigma(k), S^\sigma_k)$, and then equally shares with every incumbent her surplus/loss incurred by the corresponding positive/negative externality imposed by him, and also equally shares his surplus/loss with all his successors. The word of *concession* is used here because players concede to each other and make a compromise on assuming responsibilities of the externalities.

We first define player $\sigma(k)$'s *concession payoff for the externalities on previous players* as

$$\mathcal{P}^\sigma_{\sigma(k)}(u) = \sum_{j=1}^{k-1} \frac{u(\sigma(j), S^\sigma_k) - u(\sigma(j), S^\sigma_{k-1})}{2}$$

and his *concession payoff from the subsequent externalities* as

$$\mathcal{S}^\sigma_{\sigma(k)}(u) = \sum_{l=k+1}^{|N|} \frac{u(\sigma(k), S^\sigma_l) - u(\sigma(k), S^\sigma_{l-1})}{2}.$$

Apparently, when a player enters the game u in the very first place, he has no concession payoff for the externalities on previous players. Therefore $\mathcal{P}^\sigma_{\sigma(1)}(u) = 0$. Correspondingly, when a player enters a game in the very last place, there is no subsequent externality for him. Hence, $\mathcal{S}^\sigma_{\sigma(|N|)}(u) = 0$.

Moreover, the concession payoff from the subsequent externalities for player $\sigma(k)$ can be simplified as

$$\mathcal{S}^\sigma_{\sigma(k)}(u) = \frac{u(\sigma(k), N) - u(\sigma(k), S^\sigma_k)}{2}$$

for all $k = \{1, ..., |N| - 1\}$.

Now, formally, the concession vector is the vector in $\mathbb{R}^N$ defined by

$$C^{\sigma}_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) + \mathcal{S}^{\sigma}_{\sigma(1)}(u) & \text{if } k = 1 \\ u(\sigma(k), S^{\sigma}_k) + \mathcal{P}^{\sigma}_{\sigma(k)}(u) + \mathcal{S}^{\sigma}_{\sigma(k)}(u) & \text{if } k = \{2, ..., |N| - 1\} \\ u(\sigma(|N|), N) + \mathcal{P}^{\sigma}_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

And more explicitly,

$$C^{\sigma}_{\sigma(k)}(u) = \begin{cases} \frac{u(\sigma(1), N) + u(\sigma(1), \{\sigma(1)\})}{2} & \text{if } k = 1 \\ \mathcal{P}^{\sigma}_{\sigma(k)}(u) + \frac{u(\sigma(k), N) + u(\sigma(k), S^{\sigma}_k)}{2} & \text{if } k = \{2, ..., |N| - 1\} \\ u(\sigma(|N|), N) + \mathcal{P}^{\sigma}_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

We want to note that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{|N|} C^{\sigma}_{\sigma(k)}(u) = \sum_{k=1}^{|N|} u(\sigma(k), S^{\sigma}_{|N|}),$$

but generally,

$$\sum_{k=1}^{t} C^{\sigma}_{\sigma(k)}(u) \neq \sum_{k=1}^{t} u(\sigma(k), S^{\sigma}_t)$$

for all $t \in \{1, ..., |N| - 1\}$.

The concession value $\mathcal{C}(u)$ is defined as the average of the concession vectors, i.e.

$$\mathcal{C}(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} C^{\sigma}(u).$$

Since basically the concession value is in the same spirit as the consensus value for TU games (see chapter 2), we might call it *the consensus value* for primeval games.

Example 5.4.2 *Consider the three-household village example.*

With $\sigma = (b\ a\ c)$ which is a shorthand notation as in Example 5.4.1, we get

$$
\begin{aligned}
C^\sigma_b(u) = {} & \frac{u(b,\{b,a,c\}) + u(b,\{b\})}{2} = \frac{2+3}{2} = 2\frac{1}{2}, \\
C^\sigma_a(u) = {} & \frac{u(b,\{b,a\}) - u(b,\{b\})}{2} + \frac{u(a,\{b,a,c\}) + u(a,\{b,a\})}{2} \\
= {} & \frac{2-3}{2} + \frac{8+8}{2} = 7\frac{1}{2}, \\
C^\sigma_c(u) = {} & u(c,\{b,a,c\}) \\
+ {} & \frac{(u(b,\{b,a,c\}) - u(b,\{b,a\})) + (u(a,\{b,a,c\}) - u(a,\{b,a\}))}{2} \\
= {} & 2.
\end{aligned}
$$

Similarly, all concession vectors are given by

σ	$C^\sigma_a(u)$	$C^\sigma_b(u)$	$C^\sigma_c(u)$
$(a\ b\ c)$	$6\frac{1}{2}$	$3\frac{1}{2}$	2
$(a\ c\ b)$	$6\frac{1}{2}$	4	$1\frac{1}{2}$
$(b\ a\ c)$	$7\frac{1}{2}$	$2\frac{1}{2}$	2
$(b\ c\ a)$	$8\frac{1}{2}$	$2\frac{1}{2}$	1
$(c\ a\ b)$	6	4	2
$(c\ b\ a)$	$8\frac{1}{2}$	$1\frac{1}{2}$	2

Then, we get $\mathcal{C}(u) = \left(7\frac{1}{4}, 3, 1\frac{3}{4}\right)$. Thus, in respect of compensation for externalities, a needs to pay $\frac{3}{4}$ to b, and c will pay $\frac{1}{4}$ to b. Compared to the Shapley value, both a and c give less compensations to b in this case.

Theorem 5.4.3 *The consensus value is the average of the status quo payoff and the Shapley value. That is, for any game $u \in PRI^N$, it holds that*

$$\mathcal{C}_i(u) = \frac{1}{2}u(i,N) + \frac{1}{2}\Phi_i(u)$$

for all $i \in N$.

Proof. Given a game $u \in PRI^N$ and $\sigma \in \Pi(N)$, let $i = \sigma(k)$, where $k \in \{1, 2, ...|N|\}$.

By definition, we know for $k \in \{2, ..., |N|-1\}$

$$
\begin{aligned}
C_i^\sigma(u) = \quad & C_{\sigma(k)}^\sigma(u) \\
= \quad & \frac{1}{2}\left(u(i,N) + u(i,S_k^\sigma)\right) + \frac{1}{2}\sum_{j=1}^{k-1}\left(u(\sigma(j),S_k^\sigma) - u(\sigma(j),S_{k-1}^\sigma)\right) \\
= \quad & \frac{1}{2}(u(i,N) + \frac{1}{2}\left(u(i,S_k^\sigma) + \sum_{j=1}^{k-1}\left(u(\sigma(j),S_k^\sigma) - u(\sigma(j),S_{k-1}^\sigma)\right)\right) \\
= \quad & \frac{1}{2}u(i,N) + \frac{1}{2}m_{\sigma(k)}^\sigma(u).
\end{aligned}
$$

Moreover, it is obvious for the cases that $k = 1$ or $|N|$. Hence, $\mathcal{C}_i(u) = \frac{1}{2}u(i,N) + \frac{1}{2}\Phi_i(u)$. ■

5.4.3 The primeval value

We now propose an alternative solution concept, which is more context-specific and therefore seems more appropriate for primeval games. The basic idea of this solution concept is that the losses due to negative externalities should be compensated whereas the benefits from the positive externalities are enjoyed for free. This is a somehow general and natural attitude when people face externalities in reality. Thus, the rule based on this idea might be easy to be accepted and implemented in practice.

The value defined below and corresponding to the above idea can be named as the *chargeable negative (externalities) and free positive (externalities) value.* In short and since we believe that it provides more suitable and practical solution for primeval games, we may directly call it *the primeval value.*

For a primeval game $u \in PRI^N$ and an ordering $\sigma \in \Pi(N)$ and $k \in \{1, 2, ..., |N|\}$, we construct the *primeval vector* $B^\sigma(u)$, which corresponds to the situation where the players enter the game one by one in the order $\sigma(1), \sigma(2), ..., \sigma(|N|)$ and where each player $\sigma(k)$ compensates the losses of his predecessors but enjoys positive externalities from his successors freely.

We now define player $\sigma(k)$'s *loss for compensating negative externalities* as

$$L_{\sigma(k)}^\sigma(u) = \sum_{j=1}^{k-1}\left(u(\sigma(j),S_{k-1}^\sigma) - u(\sigma(j),S_k^\sigma)\right)^+$$

and his *gain from subsequent positive externalities* as

$$G^{\sigma}_{\sigma(k)}(u) = \sum_{l=k+1}^{|N|} \left(u(\sigma(k), S^{\sigma}_{l}) - u(\sigma(k), S^{\sigma}_{l-1})\right)^{+},$$

where $a^{+} = \max\{0, a\}$.

Apparently, when a player enters the game u in the very first place, he assumes no responsibility to the others. Therefore $L^{\sigma}_{\sigma(1)}(u) = 0$. Correspondingly, when a player enters a game in the very last place, he could not enjoy any subsequent positive externality from the others. Hence, $G^{\sigma}_{\sigma(|N|)}(u) = 0$.

Formally, the primeval vector is the vector in $\mathbb{R}^N$ defined by

$$B^{\sigma}_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) + G^{\sigma}_{\sigma(1)}(u) & \text{if } k = 1 \\ u(\sigma(k), S^{\sigma}_{k}) - L^{\sigma}_{\sigma(k)}(u) + G^{\sigma}_{\sigma(k)}(u) & \text{if } k \in \{2, ..., |N| - 1\} \\ u(\sigma(|N|), N) - L^{\sigma}_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

Similar to the consensus value, here one can check that for a primeval game $u \in PRI^N$ and an order $\sigma \in \Pi(N)$,

$$\sum_{k=1}^{|N|} B^{\sigma}_{\sigma(k)}(u) = \sum_{k=1}^{|N|} u(\sigma(k), S^{\sigma}_{|N|}),$$

but generally,

$$\sum_{k=1}^{t} B^{\sigma}_{\sigma(k)}(u) \neq \sum_{k=1}^{t} u(\sigma(k), S^{\sigma}_{t})$$

for all $t \in \{1, ..., |N| - 1\}$.

The primeval value $\zeta(u)$ is defined as the average of the primeval vectors, i.e.

$$\zeta(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} B^{\sigma}(u).$$

Example 5.4.4 *Consider the three-household village example.*

With $\sigma = (b\ a\ c)$, *we get*

$$\begin{aligned} B^{\sigma}_{b}(u) &= u(b, \{b\}) = 3, \\ B^{\sigma}_{a}(u) &= u(a, \{b, a\}) - (u(b, \{b\}) - u(b, \{b, a\}) = 8 - (3 - 2) = 7, \\ B^{\sigma}_{c}(u) &= u(c, \{b, a, c\}) = 2. \end{aligned}$$

Similarly, all primeval vectors are given by

σ	$B_a^\sigma(u)$	$B_b^\sigma(u)$	$B_c^\sigma(u)$
$(a\ b\ c)$	8	2	2
$(a\ c\ b)$	8	2	2
$(b\ a\ c)$	7	3	2
$(b\ c\ a)$	7	3	2
$(c\ a\ b)$	7	2	3
$(c\ b\ a)$	7	1	4

Then, we get $\zeta(u) = \left(7\frac{1}{3}, 2\frac{1}{6}, 2\frac{1}{2}\right)$. *Thus, in respect of compensation for externalities, a needs to pay* $\frac{1}{6}$ *to b and* $\frac{1}{2}$ *to c. Note that in this case c even becomes a compensation receiver instead of a provider like in the previous two cases. This is due to the rule that positive externalities are free.*

5.5 Unanimity games

This section introduces unanimity games for the class of primeval games as a generalization of unanimity games for the class of TU games. A decomposition theorem is established, which states that every primeval game can be written in a unique way as a linear combination of unanimity games.

Recall that the unanimity games $\{(N, u_T)|T \in 2^N\backslash\{\emptyset\}\}$, where for each $T \in 2^N\backslash\{\emptyset\}$ the unanimity game $u_T \in TU^N$ is defined by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subset S \\ 0, & \text{otherwise} \end{cases}$$

for all $S \in 2^N$, form a basis for the class of all TU games with player set N. Below we will define *unanimity games* for primeval games.

Definition 5.5.1 *Let* $(j, T) \in \mathcal{E}(N)$ *be an embedded player. The* unanimity game $w_{(j,T)}$, *corresponding to* (j, T), *is given by*

$$w_{(j,T)}(i, S) = \begin{cases} 1, & \textit{if } j = i \textit{ and } T \subset S \\ 0, & \textit{otherwise} \end{cases}$$

for every $(i, S) \in \mathcal{E}(N)$.

Example 5.5.2 *Consider the three-household village game and denote the player set with* $N = \{1, 2, 3\}$ *instead of* $\{a, b, c\}$*. The following table gives the values of* $w_{(j,T)}(i, S)$ *for all embedded players* (j, T) *and* (i, S)*.*

For saving spaces, we use the following notations. $\tau_1 = (1, \{1\})$, $\tau_2 = (2, \{2\})$, $\tau_3 = (3, \{3\})$, $\tau_4 = (1, \{1, 2\})$, $\tau_5 = (2, \{1, 2\})$, $\tau_6 = (1, \{1, 3\})$, $\tau_7 = (3, \{1, 3\})$, $\tau_8 = (2, \{2, 3\})$, $\tau_9 = (3, \{2, 3\})$, $\tau_{10} = (1, \{1, 2, 3\})$, $\tau_{11} = (2, \{1, 2, 3\})$, $\tau_{12} = (3, \{1, 2, 3\})$.

$(i, S)\backslash(j, T)$	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}	τ_{11}	τ_{12}
τ_1	1	0	0	0	0	0	0	0	0	0	0	0
τ_2	0	1	0	0	0	0	0	0	0	0	0	0
τ_3	0	0	1	0	0	0	0	0	0	0	0	0
τ_4	1	0	0	1	0	0	0	0	0	0	0	0
τ_5	0	1	0	0	1	0	0	0	0	0	0	0
τ_6	1	0	0	0	0	1	0	0	0	0	0	0
τ_7	0	0	1	0	0	0	1	0	0	0	0	0
τ_8	0	1	0	0	0	0	0	1	0	0	0	0
τ_9	0	0	1	0	0	0	0	0	1	0	0	0
τ_{10}	1	0	0	1	0	1	0	0	0	1	0	0
τ_{11}	0	1	0	0	1	0	0	1	0	0	1	0
τ_{12}	0	0	1	0	0	0	1	0	1	0	0	1

We can prove, similar to the case of TU games, that the unanimity games form a basis for the class of primeval games.

Lemma 5.5.3 *If* (N, u) *is a primeval game, then there exist uniquely determined real numbers* $d_{(j,T)}$, $(j, T) \in \mathcal{E}(N)$*, such that*

$$u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}.$$

These numbers are given by $d_{(j,T)} = \sum_{(j',T'):j'=j;T'\subset T} (-1)^{|T|-|T'|} u(j', T')$.

Proof. It suffices to show for the $d_{(j,T)}$, specified in the lemma, that $u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}$.

Let $(i,S) \in \mathcal{E}(N)$, we have

$$\begin{aligned}
\sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}(i,S) &= \sum_{(j,T):j=i;T\subset S} d_{(j,T)} \\
&= \sum_{(j,T):j=i;T\subset S} \left(\sum_{(j',T'):j'=j;T'\subset T} (-1)^{|T|-|T'|} u(j',T') \right) \\
&= \sum_{(j',T'):j'=i;T'\subset S} \left(\sum_{(j,T):j'=j=i;T'\subset T\subset S} (-1)^{|T|-|T'|} \right) u(j',T').
\end{aligned}$$

As we know,

$$\begin{aligned}
\sum_{(j,T):j'=j=i;T'\subset T\subset S} (-1)^{|T|-|T'|} &= \sum_{k=0}^{|S|-|T'|} (-1)^k \binom{|S|-|T'|}{k} \\
&= \begin{cases} 1, & \text{if } S = T' \\ 0, & \text{if } S \neq T' \end{cases}
\end{aligned}$$

Therefore, we conclude that $\sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)}(i,S) = u(i,S)$ for all $(i,S) \in \mathcal{E}(N)$, which finishes the proof. ∎

The following example shows the linear expansion of a primeval game (N,u) with respect to the unanimity games $w_{(j,T)}$.

Example 5.5.4 *Consider the primeval game (N,u) in the three-household village example. Calculating the numbers $d_{(j,T)}$, we have*

$$\begin{aligned}
&d_{(1,\{1\})} = 5,\ d_{(2,\{2\})} = 3,\ d_{(3,\{3\})} = 2,\ d_{(1,\{1,2\})} = 3,\ d_{(2,\{1,2\})} = -1, \\
&d_{(1,\{1,3\})} = 0,\ d_{(3,\{1,3\})} = -1,\ d_{(2,\{2,3\})} = 0,\ d_{(3,\{2,3\})} = -2, \\
&d_{(1,\{1,2,3\})} = 0,\ d_{(2,\{1,2,3\})} = 0,\ d_{(3,\{1,2,3\})} = 3.
\end{aligned}$$

Thus, the decomposition of u is given by

$$\begin{aligned}
u = \quad & 5w_{(1,\{1\})} + 3w_{(2,\{2\})} + 2w_{(3,\{3\})} + 3w_{(1,\{1,2\})} - w_{(2,\{1,2\})} - w_{(3,\{1,3\})} \\
- \quad & 2w_{(3,\{2,3\})} + 3w_{(3,\{1,2,3\})}.
\end{aligned}$$

5.6 Properties and characterizations

This section discusses the possible properties of a solution concept for primeval games, which are derived from the generally accepted beliefs or specific guide lines applied in practice. We then provide characterizations using those properties.

As the status quo of a primeval game always exists (in fact, it is the only real situation that may happen) and we assume that the players have no revenues from outside of the game, immediately, we have the balanced-budget property for a solution concept that the sum of all the players' values according to the solution concept can not be greater than the sum of their status quo payoffs, and should not be less than that as well (because there is no channel to let the internal payoffs leak outside the game). For terminological consistency (in the same line as the property for TU games, etc.) here we still call it the efficiency property.

- Property 1 *(Efficiency)*: $\sum_{i \in N} f_i(u) = \sum_{i \in N} u(i, N)$ for all $u \in PRI^N$.

A second property is *symmetry.* For a primeval game (N, u), we say that two players $i, j \in N$ are *symmetric* if for all $S \subset N \backslash \{i, j\}$,

$$u(i, S \cup \{i\}) + \sum_{k \in S} u(k, S \cup \{i\}) = u(j, S \cup \{j\}) + \sum_{k \in S} u(k, S \cup \{j\}).$$

It implies that in terms of total payoffs, the showing up of i has the same effect as that of j for any group of players without i and j.

- Property 2 *(Symmetry)*: $f_i(u) = f_j(u)$ for all $u \in PRI^N$, and for all symmetric players i, j in (N, u).

We now turn to a third property, which focuses on the externality side of a primeval game.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *immune player* if $u(i, S) = u(i, \{i\})$ for all $S \subset N$ and $i \in S$. Thus, an immune player is a player who is not affected by the presence of the others.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *uninfluential player* if $u(j, S \cup \{i\}) = u(j, S)$ for all $S \subset N \backslash \{i\}$ and $j \in S$. Thus, an uninfluential player is a player who never affects the others.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *neutral player* if it is both an immune player and an uninfluential player in u.

Here, one can imagine that the third property requires that a neutral player in a game should get his R-C payoff.

- Property 3 *(The neutral player property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any neutral player i in (N, u).

For any primeval game (N, u), a player $i \in N$ is called a *dummy* if

$$\sum_{j \in S} u(j, S \cup \{i\}) + u(i, S \cup \{i\}) = \sum_{j \in S} u(j, S) + u(i, \{i\})$$

for all $S \subset N \backslash \{i\}$.

- Property 4 *(The dummy property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any dummy player i in (N, u).

We now introduce the following property.

- Property 5 *(Additivity)*: $f(u_1 + u_2) = f(u_1) + f(u_2)$ for all $u_1, u_2 \in PRI^N$, where $u_1 + u_2$ is defined by $(u_1 + u_2)(i, S) = u_1(i, S) + u_2(i, S)$ for every $(i, S) \in \mathcal{E}(N)$.

Theorem 5.6.1 *The Shapley value satisfies efficiency, symmetry, the neutral player property, the dummy property and additivity.*

Proof.
(i) Efficiency: Clearly, by construction, $m^\sigma(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Symmetry: Let i_1, i_2 be two symmetric players in $u \in PRI^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i_1$, $\sigma(h) = i_2$, where $i_1, i_2 \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from σ by interchanging the positions of i_1 and i_2, i.e.

$$\bar{\sigma}(w) = \begin{cases} \sigma(w) & \text{if } w \neq k, h \\ i_1 & \text{if } w = h \\ i_2 & \text{if } w = k. \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $m^\sigma_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
Case 1: $1 < k < h$.
By definition, we know

$$\begin{aligned} m^\sigma_{i_1}(u) = m^\sigma_{\sigma(k)}(u) &= \sum_{l=1}^{k} u(\sigma(l), S^\sigma_k) - \sum_{j=1}^{k-1} u(\sigma(j), S^\sigma_{k-1}) \\ m^{\bar{\sigma}}_{i_2}(u) = m^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) &= \sum_{l=1}^{k} u(\bar{\sigma}(l), S^{\bar{\sigma}}_k) - \sum_{j=1}^{k-1} u(\bar{\sigma}(j), S^{\bar{\sigma}}_{k-1}). \end{aligned}$$

Obviously, $u(\sigma(j), S^{\sigma}_{k-1}) = u(\bar{\sigma}(j), S^{\bar{\sigma}}_{k-1})$ for all $j \in \{1, ..., k-1\}$. Moreover, by symmetry, $\sum_{l=1}^{k} u(\sigma(l), S^{\sigma}_{k}) = \sum_{l=1}^{k} u(\bar{\sigma}(l), S^{\bar{\sigma}}_{k})$. Therefore, $m^{\sigma}_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
Case 2: $1 < h < k$. The proof is analogous to the above.
Case 3: $1 = k < h$. Apparently,

$$m^{\sigma}_{i_1}(u) = m^{\sigma}_{\sigma(1)}(u) = u(i_1, \{i_1\}) = u(i_2, \{i_2\}) = m^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = m^{\bar{\sigma}}_{i_2}(u).$$

Case 4: $1 = h < k$. Analogously, the proof is easy to be established.
As a consequence, $m^{\sigma}_{i_1}(u) = m^{\bar{\sigma}}_{i_2}(u)$.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $m^{\sigma}_{i}(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The dummy property: Obvious.
(v) Additivity: It follows from the fact that $m^{\sigma}_{\sigma(k)}(u_1 + u_2) = m^{\sigma}_{\sigma(k)}(u_1) + m^{\sigma}_{\sigma(k)}(u_2)$ for all $u_1, u_2 \in PRI^N$ and for all $k \in \{1, 2, ..., |N|\}$. ■

However, the consensus value and the primeval value do not satisfy the dummy property.

Example 5.6.2 *Here, the three-household village game is manipulated into two new primeval games (N, u_1) and (N, u_2) with $N = \{a, b, c\}$ such that player c is a dummy in game u_1 and is a null player*[2] *in game u_2.*

	(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
u_1	(5)	(3)	(2)	$(8,2)$	$(3,4)$	$(4,1)$	$(6,0,6)$
u_2	(5)	(3)	(0)	$(8,2)$	$(3,2)$	$(2,1)$	$(6,0,4)$

The solutions for the above two games are given as follows.

$\Phi(u_1)$	$=$	$(6,$	$4,$	$2)$	$\Phi(u_2)$	$=$	$(6,$	$4,$	$0)$
$\mathcal{C}(u_1)$	$=$	$(6,$	$2,$	$4)$	$\mathcal{C}(u_2)$	$=$	$(6,$	$2,$	$2)$
$\zeta(u_1)$	$=$	$(5\frac{1}{2},$	$2,$	$4\frac{1}{2})$	$\zeta(u_2)$	$=$	$(6\frac{1}{6},$	$1\frac{2}{3},$	$2\frac{1}{6})$

Theorem 5.6.3 *There is a unique solution on PRI^N satisfying efficiency, symmetry, the dummy property and additivity. This solution is the Shapley value.*

[2] *A player is a null player if it is a dummy player and has zero R-C payoff.*

Proof. From Theorem 5.6.1, it follows that the Shapley value Φ satisfies efficiency, symmetry, the dummy property and additivity.

Conversely, suppose that a solution concept f satisfies these four properties. We have to show that $f = \Phi$. Let u be a primeval game on N. Then,

$$u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)} \text{ with } d_{(j,T)} = \sum_{(j',T'):j'=j;T'\subset T} (-1)^{|T|-|T'|} u(j',T').$$

By the additivity property,

$$f(u) = \sum_{(j,T)\in\mathcal{E}(N)} f(d_{(j,T)} w_{(j,T)}) \text{ and } \Phi(u) = \sum_{(j,T)\in\mathcal{E}(N)} \Phi(d_{(j,T)} w_{(j,T)}).$$

Thus, it suffices to show that for all $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$ we have

$$f(d_{(j,T)} w_{(j,T)}) = \Phi(d_{(j,T)} w_{(j,T)}).$$

Let $(j,T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$. For any $i \notin T$, one readily verifies that i is a dummy player of game $(N, d_{(j,T)} w_{(j,T)})$. Therefore, by the dummy property,

$$f_i(d_{(j,T)} w_{(j,T)}) = \Phi_i(d_{(j,T)} w_{(j,T)}) = 0 \text{ for all } i \notin T. \tag{5.1}$$

Then, for any two players $i, k \in T$, we can easily see that i and k are symmetric player in $(N, d_{(j,T)} w_{(j,T)})$. By symmetry,

$$f_i(d_{(j,T)} w_{(j,T)}) = f_k(d_{(j,T)} w_{(j,T)}) \text{ for all } i, k \in T; \tag{5.2}$$

and similarly,

$$\Phi_i(d_{(j,T)} w_{(j,T)}) = \Phi_k(d_{(j,T)} w_{(j,T)}) \text{ for all } i, k \in T. \tag{5.3}$$

Therefore, efficiency and (5.1)-(5.3) imply that

$$f_i(d_{(j,T)} w_{(j,T)}) = \Phi_i(d_{(j,T)} w_{(j,T)}) = \frac{1}{|T|} d_{(j,T)} \text{ for all } i \in T.$$

■

Consider the dummy property which takes a marginal contribution perspective and assigns a dummy player his R-C payoff. As we know, without taking compensation into account, a dummy player i will get his status quo payoff in game u, i.e., $u(i,N)$. As $u(i,\{i\})$ and $u(i,N)$ represent two polar opinions, one may argue that taking the average could be a fair compromise. Therefore, the so-called quasi dummy property makes sense.

- Property 6 *(The quasi dummy property)*: $f_i(u) = \frac{u(i,\{i\})+u(i,N)}{2}$, for all $u \in PRI^N$ and for any dummy player i in (N, u).

Now we introduce the property of *adjusted symmetry.* Similar to the quasi dummy property, one may have the following argument. The property of symmetry requires the same value for symmetric players in a game, which can be denoted as $\alpha(u)$. Since symmetric players are not the same, their differences should be taken into account. An immediate and easy way to deal with this problem is to adjust the values by their status quo payoffs.

- Property 7 *(Adjusted symmetry)*: For any primeval game (N, u), there is an $\alpha(u) \in \mathbb{R}$ such that

$$f_i(u) = \frac{\alpha(u) + u(i,N)}{2} \text{ and } f_j(u) = \frac{\alpha(u) + u(j,N)}{2}$$

for all symmetric players $i, j \in N$, where $\alpha(u)$ is called the standard value for symmetric players in (N, u).

Theorem 5.6.4 *The consensus value satisfies efficiency, adjusted symmetry, the neutral player property, the quasi dummy property and additivity.*

Proof.
(i) Efficiency: Clearly, by construction, $C^\sigma(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Adjusted symmetry: By Theorem 5.4.3, the proof is readily established.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $C_i^\sigma(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The quasi dummy property: Given a game $u \in PRI^N$ and $\sigma \in \Pi(N)$, let player i be a dummy player in u and $i = \sigma(k)$. By definition, one can readily check that for all $k \in \{2, ..., |N|\}$,

$$\begin{aligned}
\mathcal{P}^\sigma_{\sigma(k)}(u) &= \frac{1}{2}\sum_{j=1}^{k-1}\left(u(\sigma(j), S_k^\sigma) - u(\sigma(j), S_{k-1}^\sigma)\right) \\
&= \frac{1}{2}\left(u(i,\{i\}) - u(i, S_k^\sigma)\right).
\end{aligned}$$

Then, by the definition of the concession vector, we know

$$C^\sigma_{\sigma(k)}(u) = \frac{u(i,\{i\}) + u(i,N)}{2}$$

for all $k \in \{1, 2, ..., |N|\}$.
Hence, what remains is obvious.
(v) Additivity: It is immediate, by definition, to see that $C^{\sigma}_{\sigma(k)}(u_1 + u_2) = C^{\sigma}_{\sigma(k)}(u_1) + C^{\sigma}_{\sigma(k)}(u_2)$ for all $u_1, u_2 \in PRI^N$ and for all $k \in \{1, 2, ..., |N|\}$. ■

Theorem 5.6.5 *There is a unique solution on PRI^N satisfying efficiency, adjusted symmetry, the quasi dummy property and additivity. This solution is the consensus value.*

Proof. From Theorem 5.6.4, it follows that the consensus value $\mathcal{C}$ satisfies efficiency, adjusted symmetry, the quasi dummy property and additivity.

Conversely, suppose a solution concept f satisfies these four properties. We have to show that $f = \mathcal{C}$. Let u be a primeval game on N. Then,

$$u = \sum_{(j,T)\in\mathcal{E}(N)} d_{(j,T)} w_{(j,T)} \text{ with } d_{(j,T)} = \sum_{(j',T'):j'=j;T'\subset T} (-1)^{|T|-|T'|} u(j', T').$$

By the additivity property,

$$f(u) = \sum_{(j,T)\in\mathcal{E}(N)} f(d_{(j,T)} w_{(j,T)}) \text{ and } \mathcal{C}(u) = \sum_{(j,T)\in\mathcal{E}(N)} \mathcal{C}(d_{(j,T)} w_{(j,T)}).$$

Thus, it suffices to show that for all $(j, T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$ we have

$$f(d_{(j,T)} w_{(j,T)}) = \mathcal{C}(d_{(j,T)} w_{(j,T)}).$$

Let $(j, T) \in \mathcal{E}(N)$ and $d_{(j,T)} \in \mathbb{R}$. For any $i \notin T$, one readily verifies that i is a dummy player of game $(N, d_{(j,T)} w_{(j,T)})$. Therefore, by the quasi dummy property,

$$f_i(d_{(j,T)} w_{(j,T)}) = \mathcal{C}_i(d_{(j,T)} w_{(j,T)}) = 0 \text{ for all } i \notin T. \tag{5.4}$$

Moreover, we know that all players in group T are symmetric players in $(N, d_{(j,T)} w_{(j,T)})$. By adjusted symmetry,

$$f_i(d_{(j,T)} w_{(j,T)}) = \frac{\alpha_f}{2} \text{ for all } i \in T\backslash\{j\} \text{ and some } \alpha_f \in \mathbb{R}, \tag{5.5}$$

and

$$\mathcal{C}_i(d_{(j,T)} w_{(j,T)}) = \frac{\alpha_{\mathcal{C}}}{2} \text{ for all } i \in T\backslash\{j\} \text{ and some } \alpha_{\mathcal{C}} \in \mathbb{R}. \tag{5.6}$$

And for player j, by adjusted symmetry as well, we have

$$f_j(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_f + d_{(j,T)}}{2} \text{ and } \mathcal{C}_j(d_{(j,T)}w_{(j,T)}) = \frac{\alpha_\mathcal{C} + d_{(j,T)}}{2}. \tag{5.7}$$

Thus, efficiency and (5.4)-(5.7) imply that

$$\alpha_f = \alpha_\mathcal{C} = \frac{1}{|T|}d_{(j,T)}.$$

■

Before introducing the next property, we first define completely symmetric players. Given a primeval game (N, u), we say that two players $i, j \in N$ are *completely symmetric* if for all $S \subset N \backslash \{i, j\}$,

$$u(i, S \cup \{i\}) = u(j, S \cup \{j\}) \text{ and } u(i, S \cup \{j\} \cup \{i\}) = u(j, S \cup \{j\} \cup \{i\})$$

and for all $k \in S$

$$u(k, S \cup \{i\}) = u(k, S \cup \{j\}).$$

It is natural to require that two complete symmetric players get the same value in a primeval game as their emergences generate the same influence to other players while getting the same influence from the emergences of the others.

- Property 8 *(Complete symmetry)*: $f_i(u) = f_j(u)$ for all $u \in PRI^N$, and for all completely symmetric players $i, j \in N$.

Apparently, the Shapley value and the consensus value satisfy complete symmetry.

Now we discuss another property which pays more attention to the compensation aspect and therefore seems important in the context of primeval games.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *harmful player* if $u(j, S \cup \{i\}) < u(j, S)$ for all $S \subset N \backslash \{i\}$ and $j \in S$. Thus, a harmful player is a player who always generates negative externalities to others.

Given a game $u \in PRI^N$, a player $i \in N$ is called a *harmless player* if $u(j, S \cup \{i\}) \geq u(j, S)$ for all $S \subset N \backslash \{i\}$ and $j \in S$. Thus, a harmless player is a player who never produces negative externalities to other players.

Given a game $u \in PRI^N$, a player $i \in N$ is called an *immune-harmful player* if it is both an immune player and a harmful player in u; or is called an *immune-harmless player* if it is both an immune player and a harmless player in u.

- Property 9 *(The immune-harmless player property)*: $f_i(u) = u(i, \{i\})$, for all $u \in PRI^N$ and for any immune-harmless player i in (N, u).

Theorem 5.6.6 *The primeval value satisfies efficiency, complete symmetry, the neutral player property and the immune-harmless player property.*

Proof.
(i) Efficiency: Clearly, by construction, $B^\sigma(u)$ is efficient for all $\sigma \in \Pi(N)$.
(ii) Complete symmetry: Let i_1, i_2 be two completely symmetric players in $u \in PRI^N$. Consider $\sigma \in \Pi(N)$, and without loss of generality, $\sigma(k) = i_1$, $\sigma(h) = i_2$, where $i_1, i_2 \in N$. Let $\bar{\sigma} \in \Pi(N)$ be the permutation which is obtained from σ by interchanging the positions of i_1 and i_2, i.e.

$$\bar{\sigma}(w) = \begin{cases} \sigma(w) & \text{if } w \neq k, h \\ i_1 & \text{if } w = h \\ i_2 & \text{if } w = k. \end{cases}$$

As $\sigma \mapsto \bar{\sigma}$ is bijective, it suffices to prove that $B^\sigma_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.
Case 1: $1 < k < h$.
By definition, we know

$$\begin{aligned} B^\sigma_{i_1}(u) = B^\sigma_{\sigma(k)}(u) &= u(\sigma(k), S^\sigma_k) - L^\sigma_{\sigma(k)}(u) + G^\sigma_{\sigma(k)}(u) \\ B^{\bar{\sigma}}_{i_2}(u) = B^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) &= u(\bar{\sigma}(k), S^{\bar{\sigma}}_k) - L^{\bar{\sigma}}_{\bar{\sigma}(k)}(u) + G^{\bar{\sigma}}_{\bar{\sigma}(k)}(u). \end{aligned}$$

Obviously, $u(\sigma(k), S^\sigma_k) = u(\bar{\sigma}(k), S^{\bar{\sigma}}_k)$. Moreover, since i_1, i_2 are completely symmetric players, $L^\sigma_{\sigma(k)}(u) = L^{\bar{\sigma}}_{\bar{\sigma}(k)}(u)$ and $G^\sigma_{\sigma(k)}(u) = G^{\bar{\sigma}}_{\bar{\sigma}(k)}(u)$. Therefore, $B^\sigma_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.
Case 2: $1 < h < k$. The proof is analogous to the above.
Case 3: $1 = k < h$. Apparently,

$$B^\sigma_{i_1}(u) = u(\sigma(1), \{\sigma(1)\}) + G^\sigma_{\sigma(1)}(u) = u(\bar{\sigma}(1), \{\bar{\sigma}(1)\}) + G^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = B^{\bar{\sigma}}_{\bar{\sigma}(1)}(u) = B^{\bar{\sigma}}_{i_2}(u).$$

Case 4: $1 = h < k$. Analogously, the proof is easy to be established.
As a consequence, $B^\sigma_{i_1}(u) = B^{\bar{\sigma}}_{i_2}(u)$.
(iii) The neutral player property: If player i is a neutral player in (N, u) then $B^\sigma_i(u) = u(i, \{i\})$ for any $\sigma \in \Pi(N)$.
(iv) The immune-harmless player property: Obvious. ■

Note that the primeval value does not satisfy symmetry or additivity, as an example of violating these two properties is easy to be found.

Example 5.6.7 *Consider the following two games.*

	(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
u_1	(1)	(1)	(5)	$(2,3)$	$(2,4)$	$(0,6)$	$(3,4,2)$
u_2	(6)	(4)	(7)	$(10,5)$	$(7,6)$	$(3,6)$	$(11,6,4)$

In game u_1, a and b are symmetric players. The Shapley value of u_1 is $\Phi(u_1) = \left(2\frac{1}{6}, 2\frac{1}{6}, 4\frac{2}{3}\right)$. However, $\zeta(u_1) = \left(1\frac{1}{2}, 3\frac{1}{2}, 4\right)$. So players a and b have different primeval values.

Game u_2 is obtained by adding u_1 to the three-household village game. The primeval value is $\zeta(u_2) = \left(10\frac{1}{6}, 5\frac{1}{3}, 5\frac{1}{2}\right)$, which is not equal to the sum of the primeval values of u_1 and the three-household village game.

Comparing the primeval value with the other two solution concepts, one can find that the primeval value indeed fits well in the framework of primeval games.

We first consider the following corollary which discusses the gains of an uninfluential player according to the primeval value and the Shapley value. The result is consistent with our intuition: As an uninfluential player, he need not compensate the others while he could benefit from the positive externalities from the others. So, the primeval value of an uninfluential player is always no less than its Shapley value in a primeval game.

Corollary 5.6.8 *For any game $u \in PRI^N$ and any uninfluential player $i \in N$, it holds that*

$$\zeta_i(u) \geq \Phi_i(u) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} u(i, S \cup \{i\}).$$

Proof. Given a game $u \in PRI^N$ and let $i \in N$ be an uninfluential player. Given $\sigma \in \Pi(N)$, let $i = \sigma(k)$, it suffices to show that $B_i^\sigma(u) \geq m_i^\sigma(u)$. As we know

$$B_i^\sigma(u) = B_{\sigma(k)}^\sigma(u) = \begin{cases} u(i, \{i\}) + G_{\sigma(1)}^\sigma(u) & \text{if } k = 1 \\ u(i, S_k^\sigma) + G_{\sigma(k)}^\sigma(u) & \text{if } k \in \{2, ..., |N|-1\} \\ u(i, N) & \text{if } k = |N| \end{cases}$$

and

$$m_i^\sigma(u) = m_{\sigma(k)}^\sigma(u) = u(i, S_k^\sigma).$$

So, $B_i^\sigma(u) \geq m_i^\sigma(u)$. ■

For an immune-harmful player, since he could not get any positive externality but always does harm to others, he then must pay compensation to the others. So, the primeval value is equivalent to the Shapley value.

Corollary 5.6.9 *For any game $u \in PRI^N$ and any immune-harmful player $i \in N$, it holds that*

$$\Phi_i(u) = \zeta_i(u) < \mathcal{C}_i(u) < u(i, \{i\}).$$

Proof. Given $\sigma \in \Pi(N)$, let $i = \sigma(k)$ for $k \in \{1, 2, ..., |N|\}$.

First, in order to prove $\Phi_i(u) = \zeta_i(u)$, it suffices to show that $m_i^\sigma(u) = B_i^\sigma(u)$.

Apparently, when $k = 1$, $m_i^\sigma(u) = B_i^\sigma(u) = u(i, \{i\})$. When $k \in \{2, ..., |N|\}$, we get

$$\begin{aligned} m_i^\sigma(u) &= \sum_{l=1}^{k} u(\sigma(l), S_k^\sigma) - \sum_{j=1}^{k-1} u(\sigma(j), S_{k-1}^\sigma) \\ &= u(i, S_k^\sigma) + \sum_{j=1}^{k-1} u(\sigma(j), S_k^\sigma) - \sum_{j=1}^{k-1} u(\sigma(j), S_{k-1}^\sigma) \\ &= u(i, S_k^\sigma) - \sum_{j=1}^{k-1} \big(u(\sigma(j), S_{k-1}^\sigma) - u(\sigma(j), S_k^\sigma)\big) \\ &= B_i^\sigma(u). \end{aligned}$$

Moreover, since $\sum_{j=1}^{k-1} \big(u(\sigma(j), S_{k-1}^\sigma) - u(\sigma(j), S_k^\sigma)\big) > 0$, we know $m_i^\sigma(u) = B_i^\sigma(u) < u(i, \{i\})$ for all $k \in \{2, ..., |N|\}$. Then, $\Phi_i(u) = \zeta_i(u) < u(i, \{i\})$.

By Theorem 5.4.3, we have

$$\begin{aligned} \mathcal{C}_i^\sigma(u) &= \frac{1}{2}u(i, N) + \frac{1}{2}\Phi_i(u) \\ &= \frac{1}{2}u(i, \{i\}) + \frac{1}{2}\Phi_i(u) \\ &> \Phi_i(u). \end{aligned}$$

■

Corollary 5.6.10 *For any game $u \in PRI^N$ and any immune-harmless player $i \in N$, it holds that*

$$\Phi_i(u) \geq \mathcal{C}_i(u) \geq \zeta_i(u) = u(i, \{i\}).$$

Proof. By definition and analogous to Corollary 5.6.9, the proof is easy to be established. ■

This corollary further describes the appropriateness of the primeval value for solving the compensation problems in primeval games. Generally, we would expect that an immune-harmless player obtains his R-C payoff: He does not do anything harmful to others and then has nothing to do with compensating them. Meanwhile, he need not any compensation because nobody could affect him and his payoff remains the R-C payoff in all cases. The primeval value is consistent with this belief while the other two solution concepts may give extra payoff to such a player as they take a different perspective such that the positive externalities are not free.

Corollary 5.6.11 *For any game $u \in PRI^N$ and any harmless player $i \in N$ with $u(i, N) \geq u(i, \{i\})$, it holds that*

$$\zeta_i(u) \geq u(i, \{i\}).$$

Proof. For a primeval game $u \in PRI^N$, let i be a harmless player in u. For an ordering $\sigma \in \Pi(N)$, let $i = \sigma(k)$, $k \in \{1, ..., |N|\}$. By definition and since $u(i, N) \geq u(i, \{i\})$, we know $G_i^\sigma(u) \geq 0$ if $k = 1$; $L_i^\sigma(u) = 0$ for all $k \in \{2, ..., |N|\}$; and

$$u(i, S_k^\sigma) + G_i^\sigma(u) \geq u(i, N) \geq u(i, \{i\})$$

for all $k \in \{2, ..., |N| - 1\}$. Hence, $B_i^\sigma(u) \geq u(i, \{i\})$. ■

Note that Corollary 5.6.11 can be understood as the property of individual rationality for harmless players: If a player's presence never does harm to others and his status quo payoff is greater than his R-C payoff, he should at least get his R-C payoff.

However, the Shapley value and the consensus value do not satisfy this property.

Example 5.6.12 *Consider the following game u with three players, a, b and c.*

(a)	(b)	(c)	(a, b)	(a, c)	(b, c)	(a, b, c)
(3)	(1)	(5)	$(0, 1)$	$(0, 5)$	$(0, 6)$	$(3, 1, 6)$

Here a is a harmless player. His Shapley value is $\Phi_a(u) = 2\frac{1}{3}$ and his consensus value is $\mathcal{C}_a(u) = 2\frac{2}{3}$. Both are less than a's R-C payoff of 3. However, his primeval value is $\zeta_a(u) = 4$.

5.7 A generalization of the consensus value

By relaxing the way of defining concession, we get a generalization of the consensus value: the *generalized consensus value*.

Let $u \in PRI^N$, we will construct the generalized concession vector $C^{\sigma}(u)$, which corresponds to the situation where players enter the game u one by one in an order $\sigma \in \Pi(N)$ and where every new entrant, say $\sigma(k)$, first obtains the payoff when entering, $u(\sigma(k), S_k^{\sigma})$, and then shares the surplus or loss with every incumbent according to a concession parameter $\theta \in [0,1]$, and also shares his surplus or loss with all his successors in a consistent way, i.e. according to $(1-\theta)$. That is, θ measures entrants' rights or responsibilities for the externalities.

We first define player $\sigma(k)$'s *generalized concession payoff for the externalities on previous players* as

$$(\mathcal{P}_{\theta}^{\sigma})_{\sigma(k)}(u) = \sum_{j=1}^{k-1} \theta \left(u(\sigma(j), S_k^{\sigma}) - u(\sigma(j), S_{k-1}^{\sigma}) \right)$$

and his *generalized concession payoff from the subsequent externalities* as

$$(\mathcal{S}_{\theta}^{\sigma})_{\sigma(k)}(u) = \sum_{l=k+1}^{|N|} (1-\theta) \left(u(\sigma(k), S_l^{\sigma}) - u(\sigma(k), S_{l-1}^{\sigma}) \right).$$

Apparently, when a player enters the game u in the very first place, he has no concession payoff for the externalities on previous players. Therefore $(\mathcal{P}_{\theta}^{\sigma})_{\sigma(1)}(u) = 0$. Correspondingly, when a player enters a game in the very last place, there is no subsequent externality for him. Hence, $(\mathcal{S}_{\theta}^{\sigma})_{\sigma(|N|)}(u) = 0$.

Moreover, the concession payoff from the subsequent externalities for player $\sigma(k)$ can be simplified as

$$(\mathcal{S}_{\theta}^{\sigma})_{\sigma(k)}(u) = (1-\theta) \left(u(\sigma(k), N) - u(\sigma(k), S_k^{\sigma}) \right)$$

for all $k = \{1, ..., |N|-1\}$.

Now, formally, the generalized concession vector is the vector in $\mathbb{R}^N$ defined by

$$(C_{\theta}^{\sigma})_{\sigma(k)}(u) = \begin{cases} u(\sigma(1), \{\sigma(1)\}) + (\mathcal{S}_{\theta}^{\sigma})_{\sigma(1)}(u) & \text{if } k = 1 \\ u(\sigma(k), S_k^{\sigma}) + (\mathcal{P}_{\theta}^{\sigma})_{\sigma(k)}(u) + (\mathcal{S}_{\theta}^{\sigma})_{\sigma(k)}(u) & \text{if } k = \{2, ..., |N|-1\} \\ u(\sigma(|N|), N) + (\mathcal{P}_{\theta}^{\sigma})_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

And more explicitly,

$$(C_\theta^\sigma)_{\sigma(k)}(u) = \begin{cases} \theta u(\sigma(1), \{\sigma(1)\}) + (1-\theta)u(\sigma(1), N) & \text{if } k = 1 \\ (\mathcal{P}_\theta^\sigma)_{\sigma(k)}(u) + \theta u(\sigma(k), S_{\sigma(k)}^\sigma) \\ +(1-\theta)u(\sigma(k), N) & \text{if } k = \{2, ..., |N|-1\} \\ u(\sigma(|N|), N) + (\mathcal{P}_\theta^\sigma)_{\sigma(|N|)}(u) & \text{if } k = |N|. \end{cases}$$

The generalized consensus value $\mathcal{C}_\theta(u)$ is defined as the average of the generalized concession vectors, i.e.

$$\mathcal{C}_\theta(u) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} C_\theta^\sigma(u).$$

Example 5.7.1 *Consider the three-household village example. Taking θ equal to $\frac{2}{3}$, one can readily check that $\mathcal{C}_{\frac{2}{3}}(u) = \left(7, 3\frac{1}{3}, 1\frac{2}{3}\right)$.*

For a one-point solution concept $f : PRI^N \longrightarrow \mathbb{R}^N$, defining the θ-dummy property by $f_i(u) = (i-\theta)u(i, N) + \theta u(i, \{i\})$, for all $u \in PRI^N$ and for any dummy player i in (N, u); and defining θ-symmetry by there exists an $\alpha \in \mathbb{R}$ such that

$$f_i(u) = \theta\alpha + (1-\theta)u(i, N) \text{ and } f_j(u) = \theta\alpha + (1-\theta)u(j, N)$$

for all $u \in PRI^N$, and for all symmetric players $i, j \in N$, one can readily obtain the following theorem.

Theorem 5.7.2 *For $\theta \in [0, 1]$:*

(a) There is a unique solution on PRI^N satisfying efficiency, θ-symmetry, the θ-dummy property and additivity. This solution is the generalized consensus value.

(b) The generalized concession value is a convex combination of the status quo payoff and the Shapley value. That is, for any game $u \in PRI^N$, it holds that

$$(\mathcal{C}_\theta)_i(u) = (1-\theta)u(i, N) + \theta\Phi_i(u)$$

for all $i \in N$.

5.8 From primeval games to cooperation

This section suggests a first attempt to combine primeval games with TU games. Here, a major issue is still about the sharing of the joint gains or compensation for externalities. We introduce the possible situations and the approaches to solve the associated externality compensation and surplus sharing problems. However, for a more formal and general analysis, we leave it for future work.

5.8.1 An example: cooperation in the three-household village

To illustrate the possible issues, we consider the three-household village example introduced in the second section, which becomes more complicated by taking cooperative behavior into account. Imagine that if the three households cooperate they obtain a joint payoff 20. Then, how to share it among the households a, b, and c?

We may have the following two cases regarding the realization of 20.

- Case 1: Cooperation after the primeval game

This case takes the current structure of the primeval game as a starting point. That is, a, b, and c, based on their individual values 8, 2, 2, respectively, plan to cooperate with each other. There could exist many types of cooperation such as coordination behavior with each other concerning the externalities, working together for a joint project, and so on. However, we are not interested in the specific types of cooperation but would focus on the sharing rules of the joint gains. Here, undoubtedly, the cooperation between two households still may generate externalities to the third one. Therefore, this case well relates to a partition function form game. Consider the following table as a numerical example for this case.

(a,b,c)	(ab,c)	(ac,b)	(bc,a)	(abc)
$(8,2,2)$	$(13,1)$	$(11,3)$	$(4,12)$	(20)

Figure 5.2: (Case 1) Partition function form cooperation after the primeval game

- Case 2: Cooperation "beyond" the primeval game

Another possible way to get the joint gain of 20 is that those households may go beyond the primeval game and directly arrive at a cooperating outcome. More specifically, for instance, in the order of (b,a,c), household b first enters the village and gets payoff/utility 3; then a follows and has two options: either directly cooperating with b and obtaining a joint payoff 12 or standing alone to play the primeval game so that a gets 8 and b gets 2; finally, c also has two options: either collaborating with b and a and obtaining the joint value 20 or standing alone so that c gets 1 and $\{b, a\}$ together (if they cooperate) get 13 or, if all of them stand alone, c gets 2 while b and a get 2 and 8, respectively. This situation corresponds to the following table.

a	b	c	a, b	a, c	b, c	a, b, c	ab	ac	bc	ab, c	ac, b	bc, a	abc
5	3	2	8, 2	5, 1	3, 0	8, 2, 2	12	8	5	13, 1	11, 3	4, 12	20

Figure 5.3: (Case 2) Cooperation beyond the primeval game

The obvious next question is how to solve the problems of surplus sharing or compensation for externalities in the presence of cooperation.

5.8.2 Two-stage approach versus combined approach

A two-stage approach is suitable for Case 1. That is, we first calculate the solution for the primeval game according to some solution concept; then, solve the associated partition function form game according to some solution concept; finally, sum up the two parts and subtract the status quo payoff (because it is calculated twice), which serves as the final solution.

This approach has the advantage that it clearly and logically distinguishes the compensation stage that models the individual externalities and the surplus sharing stage that models the cooperation and coalitional externalities, which is in the same spirit as in chapter 4.

To illustrate this approach, consider the consensus value and the Shapley value of the following example.

Example 5.8.1 *The consensus value γ of the partition function form game w which is described by Figure 5.2 can be calculated as follows.*

σ	s_a^σ	s_b^σ	s_c^σ
$(a\ b\ c)$	11	5	4
$(a\ c\ b)$	10	6	4
$(b\ a\ c)$	11	5	4
$(b\ c\ a)$	14	3	3
$(c\ a\ b)$	10	6	4
$(c\ b\ a)$	14	3	3

Then, we get the consensus value $\gamma(w) = \left(11\frac{2}{3}, 4\frac{2}{3}, 3\frac{2}{3}\right)$*, which solves the surplus sharing problem in the cooperation situation. As we know, the primeval value for the primeval game is* $\zeta(u) = \left(7\frac{1}{3}, 2\frac{1}{6}, 2\frac{1}{2}\right)$*, which solves the externality incurred compensation problem. Hence, taking both parts into account, the final solution for the whole situation is* $\left(11, 4\frac{5}{6}, 4\frac{1}{6}\right)$.

Another possible solution is by the Shapley value. Here, the Shapley value (by Pham Do and Norde (2002)) of this partition function form game is $\Phi(w) = \left(11\frac{1}{3}, 4\frac{5}{6}, 3\frac{5}{6}\right)$*. The Shapley value for the primeval game is* $\Phi(u) = \left(6\frac{1}{2}, 4, 1\frac{1}{2}\right)$*. Then, the final solution is* $\left(9\frac{5}{6}, 6\frac{5}{6}, 3\frac{1}{3}\right)$.

Due to the various possibilities in Case 2, one can imagine that there could also exist different ways dealing with the corresponding compensation and surplus sharing problem. Here, we focus on a combined approach which seems suitable for this case. The term of "combined" means that we take the case as one game[3] and apply some solution concept.

For example, Case 2 can be directly modelled as the following TU game so that we can calculate the Shapley value.

Example 5.8.2 *A TU game derived from Case 2.*

a	b	c	ab	ac	bc	abc
5	3	2	12	8	5	20

Then, one can get the Shapley value as $\left(9\frac{1}{6}, 6\frac{2}{3}, 4\frac{1}{6}\right)$.

A more complicated value is the consensus value, which is different from the one in TU games. Here, we take players' standing alone options as their reservation values and apply the consensus idea to derive the corresponding solution.

Example 5.8.3 *The consensus value*

Consider the order of (b, a, c)*.* b *first gets* 3*. When* a *enters,* b *will get* 2 *due to the negative externality. However,* a *can ensure himself with* 8*. If they cooperate with each other,* 12 *will be generated. So,* b *and* a *should get* 3 *and* 9*, respectively. Finally,* c *enters the scene. Without cooperation, coalition* $\{ab\}$ *can enjoy* 13 *while* c *gets* 1*. So,*

[3] Note that it is not a regular game which we defined previously. Here we do not give formal definition but only focus on the idea that may handle it.

they should share the joint surplus $6(= 20 - 13 - 1)$ *equally. Therefore,* c *gets* 4 *in the end;* b *and* a *equally share their surplus* 4 *again and finally get* 5 *and* 11*, respectively.*

σ	s_a^σ	s_b^σ	s_c^σ
$(a\ b\ c)$	11	5	4
$(a\ c\ b)$	9	6	5
$(b\ a\ c)$	11	5	4
$(b\ c\ a)$	14	$4\frac{1}{2}$	$1\frac{1}{2}$
$(c\ a\ b)$	9	6	5
$(c\ b\ a)$	14	$4\frac{1}{2}$	$1\frac{1}{2}$

Then, we get the consensus value $\left(10\frac{1}{3}, 5\frac{1}{6}, 3\frac{1}{2}\right)$.

5.8.3 An application: externality, efficiency, and compensation

Based on the above discussion, we now apply the approaches to solve the compensation problem in an economic situation characterized by externality and coordination for efficiency.

Suppose that three economic agents are located in an area, as depicted in Figure 5.1. Imagine that a is a polluting firm, b is a household who loves loud music, and c is a gardener who makes slot 3 full of beautiful flowers.

Agent a produces output x so as to maximize profit; agent b's utility depends on the volume of music y; c's utility or profit comes from the amount of flowers z he plants. We further assume that the activity of each agent imposes externalities on the others. In particular, their profit or utility functions are

$$\begin{aligned}\pi_{a,\{a,b,c\}} &= 6x - 0.5x^2 - 2y + z\\ \pi_{b,\{a,b,c\}} &= 20y - 5y^2 - 2x + 3z\\ \pi_{c,\{a,b,c\}} &= 10z - z^2 - x + 2y.\end{aligned}$$

Here, $\pi_{a,\{a,b,c\}}$ denotes agent a's utility in the situation where all agents are located in their slots without any collaboration. Now we can see that a's choice of output imposes negative externalities on b and c; b produces negative externality on a but positive externality on c; both a and b benefit from c's gardening activity. Agents would maximize their profits or utilities, which yields that $\pi^*_{a,\{a,b,c\}} = 19$, $\pi^*_{b,\{a,b,c\}} = 23$, and $\pi^*_{c,\{a,b,c\}} = 23$.

In order to attain efficiency for the whole society, those agents may coordinate their behavior. Then, they will have the following objective functions.

$$\begin{aligned}
\pi_{ab,\{ab,c\}} &= 4x - 0.5x^2 + 18y - 5y^2 + 4z \\
\pi_{c,\{ab,c\}} &= 10z - z^2 - x + 2y \\
\pi_{ac,\{ac,b\}} &= 5x - 0.5x^2 + 11z - z^2 \\
\pi_{b,\{ac,b\}} &= 20y - 5y^2 - 2x + 3z \\
\pi_{bc,\{bc,a\}} &= 22y - 5y^2 + 13z - z^2 - 3x \\
\pi_{a,\{bc,a\}} &= 6x - 0.5x^2 - 2y + z \\
\pi_{abc,\{abc\}} &= 3x - 0.5x^2 + 20y - 5y^2 + 14z - z^2.
\end{aligned}$$

In particular, $\pi^*_{abc,\{abc\}} = 73.5$.

The main problem involved in this situation is how to share the value 73.5 as the result of cooperation?

The two-stage approach takes compensation for externalities and surplus sharing for cooperation into consideration and provides an appropriate solution.

We first deduce those agents' profit or utility functions based on their stand-alone situations (i.e. only when one agent exists in that area):

$$\begin{aligned}
\pi_{a,\{a\}} &= 6x - 0.5x^2 \\
\pi_{b,\{b\}} &= 20y - 5y^2 \\
\pi_{c,\{c\}} &= 10z - z^2.
\end{aligned}$$

Similarly, we have the information about the other cases such as $\pi_{b,\{b,c\}}$ and $\pi_{c,\{b,c\}}$, i.e., when b and c appear while a is absent. Thus, we obtain the corresponding primeval game (N, u).

(a)	(b)	(c)	(a,b)	(a,c)	(b,c)	(a,b,c)
(18)	(20)	(25)	$(14,8)$	$(23,19)$	$(35,29)$	$(19,23,23)$

The Shapley value of this primeval game is $\Phi(u) = (9.5, 21.5, 34)$. The consensus value is $\mathcal{C}(u) = (14.25, 22.25, 28.5)$. The primeval value is $\zeta(u) = (12, 27, 26)$.

Then, we consider the partition function form game (N, w).

(a,b,c)	(ab,c)	(ac,b)	(bc,a)	(abc)
$(19,23,23)$	$(44.2,24.6)$	$(42.75,26.5)$	$(48.45,20.1)$	(73.5)

The Shapley value of w is $\Phi(w) = (21.51, 26.36, 25.63)$. The consensus value is $\gamma(w) = (21.43, 26.45, 25.62)$.

Finally, the two-stage approach may yield two solutions, one is $(12.01, 24.86, 36.63)$ by the Shapley value, and the other is $(14.43, 30.45, 28.62)$ by the primeval value of game (N, u) and the consensus value of game (N, w).

According to the combined approach, one can model the above situation as the following TU game (N, v).

a	b	c	ab	ac	bc	abc
18	20	25	24.2	42.75	66.45	73.5

One can readily check that the Shapley value yields $\Phi(v) = (12.01, 24.86, 36.63)$ and the consensus value is $\gamma(v) = (16.755, 24.18, 32.565)$. If applying the method illustrated in Example 5.8.3 to this case, we will get the consensus value $(24.43, 25.79, 23.28)$.

Chapter 6

Transition by experimentation: the gradualist reform in China's banking sector

"*Crossing the river by touching the stones.*"

Deng Xiaoping, Chinese leader

6.1 Introduction

The transition from plan to market in former planned economies is one of the main economic events of the 20th century. Not only does it affect the lives of approximately 1.65 billion people, but it is contributing to a shift in emphasis in economics from standard price and monetary theory to contracting and its institutional environment.

In the study of economic transition, one of the most important issues is on transition approaches or strategies. So far there are mainly two alternative transition approaches, regarding the speed and sequence of transition reforms, being implemented in practice: "shock therapy" (also termed the "big bang" or "radical approach") and "gradualism" (or "conservative approach"). For the comparison between those two approaches, we refer to Roland (2000). Despite the substantial studies on this issue, especially on the "big bang" approach, we observe that there is no concise, yet deep-going summary of the second approach: What are the key and fundamental features of gradualism oriented transition? Since China is a major transition economy adopting such an approach, this chapter[1] focuses on Chinese experience and aims to give a satisfying

[1]This chapter is based on Ju (2003).

answer. Moreover, instead of using the single word "gradualism" to name China's transition approach, we would call it the strategy of "transition by experimentation". As discussed later in the chapter, gradualism is only one element of the whole strategy.

Meanwhile, a practical role that the chapter can play is to provide the background (knowledge) for chapter 7 which analyses the compensation problems in China's transition period and proposes an alternative transition approach, namely, "transition by compensation".

China's economic reform towards a market-oriented economy, starting from 1978, adopted a distinguished gradualist approach and has been recognized as essentially successful. The average rate of growth of real GDP in the first two decades of reform were about 9.6 percent annually according to official statistics. To see how reform worked in China is not only necessary for understanding the reality of the world economy but is also helpful for researchers to make theoretical explorations, as pointed by Chow (1997), the gradualist reform and transition process in China issue ample challenges for economic research. However, it is almost impossible to comprehensively study and answer this "big" problem by writing one or two papers only. Therefore, this chapter does not aim to provide a full view about China's economic reform, but will focus on the banking sector to partially illustrate the measures of the reform and the process of the transition so as to show exactly how the reform goes along a unique way and reveal the ideas and features behind that.

Why is the banking sector chosen here? The main reason lies in its importance in the economy and the urgency of reforming. The bottom-up approach and the sequence of the economic reform—rural reform → enterprise/city reform → financial reform—resulted in a long-term transition period. More and more challenges, even difficulties have crowded into the road to further development since the mid-1990s: unemployment, deflation, stagnating in the reform of the stated-owned enterprises (SOE) and financial reform, widening wealth disparity, disparity in district developments, and so on. The banking sector, as the crucial component and late-reformed[2] part of the economy, attracts more attention nowadays. People expect the reform in this sector would change its low efficiency and diminish the huge amount of bad debts or non-performing loans (NPLs), and so foster the economic development. Since the banking sector is a good representative of the economy, it helps to portray the profile of the

[2]The related indicators of structural reform and institutional quality can be found in EBRD Report 1999 and IMF Report 2000.

reform. Besides, the clear boundary of the sector makes it easier to analyse and grasp the essential of the problem.

Much work has been done in the field of China's reform and transition. Among others, some important works which may closely relate to this chapter are as follows. Gregory C. Chow addresses every detail of China's economic reform in his book *China's Economic Transformation* (2002). Roland (2000) provides an overview of current research raised by transition and concludes the transition economics in his book *Transition and Economics*, especially, of which the third part looks at the economic behavior of firms in the transition from state to private ownership and compares the effects of privatization, restructuring, and financial reform. Dewatripont and Roland (1992) analyse the virtues of gradualism and legitimacy in the transition and extensively examine the impact of political constraints on economic reform plans in a next paper in the same year. Some economists regard China's gradualist approach as self-defeating (Murphy, Schleifer and Vishny (1992)). Feltenstein and Nsouli (2001) empirically analysed the strategy of gradualism in China. Other economists argue that it is neither gradualism nor experimentation but rather China's unique initial conditions—namely, a large agricultural labor force, low subsidies to the population, and a rather decentralized economic system—that have contributed to China's success (Qian and Xu (1993); Sachs and Woo (1994); Woo (1994)). Moreover and specifically, Lau, Qian and Roland (2000) develop a simple model to analyse the dual-track approach to market liberalization as a mechanism for implementing efficient Pareto-improving economic reform. To further understand how reform works in a developing and transition economy that has great growth potential, Qian (2001) suggests that one should study how feasible, imperfect institutions have evolved to complement the initial conditions and to function as stepping stones in the transition toward the goal. However, there is a lack of study on the general and basic ideas and features of China's reform which seem essential to understand the success of the transition process. This chapter offers a fundamental but comprehensive perspective on the topic although only relying on a case study of the banking sector. As for the analysis about the reform in China's banking sector, especially from the market structure point of view, few work can be found except some papers in Chinese which are helpful, e.g. Yu and Ju (1999, 2000), Pang and Ju (2000).

In addition to this section introducing the purpose of the chapter and reviewing the literature briefly, the remaining part is structured as follows. The next section presents a retrospectus about the reforms in the banking sector and provides a timeline of the market evolution. Based on explaining the issue of "the river and the stones" and showing the consistency between the banking sector reform and the whole economic transition, the third section discusses and summarizes the basic ideas and features of China's transition model. Concluding remarks follow in the last section.

6.2 From plan to market: reforms in the banking sector

The P.R.China has nationalized after 1949, when it was established, all means of production, including the banking system. It introduced a Planned Economy in which final decisions were centralized. From 1949 to 1978[3], the banking system was governed by a national monopoly, called the People's Bank of China (PBC) which operated all kinds of financial businesses with the primary function of serving government to allocate capital resources through budgetary grants. As the financial businesses were quite limited due to the planned economy itself, PBC simply provided credit needed by the SOEs for their production plans and provided cash used principally to cover labor costs and purchases of agricultural products. Obviously, the system was not modern at all and played only a limited role in promoting economic growth.

Since 1978, the policy of reform and opening up has fundamentally changed China's economy and society. As a crucial component of the economy, the banking sector has also experienced a from-plan-to-market transition process and has gone through three phases of reform.

The first phase of the banking reform, from 1979 to 1995, consisted of two parts. The main part focused on the establishment of a two-tier banking system that comprised primarily a central bank and four state-owned specialized banks (Big-four): the Agricultural Bank of China (ABC), the Bank of China (BOC), and China Construction Bank (CCB) which were established in 1979, and the Industrial and Commercial Bank of China (ICBC) established in 1984. Although the market was still a (specialized) monopolistic one because of the identical ownership (wholly state-owned) and the strict market segmentation and large market share (those four banks divided the banking market into sub-markets with clear boundaries in terms of specific banking businesses; one bank could hardly enter another bank's market and each bank

[3]The Third Plenum of the Eleventh National Party Congress, held December 18-22, 1978, is considered a major turning point in modern Chinese political history.

dominated its own market excluding the others from taking over: for instance, the Agricultural Bank of China covered more than 95% of the rural banking business), it was the first step to break the monopoly status of the PBC. Since 1986, the PBC has gradually focused on its role as central bank by transferring its commercial banking activities to the Big-four. In 1995, with the enacts of the Central Bank Law and the Commercial Bank Law, the government completed the two-tier banking system. The second part was the gradual removal of entry barrier, which resulted in an increasing number of new entrants. Before 1987, a first attempt was made in this respect but with a very limited scope. Besides the Big-four, only a few new banks were set up: the state-owned China Investment Bank (1981), the joint-stock Bank of Communications (1986), CITIC Industrial Bank (1987) owned by China Investment and Trust Cooperation, etc. From 1987, especially from 1992, three policy-lending banks, ten commercial joint-stock banks and a large number of urban credit cooperatives and rural credit cooperatives were established. As for the opening up policy in this sector, we can also see it developed step by step. Since 1979, foreign financial institutions started to set up representative agencies in China. In 1982, the government started to license foreign banks to engage in banking businesses in China on an experimental basis, and authorized a Hong Kong based bank, the Nanyang Commercial Bank, to establish a branch in Shenzhen Special Economic Zone (SEZ). In 1985, three other SEZs were also opened to foreign financial institutions. By 1995, the number of cities opening up increased to 24.

Clearly, the process of the banking reforms in this period was rather slow, especially compared to other sectors in the economy. The reason is twofold. On the one side, the role of banks in economic development was not significant and even overlooked as China just started its marketization. On the other side, the emphases of the whole reform in this phase were rural reform (1978-1986) and the SOE reform (1987-1997). However, the domestic NPL problem of the state-owned banks (SOBs) and the financial crisis in Southeast Asia made the government realize the importance and urgency of accelerating the banking reforms in the next periods.

The second phase of banking reforms also had two major parts: further institutional building and the management of NPLs. The former focused on the commercialization of specialized banks and a separation between policy and commercial lending activities, which were started from 1995. The measures included improving the management of the SOBs by granting their managers autonomy and employees profit incentives, reducing local government intervention (e.g. the Central Bank system was re-organized by merging 31 provincial branches into 9 regional ones), the removal of credit allocation, limited interest rate deregulation, entry deregulation, a narrowing of the scope of business[4] and a gradual tightening of accounting and prudential regulations. Responsibilities of financial regulation and supervision were separated into three mutually independent regulatory agencies with the PBC responsible for bank supervision, the China Securities and Regulatory Commission for equities market, and the China Insurance Regulatory Commission for the insurance industry. The latter included the recapitalization of the Big-four SOBs by injecting 270 billion RMB into them, the disposal of NPLs held by SOBs through setting up four assets management companies (as a result, 1.4 trillion RMB NPLs were carved out from the Big-four), the merger and closure of problematic banks, the transformation of urban credit cooperatives into city commercial banks, and the promotion of debt-equity swaps. Moreover, in 1999, China abolished the geographic limitations on the establishment of business institutions for foreign banks, but still with the limitation in operating specific businesses. Especially, local currency business was restricted to a few cities and banks. As of 2001, China's banking system consists of the Central Bank (PBC), the Big-four SOBs, three policy-lending banks, 110 commercial banks (99 of which are city commercial banks, one is home savings bank and the rest joint-stock commercial banks), about 3,000 urban credit cooperatives, some 42,000 rural credit cooperatives, and about 190 foreign banks with 164 branches and 233 representative offices.

As China officially joined the WTO on December 11, 2001, the third phase of banking reforms started. A detailed timeline is given as below.

- December 11, 2001. The government removed restrictions on the foreign exchange clints of foreign banks and allowed them to conduct local currency (RMB) business in four cities: Shanghai, Shenzhen, Tianjin and Dalian;

- December 1, 2002. Foreign-funded financial institutions would be allowed to conduct RMB business in five more cities: Guangzhou, Zhuhai, Qingdao, Nanjing and Wuhan;

- 2003. Foreign banks can serve Chinese enterprises;

- 2005. Foreign banks can conduct RMB business in 20 cities;

[4]In effect, the government has since adopted a Glass-Steagall Act to separate commercial banking from investment banking business.

- 2006. National treatment to foreign banks in China. Opening up in all dimensions in the banking sector.

In 2006, the Chinese banking market will be open to foreign competition. Thus, the third phase will stress the roles of the market structure and property right reform: increasing the market competition by gradually removing barriers to entry, especially for foreign banks and diversification of SOBs' ownership through introducing non-state sources of capital so as to increase their competitiveness.

From the above, although there are still many problems left unsolved, we have to admit that in the past twenty years or so, China's banking sector has evolved from a planned and undeveloped system to a market-oriented and modern system through the gradual reform.

6.3 The river, the stones and the strategy

6.3.1 A popular and perfect metaphor

China's approach of reform is indeed original and unique[5]. Although foreign advice is frequently sought, China's reforms are genuinely homegrown, based on domestic actual conditions and on learning by doing. The Chinese leader Deng Xiaoping, the general "designer" for China's reform and opening up, described the process as "crossing the river by touching the stones[6]". This perfect metaphor vividly illustrates all the key elements of the reform strategy in China.

No doubt, generally, the starting point on the one side of the river is the planned, undeveloped and agriculture-based economy, while the opposite bank is a market-oriented, developed and modern economy. The river is the division in between the two sides. Crossing it implies the transition process from plan to market, from backwardness to modernization, from the poor to the wealthy and from isolation to opening up.

However, on which exact place to start the "wading"? How to cross the river? Shall we take the old stuff as crossing? How far is the opposite side? Where is shallow to be easy for wading or where is deep and full of dangers? Which stones are firm and stable that can be relied on and which are loose? Where is the right point for berth

[5] Of course, it has influenced some other transition economies. For example, Vietnam has adopted a similar strategy since early 1990s. The spill-over effect is greater and greater.

[6] A literal (Western) interpretation would be, "if you are on an unfamiliar course, look carefully and take one step at a time."

on the other side? All these questions were not clear at all during the course of the reform, especially in the beginning.

In reality, the initial "wading" started from rural sectors, which is the relatively simple and efficient program of economic liberalization as it immediately changed the situation of shortage in agricultural products by reconstructing incentive mechanisms. One of the most well-known "effective" "stones" was the "household responsibility system". Farmers worked as a team consisting of some forty persons in the Commune system in China before 1978. A farmer could not get extra reward by working harder because all members of the team would share the additional output due to his additional labor. In the very severe situation of starvation, some farmers realized that if they farmed separately the team could produce more in total and still delivered the same amount of output required by the procurement system for government distribution of agricultural products in the economy. The Commune system was changed as the team was reorganized by distributing its land to individual households to farm separately, each getting the additional reward for additional labor after delivering a fixed amount of output to the team for delivery to the government procurement agencies. Such practice was introduced and spread in many areas of the country and resulted in a formal national policy named as above. Now when looking back at that period, we might see that a correct starting point as the result justified it. However, that was not the case in 1978. Farmers as well as local governments had no choice except this way: it was not a well-designed blueprint.

The looking while going and learning by doing propelled the reform. The next step is city and enterprise reform. As for the open door policy, in 1980, the government first established a special economic zone in Shenzhen as an experimental field for the market-based activities such as international trade, foreign investment and setting up new factories with non-state ownership. Later on, other four SEZs were set up. With the successful experiences in those districts, the reform went further by opening 14 coastal cities. Nowadays, the open-door policy applies in the hinterland.

Gradual process in SOE reforms is more obvious. In contrast to Russia and the Eastern European countries which adopted the "shock therapy" to privatize the state-owned sectors over one night, China has not completed the SOE reforms until today. The role of the dual-track approach (cf. Lau, Qian and Roland (2000)) is significant and efficient in this respect because of its Pareto-improving property: developing the market track without affecting the plan track. In such a way, the reform accomplished the goals of achieving economic development and keeping social stability simultaneously. Similar to the above, the reform also went forward by touching the "stones". The first and most common stone "touched" from the early 1980s was the so-called contract responsibility system[7], by which the management of SOEs were strengthened. As later on it ran into trouble, a second stone found in the first half of the 1990s was the separation between government and enterprises. Currently, the third stone which may be considered as an obstacle as well is the ownership reform.

To further understand the ideas and features behind China's reform strategy, we will go back to the example of the banking sector in the following to make some more detailed investigation.

6.3.2 The philosophy of the reform

As shown in the second section, the practice of banking sector reforms is in line with the whole economic reform. Investigating this sector provides a suitable window to watch the panorama of China's reform and transition.

Apparently, the banking reform aims to go across the river between a planned, backward and inefficient system and a market-oriented, developed and efficient system. In order to fulfill the transition, the reform still explores the way by touching the stones, step by step. The stones, the path and the direction show us the philosophy of the reform.

- **In process: *gradualism***

  The basic feature of China's reform is gradualism, which is an approach or policy of advancing toward the goal of constructing the market system by gradual, often slow stages. The steps of the transition process were taken one after another and not all at once, in contrast to the so called "shock therapy" in the economic literature. In fact, this gradualist strategy is naturally formed during the course of processing rather than pre-designed. The ultimate goal of the reform evolved over time and has only gradually come into sharper focus. In the very beginning of the reform, nobody, at least no one in China or in the government that formally started the reform, realized that the final target would be constructing the market-oriented economy, simply because it was a bottom-up reform instead of a top-down one. The first goal was rather direct: increasing the output of

[7] After paying a fixed tax to the government having jurisdiction over it, each SOE was allowed to keep the remaining profit for distribution to its staff and workers and for capital investment.

agricultural products to get out of starvation. The second goal was still temporary: to construct a planned-based while market-supplemented economic system, which corresponded to the dual-track system period. Only after a long time of exploration and various kind of tests, did the final target come into being clearly. The other side of the river is visible now.

As for the evidences in the banking sector, in addition to the general three phases mentioned in section 2, four lines of the reforms further demonstrate this feature.

1. Market: Big-four → new SOBs and Joint-stock banks → foreign banks;
2. Measure: management reform → ownership reform;
3. Opening up: SEZs → coastal cities → hinterland;
4. Foreign banking business: representative agency → foreign currency business → local currency business.

The gradualist approach provides ample time and space for experimenting all the measures related to reforms.

- **In method: *experimentalism***

Here, experimentalism means using experimental methods to determine the validity of the measures of the reform. Each step of the transition process was taken after drawing the experience of the previous step. The Chinese government always tests a new reform measure in some districts, enterprises or sectors before its formal and nationwide carrying out. Because it is prudential and mistake-allowed, this idea has advantage in minimizing the cost of reform.

Some examples in banking reform show this feature. After accumulating enough experience to supervisor foreign banks' activities in the SEZs, the government would open other coastal cities, even western provinces to foreign entrants. When introducing a kind of new business, the government always first issues license to some bank(s) to take an experiment and see the effect and then determine whether or not to apply and develop it. The event of the bankruptcy of China Agricultural Trust and Investment Company made the government realize the necessity of separating commercial banking from investment banking business. As there is no well-designed blueprint and no former footprints can be referred to, taking experimental methods is then a natural choice when confronting uncertainty.

- **In attitude: *pragmatism***

  Pragmatism is a practical, matter-of-fact way of approaching or assessing situations or of solving problems. Gradualism and experimentalism has characterized the practical attitude all through the reform.

  Given the initial economic and political conditions, China had no basis to carry out single-track full marketization in the banking sector. As a compromised and pragmatic choice, the dual-track system helped to foster the transition from plan to market as it minimizes political opposition to reform ex ante and maximizes political opposition to reversal of reform ex post. In such a way, the market gradually came into being without much instability or obvious obstruction. Another example lies in the gradual removal of entry barriers and business restrictions for foreign banks. This process provides enough time for the banks to make a sufficient domestic competition as the rehearsal for the future international competition through learning by doing. As the spill-over effect of know-how from foreign banks strengthened the competitiveness of domestic banks, we can anticipate that in 2006, when the door is fully open, the SOBs will grow up to a highly competitive standard. So, the strategy is successful both in developing the economy and in keeping social stability.

  This feature can be further understood from another well-known saying by Deng Xiaoping: "No matter whether it is black or white, a cat that catches mice is a good cat." It reflected that the government would explore a way by practice rather than by dogmatism.

- **In nature: *evolutionism***

  Evolutionism in reform implies a belief that reform is a gradual process in which the economy changes into a different and usually better form through a developing and continuously revising way rather than a well pre-designed or one-shot approach. Exactly speaking, this feature is not an approach but a belief and conclusion.

  Importantly and fortunately, evolutionism is not alone. The path-dependent property and self-enforcing mechanism keep the reform go forward to the market instead of backward to the planned economy. That is, once stepping on the road, then no way back.

  For instance, nobody was against the establishments of the Big-four and other new SOBs. A limited scope of experiment of joint-stock commercial banks and foreign banks did not incur much objection either, as the existing interest groups can be relatively protected. However, gradually, the competition from the new track has forced the old system to make change and approach to the market system. The entry to the WTO is a milestone. Even if someone wants to go back, that is not possible, which guarantees the direction of the reform. The external pressure and incentive provide new engines for the reform.

A natural question concerning those distinguishing features of China's reform and transition is why it initially and mainly happened in China and not in the other transition economies? Economists usually look for explanations from the initial economic and political constraints and actual conditions of the beginning of the reform in China. Indeed, that is the direct reason. However, the cultural dimension and even the historical/traditional or philosophical dimension as the fundamental factors were neglected. An excuse might be it is hard to introduce them into scientific models. However, cutting edge findings usually come from breakthroughs. Scientific research needs the spirit of "crossing the river by touching the stones" as well. In short, if taking the doctrine of the mean of the Confucian school and China's more than two thousand years' feudal history and stable social structure into consideration, we will find a more reliable answer. Put it another way, why is it easier to practise the "big bang" strategy in European countries? Fundamentally, it is easier to be understood and accepted by the minds of the people living there.

6.4 Conclusion: transition by experimentation

While one can view the "big bang" strategies followed by the Soviet Union and many former socialist countries in Eastern Europe as the approach of transition by coercion, we might summarize the gradualist strategy so far adopted in China's transition to a market-oriented economy as an approach of transition by experimentation. The characteristics of this kind of transition approach are in fact the gradualism in process, experimentalism in method, pragmatism in attitude and evolutionism in nature.

Besides the significant effects in the transition and development of China's economy and society, the gradualist reform has far reaching influences in two aspects at least. First, the success sets an example for other countries, especially the transition economies and developing countries. Vietnam took a similar gradualist strategy: the sequencing of reforms in Vietnam is quite close to the one observed in China. Some

CEE countries have already adopted the similar approaches and measures in specific sectors to slow down the radical paces. Even in North Korea, the government started to set up the SEZs to make some experiments. Moreover, Japan also realized the advantages of the strategy and started to learn from it for reforming its stagnant and depressed economy. Another effect is in academic research: both in economics and in the related social sciences, China's reform provides a platform and a new path to explore the unknown world and to further update our knowledge. The Chinese experience has shown that the conventional view, i.e. reforming a socialist system in a piecemeal manner cannot be successful, is not always true. Likewise, this experience has also falsified the claim that a political revolution is necessary for pushing the reform into the transition stage. In addition, the practice as well as the different approaches of transition in various countries call for further study and explanations, especially from the cultural and historical perspectives.

Here we want to note that this study does not make any prediction regarding the eventual success or failure of China's transition. What could be the exact meaning of "eventual" in respect of a societal evolution anyway? If prediction of events is also important in the social sciences, the first priority in economics is understanding the structures and explaining them to policy makers who shape the future events. Meanwhile, we are not in a position to assert that the gradualism or experimentation based transition is better than "shock therapy" in general although it is confirmed that so far the approach of experimentation has worked for China. Depending on different actual conditions of the transition economies, one might choose a way which is particularly suitable and effective in its own social and economic environments. Moreover, even in China, transition by experimentation is not a panacea for all stages. As more conflicts appear and social stability is jeopardized by implementing the structural and ownership related reforms, compensating losers to obtain their support seems more demanding for facilitating and finalizing the transition. Hence, the experimentation approach eventually gives way to the so-called "transition by compensation" approach.

Finally, we want to briefly discuss the different transition approaches reflecting different attitudes to the compensation problems in the transition period. As Roland (2002) puts, "we rarely observe efficiency-enhancing reforms". Transition from one economic system to another implies that someone might become a loser. Thus, compensation problems necessarily arise. It is generally observed that the strategy of "shock therapy" has hurt most people in the economy, and to compensate them was hardly possible. However, in China, since a piecemeal manner and experimental methods have been adopted, only small groups of people were hurt at every stage of the reform process. Thus, compensating them was possible. We may say that this approach circumvented serious compensation problems as the basic social stability has been secured and the general support for reforms has been obtained. But as being noted above, to complete the transition to a market economy, China has to implement the structural and ownership related reforms. It requires a more sophisticated approach. This is the subject of the last chapter.

Chapter 7

Transition by compensation: the political economy of demolition and eviction in China

7.1 Introduction

This chapter studies the problem of compensation in China's transition period. We discuss that an approach of *transition by compensation*, a method of facilitating and realizing the transition process to a market-oriented system by means of compensating (potential) losers to buy their acceptance and support for the reforms, could be particularly useful for the current stage of the transition process in China. Differing from the existing studies (Lau, Qian and Roland (1997, 2000); Roland (2001)), we focus on the micro-level of the compensation problem and propose fair compensation rules and an effective compensation system which could be well applied in the economic reforms in China. Based on a case study on the demolition and eviction in China's transition period, we show implications for the whole economy.

It is well known that China adopts a gradualist reform strategy for the transition to a market-oriented economy (see chapter 6). Although one may observe that the old planned system and the new market system have coexisted and sometimes intertwined in the economy since 1978, according to Qian (2000), the main line of the transition process is clear: China's transition process evolved in two stages of which the first one focused on improving incentives within the planned system and increasing the scope of the market in resource allocation, and the second one mainly involved the construction of the institutions supporting a market system. As shown in chapter 6, the transition reforms in the first stage are dominated by the approach of "transition by experimentation". In contrast to the "big bang" strategy, the transition by experimentation is an approach of learning by doing in economic reforms, which is featured by gradualism in process, experimentalism in method, pragmatism in attitude, and evolutionism in nature. However, as the goal of establishing the market-oriented economy has been set down, and as more conflicts have appeared due to the abolishment of the plan-track, the experimentation approach should give way to the so-called "transition by compensation" approach because it is more suitable and demanding for the second stage.

In this chapter, we first present the background for such a transition approach, and then mainly analyse the issue of demolition and eviction as a case study to show that the existing compensation rules are neither fair nor effective. We argue that the cooperative game theory provides a suitable method to look into the compensation problems. Based on that, the Shapley value (Shapley (1953)) and the consensus value (cf. chapter 2) for TU games can be applied to formulate fair compensation rules. We further discuss the other elements to compose an effective compensation system.

Here, in particular, the consensus value is not only featured by its constructive sharing process but is also characterized as the unique one-point solution concept for TU games that satisfies efficiency[1], symmetry, the quasi dummy property and additivity. The classical dummy property requiring that a dummy player (who cannot make any additional contributions to any coalition of agents except for his own individual value) only gets his individual value is in fact a utilitarian or individualist dummy property. The equal surplus solution (cf. Moulin (2003)) makes no difference between a dummy player and non-dummies: each agent equally share the joint surplus of the grand coalition, so the corresponding dummy property can be understood as an egalitarian or collectivist dummy property. As a hybrid case, the quasi dummy property is an average of the above two and provides a more socially and morally acceptable solution for determining the gain of a dummy player as it makes a fair compromise between those two contrastive cases.

More specifically, as the market economy is utilitarianism or individualism oriented while the planned economy is featured by egalitarianism or collectivism, and since the quasi dummy property well balances the trade-off between those two extreme opinions about the gain of a dummy player, the consensus value provides an appropriate theoretical foundation to determine the exact (acceptable and fair) compensation for

[1]Here, the term efficiency is a game theory based concept that can be understood as a balanced budget property: the sum of all agents/players' final payoffs in a game equals to the value that is created by their cooperation.

losers in the transition period.

In addition to this section introducing the chapter and reviewing the relevant fundamental works briefly, the remaining part has the following structure. In the next section, we present the background for implementing the approach of transition by compensation in China. In section 3, we analyse the case of demolition and eviction and its associated compensation problems. Section 4 studies the fair compensation rule and the effective compensation system. The final section discusses the implications of the rule as well as the compensation system for other sectors in the economy.

7.2 From Pareto-improvement to Pareto-neutrality

It is known that China adopts a gradualist strategy for the transition from plan to market. Differing from the so-called "big bang" path, this gradual reform process incrementally promoted liberalization and postponed privatization until recently but circumvented the issue of democratization. It is because of such a pragmatic strategy that China has made a remarkable success in transition so far. However, it is also noted that greater challenges follow.

Qian (2000) holds that China's transition process evolved in two stages. In the first stage (1979-1993), the centrally planned system was reformed incrementally to improve incentives and increase the scope of the market in resource allocation. In the second stage (since 1994), new institutions supporting a market system are being built, but before old institutions are destroyed.

In the first stage, reforms often featured schemes known as "particularistic contracting"[2], through which the incentives of economic agents were improved and at the same time existing interests could also be protected. The first fifteen years of reform changed the landscape of China's economy drastically compared to 1978 and succeeded in substantially improving people's living standards on a broad basis. Reform received solid and popular support. Moreover, the state sector was no longer the dominant part of the economy, and most old revolutionaries were fading away from the political scene. All of these changes did facilitate a strategic shift in the official ideology to completely abandon central planning and embrace a market-oriented system with private ownership. Lau, Qian and Roland (2000) attribute the success of this stage to the

[2]The main characteristic of this scheme is that instead of standardized rules to be applied uniformly to all units of the economy, ad hoc and selective arrangements evolve with the relevant unit. For more details, please refer to Shirk (1994).

"dual-track" approach (constructing the market-track while simultaneously protecting the plan-track) as a mechanism for implementing efficient Pareto-improving economic reform, that is, reform achieving efficiency without creating losers. To illustrate this, consider the banking sector reform analysed in chapter 6 that also adopted the "dual-track" approach: strengthening the existing banking system so as to achieve efficiency and well protect its existing interests, while simultaneously liberalizing the market. Therefore, no one is a loser. On the other hand, it is also due to the "dual-track" approach, or more precisely due to the gradualist reform itself, by the end of 1993, that the economic system as a whole was still a half-way house between a plan and a market economy. However, the track of plan has to be abolished in order to achieve the goal of constructing a market-oriented system. Thus, a more profound structural reform is needed.

Since 1994, particularistic contracting is being replaced by universalistic rules and market-supporting institutions based on the rule of law and incorporating international best practices are being established. The fourteenth Party Congress in September 1992 endorsed the "socialist market economy". Some further breakthroughs on ownership issues were made by the Fifteenth Party Congress in September 1997 and by the Constitutional Amendments of March 1999 and the Constitutional Amendments of March 2004, which strengthened the goal of transition to a market economy. The crucial task of this stage is to privatize and restructure state-owned enterprises. Although the economy steps forward and attains efficiency as a whole, it is no longer a Pareto-improving stage: the society is mainly differentiated into two groups, some are benefitted from the reforms in this stage while others suffer. For instance, due to the competition pressure from the foreign banks after joining the WTO and the increasingly intensive competition with domestic commercial banks, the big-four commercial banks in China have to undertake stricter and tighter reform measures like laying off redundant employees to strengthen their competitiveness. More examples can be found in the urban reforms such as the changes in the free medical care system, housing system, etc. We then find the reforms in the second stage are somehow Pareto-neutral. Here, the term of Pareto-neutrality implies a Pareto-efficient situation with a Pareto superior situation in view but the path to this superior situation is blocked. Put differently, it is a Pareto-efficient situation which can be improved if some compensation rule is applied so that all agents are willing to move. Thus, to facilitate the reforms in this stage and thereby fully establish the market economy, the losers are needed to be compensated so that superior efficiency can be achieved in the economy.

There are more facts to account for the importance of compensation for China's transition. With the largest population in the world, the society in China is very sensitive to any reforms. Compensating losers is the only way to get a relatively stable social environment as the prerequisite for implementing such a large-scale economic transition process. In addition, the market economy calls for the protection of private properties, which implies that compensation has to be given in case of any expropriation of private properties during the transition period, although this was not explicitly specified in the first stage of the transition. We want to note that there exist a special group of losers in the transition period: people with vested interests from the pre-reform system will lose their power and rents in the new system such as conservative political leaders and bureaucracies with established interests. To make reform acceptable, it is necessary to buy off those people in power.

The issue of compensation in transition has been noted by the State (e.g. various regulations of specific reform measures). Especially, in the most recent Constitutional Amendments, i.e., the Constitutional Amendments of March 2004, the inviolability of legal private property was specified for the first time since 1949. Moreover, the amendment adds "the State gives compensation" to the original stipulation that "for public use, the State has the right to expropriate urban and rural land and citizens' private properties."

Existing studies on the issue of compensation in transition mainly focus on the macro-level. Among others, Lau, Qian and Roland (1997, 2000) show that the dual-track strategy provides implicitly a set of feasible lump-sum transfers (transfers that are independent of the actions of the individual economic agents) to compensate the losers of the economic reforms. Roland (2001) addresses a strategy for easing political constraints so that reforms can be enacted: building reform packages that give compensating transfers to losers from reforms. He further explains why such transfers can be difficult to enact in the real world. A first difficulty is that redistributive transfers must be financed by collecting revenues, which usually involves distortional costs. A second difficulty involves asymmetric information about the losers from reforms. A third difficulty is related to weak commitment power of decision-makes. Laffont and Qian (1999) stress the constraints of enacting compensations in China: lack of commitment and liquidity.

However, those difficulties or constraints are not the main problems any more. Generally speaking, after more than twenty years of reform and development, China has gained a much better economic situation which makes the potential transfer feasible[3]. Meanwhile, market-oriented institutions built in the past ten years provide guidelines for enacting the potential compensations and helps to overcome any commitment problem. Therefore, the issue of compensation deserves more attention than before.

Moreover, one may notice that there also lack studies on the micro-level of compensations in the transition period. In the following sections, we focus on the questions how to design a fair rule to compensate the potential losers from a reform and thereby to help in enacting such a reform, and more systematically, what an effective compensation system should be.

In principle, any sector in the economy where the reforms generate losers needs compensations. Taking the case of demolition and eviction, we investigate the associated compensation problems and draw policy implications for the issue of compensation in transition as a whole.

7.3 Demolition and eviction, and the associated compensation problems

7.3.1 (Forced) demolition and eviction

(Forced) demolition and eviction are the destruction of people's homes or the expropriation of their lands and the removal of those people from their places, which are usually against their will. Governments are often actively involved in those activities. Demolitions and evictions conducted or tolerated by governments are carried out in a variety of circumstances and for a number of different reasons. In China, the more common rational of the central and local governments for demolition and eviction lies amongst others in

- development and infrastructure projects (e.g. construction of dams and other energy projects);
- prestigious international events (e.g. Olympics);
- urban redevelopment or city beautification project;
- commercial development (e.g. a foreign company invests and constructs a factory in a city);

[3] As an indicator, China's foreign exchange reserve amounted to 439.8 billion USD in March 2004.

- conflict over land rights.

In fact, the issue of demolition and eviction in China has begun to receive attention in official and academic circles only since the 1990s. In the decades following the establishment of the People's Republic of China in 1949, all lands were owned by the state and ownership could not be transferred to private individuals or companies. Individual citizens did not own private homes or work for private companies, but were required to live in apartments located within the compound of the government work unit where they were employed. Since no private property existed in the country, basically governments had the right to allocate houses to people and also had the right to remove them to other places when it was necessary to do so. Hence, there were no real compensation problems involved even if people were asked to leave for somewhere else; and moreover, it was not difficult for people to accept such removals.

Since the late 1980s, China's rapid shift toward a market economy, the increasing demand for private home ownership, the budget constraints of local governments for providing houses, and the need of local governments for revenue, have resulted in a thriving real estate market. More and more people bought their own apartments or houses. Meanwhile, economic development called for various infrastructure constructions and more activities that would lead to restructure cities and demolition and eviction. Naturally, the associated compensation problem came into the scene. The basic reason for compensation is clear: demolition and eviction expropriate citizens' private property and cause losses on them.

A typical process of demolition and eviction and the actors who might be involved in such a process can be described as follows.

- **Developers** who acquire a parcel of land and wish to build on a site must apply for and obtain a series of permits from demolition and eviction management departments;

- The **demolition and eviction management departments** in municipal governments process the applications for demolition permits, collect the necessary fees, and are responsible for the process of demolition and eviction;

- The developers, or the government departments acting on their behalf, are required by law to approach the existing **residents** at each site, whether homeowners or tenants, to advise them of their eviction and negotiate compensation;

- Developers subcontract a private **demolition company**, which specializes in the demolition and clearing of sites for construction.

According to the law, once all parties have signed a compensation agreement, the resident must relocate. The demolition and eviction management department can also arbitrate disputes between developers and residents over compensation, and may give developers approval to proceed with "forced demolition and eviction" if the resident refuses to move. China's *Regulations for the Management of Urban Residential Demolition and Eviction* specifies the procedures through which cities may evict residents.[4] And most local legislatures have passed implementing regulations that generally copy the language of national regulations with only minor modifications. Here, in particular, when developers and residents fail to reach a compensation agreement, regulations permit developers to apply for permission from the demolition and eviction department to proceed with forced eviction.[5] Needless to say, this permission is critical for generating so many cases of forced evictions and the corresponding consequences like protests, violence, and other social problems.

This is a brief introduction about the issue of demolition and eviction in China. For a detailed summary, we refer to Human Rights Watch (2004).

7.3.2 Current compensation rules

Disputes between developers and residents often arise over low rates of compensation and poor resettlement options. Once they learn that their home will be demolished, residents generally have little option to prevent it, and instead attempt to negotiate with the development company over the amount of compensation. According to the national demolition regulations, developers must pay evictees compensation equal to the full market value of their properties, with an added (although unspecified) amount of compensation for business loss in the case of non-residential properties. Regardless of the regulations, it is also noted that the amount of compensation may in some instances be unilaterally decided by the developers or the demolition companies. It may be set far below market value, with little or no account taken for loss of income in the case of properties used for family businesses.[6]

[4] Chengshi Fangwu Chaiqian Guanli Tiaoli (Regulations for the Management of Urban Residential Demolition and Eviction), published by the State Council on March 22, 1991; and a modified new version with the same title was published on June 13, 2001, and implemented from November 1, 2001.

[5] Regulations for the Management of Urban Residential Demolition and Eviction, art. 17.

[6] Wang Xiaoxia, Chaiqian Cheng Raomin Gongcheng, Zhuanjia Jianyi Tigao Buchang Biaozhun (Chaiqian has become the harassment to people and experts suggest raising the compensation stan-

From the third chapter of the Regulations for the Management of Urban Residential Demolition and Eviction, one could find that it actually presents a guideline for compensation to evictees. In the practice of demolition and eviction, there are three specific types of compensation rules that are applied.

- Generic rule.

  The generic rule is determined by will or by arbitrary preference or based on general judgement, but not by reason or principle. Since the current regulations leave too much room to the authorities, they can arbitrarily determine some part of the compensation for resettlement. Sometimes the authorities even manipulate circumstances in order to get a generic rule. According to an article in *Nanfang Zhoumo* (*Southern Weekend*), residents in Jinhua city, Zhejiang, said authorities claimed in a demolition notice that they were being relocated to make way for a "green belt". Authorities used this reason to justify low compensation for the eviction and the refusal of residents' request that they be resettled in the same neighborhood.[7] Although it is clear that this rule is not fair mainly because it does not cover the true losses of the involved parties and it does not distinguish the specific individual situations, it is widely applied in practice since it is relatively easy and costless to determine.

- Cost-based compensation rule.

  The cost-based compensation rule is exactly in the spirit of the Regulations for the Management of Urban Residential Demolition and Eviction. In the practice of demolition and resettlement, some specific measures follow Article 24 such that the sum of the compensation money will be determined based on the location, use, construction area and other factors, and by using the appraised real estate market price of the demolished home.

- Negotiation rule.

  This rule means that the involved parties can negotiate with each other on the amount of compensation. In cases that the evictee party is a strong group (like the government itself), the developer (like a private company) may offer a very attractive compensation proposal with the standard much higher than the cost.

dard), *China Economic Times*, November 12, 2003.

[7] Cheng Gong, Zhi Chaiqian Zhi Tong (Treat the Pains of Demolition and Eviction), *Southern Weekend*, December 31, 2003.

> However, in most cases when the evictee party is composed of weak groups like low income social groups, the compensation could be very low.

Unfortunately but clearly, the current regulations of demolition and eviction do not work well although there are some compensation rules as above. The following figures may account for this fact. According to the national Ministry of Construction, of 1,730 petitions filed from January to August 2003, about 70 percent were about problems with forced evictions.[8] In some cases, tenants' rights advocates organized petitions signed by large numbers of people. For example, in 2000, over 10,000 petitioners filed a civil suit against demolition and eviction at the Intermediate People's Court in Beijing.[9]

A fair compensation rule as well as an effective compensation system for demolition and eviction are in great need.

7.4 Towards an effective compensation system

Some scholars summarized the key problems of demolition and eviction as lack of rights for evictees, lack of any organized system for resettlement, generally low compensation, and difficulties in obtaining legal redress.[10] Instead of addressing every possible problem in demolition and eviction, we aim to find the crucial step in the whole process and propose a systematic remedy.

In our opinion, the main problem is a lack of an effective compensation system while all the other problems are either derived from it or relatively marginal. That is, once an effective compensation system is constructed, evicted people will get fair compensation, and the whole process will be facilitated. The reason is simple, almost all problems involved in demolition and eviction in the transition period in China are related to the low or unfair compensations to evictees.

An effective compensation system means a fair compensation rule which can be generally accepted by all parties involved in demolition and eviction so that it smoothes out the possible conflicts, plus a credible procedure to implement this rule. Therefore, in order to set up such an effective compensation system, at least three objectives should be achieved:

[8]Zhao Ling, Bude Bu Zhongshi De Wenti (A Problem That Merits Serious Attention), *Southern Weekend*, September 4, 2003.

[9]Zhao Ling, A Problem That Merits Serious Attention, *Southern Weekend*, September 4, 2003.

[10]cf. Zhao Ling, Chaiqian Shinian Beixiju (The Decade-long Drama of Demolition and Eviction), *Southern Weekend*, September 4, 2003.

- Formulating and implementing **a fair compensation rule**;
- Establishing **an independent and impartial agency** that evaluates the losses of the evictees and other factors determining compensations and supervises the implementation of the rule;
- **Revising the current regulations** so that they match for the new compensation rule and the associated measures and **strengthening judicial independence** in China's court system.

Among those three objectives, the fundamental issue is to set up and implement a fair compensation rule. Here, we first analyse the inappropriateness of the current compensation rules, and then propose a new rule.

Obviously, the generic compensation rule is too simple and does not work right as it does not rely on a reasonable basis.

The cost-based rule is more appealing. However, we still observe resistances for reforms even if the evictees get compensated according to their costs, or more exactly, get the compensation based on the market price of their homes and other cost factors. Then, a question arises naturally: why is the cost-based rule not effective?

There are at least two reasons that may account for the failure of the cost-based compensation rule. Firstly, the options for resettling in the direct neighborhood are too few. For instance, there are many cases that the new buildings only offer higher standard and larger apartments or houses while the evictee is unable to afford it. As a result, he may be forced to resettle in the developing suburbs where the living standard is low, employment is difficult to find, traveling cost becomes a burden, etc. Secondly, this cost-based rule is still relatively simple and only cares about the physical costs that the eviction might incur. In fact, the human cost and trauma of (forced) demolition and eviction on individuals, families and communities can not be over-emphasized. Evicted persons not only lose their homes and neighborhoods but sometimes are also forced to relinquish personal possessions. Evictees often also lose key relationships, those which provide a social safety net or survival network of protection and which allow many daily tasks to be shared. In most cases, evictees find themselves in worse conditions than before the eviction even if their conditions were less than ideal in the first place.

As for the bargaining rule, it is affected by the comparison of the strength between the involved parties. Then, it is far from fairness and becomes unacceptable.

In order to find a fair compensation rule, we first turn back to see the nature of demolition and eviction. The three main actors in the process are the developer, the resident, and the management agency. The management agency can be seen as a court to judge and supervise the activity. The whole process is in fact a cooperative game. The developer and the resident have their own reservation or stand-alone values, which are the payoffs or utilities when they do not cooperate. If they cooperate with each other (the resident accepts the demolition of his house and moves to somewhere else, and thus the developer can carry out his planned project there), the value of the coalition will be generated. Such a game is supposed to be superadditive, i.e., the joint payoff generated by the collaboration between two agents is greater than the sum of their stand-alone payoffs, otherwise there is no reason for cooperation.

The two well defined and justified solution concepts for cooperative games, the Shapley value and the consensus value, provide a rationale to determine the compensation. Since both of the two values satisfy the individual rationality for superadditive games, the corresponding solutions ensure that every player at least gets his reservation value. Thus, the resident's interest is protected as his cost can be fully covered. Moreover, the resident will get a share of the joint surplus of the game, which implies that the final total compensation is greater than his loss/cost. This can be well justified: if we put the developer and the resident in equal positions, there is no reason that the former should get the whole surplus of the cooperation.

As the two values yield different solutions for n-person games with $n \geq 3$, one might wonder which one is more appropriate.

Since the Shapley value is characterized as the unique function that satisfies efficiency, symmetry, the dummy property and additivity for TU games, it is more suitable for the competitive environments where the involved parties are all commercial entities.

The consensus value is featured by its constructive process for sharing the joint gains and is characterized as the unique one-point solution concept for TU games that satisfies efficiency, symmetry, the quasi dummy property and additivity. By the quasi dummy property, the consensus value makes a fair compromise between utilitarianism/individualism and egalitarianism/collectivism, which makes the solution socially and morally more acceptable. Therefore, this solution concept could be appropriate for the regular cases of demolition and eviction in which the evictees are individual citizens while the developers are for-profit entities. Of course, depending on the degree to which the utilitarianism/individualism or egalitarianism/collectivism is preferred by the society, a specific generalized consensus value can be applied.

Obviously, in the case that the demolition and eviction are caused by public uses, it is supposed that no extra profit is generated, and thus the cost-based rule coincides with the Shapley value or the consensus value based rule, which implies that the cost-based compensation is still useful in certain environments.

Similar to the coercing party that decides the sharing rule of the game, an impartial agency is indispensable for evaluating the costs of the residents and the potential gain of the developer so as to determine the compensation. The members of the agency should be constituted by experts in the specific cases and are detached from the interest groups involved in the process. Besides, such an agency can help to solve the asymmetric information problem about the losses from reforms.

Moreover, to ensure the implementation of the compensation rule, we need an independent and well functioning judicial system, which will strengthen the credibility of the compensation system.

Here we provide an illustrative example to show how much a loser can get based on the compensation rule introduced above.

Imagine that a city designs an economic development zone in its suburb. For simplicity, we assume that two potential firms are interested in entering this area to construct new plants to make profits. Then, there are three economic agents: firm a, firm b, and a group of farmers or households understood as agent c living in that area. Firm a would expand its scale of production, which requires the appropriation of the land owned by agent c. Firm b also plans to initiate a project there. In any case, agent c is confronted with the issue of demolition and eviction if the city government decides to carry out the development plan. We can model this situation by the following TU game.

S	$\{a\}$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{a,c\}$	$\{b,c\}$	$\{a,b,c\}$
$v(S)$	80	30	20	160	120	70	280

In practice, it is very common that agent c will only get 20 to compensate the loss of his house by the generic rule or the cost-based compensation rule. However, think of a situation that the cheapest house that c might find on the market that could give him the same utility as his old house costs him 25. Then no doubt c is a loser in this reform. Even if under the bargaining rule c might get 25, it is not necessarily a fair amount in terms of compensation since one can ask a question like: why the firms can grab all the surplus?

According to the Shapley value, we have $\Phi(v) = (135, 85, 60)$. The consensus value yields that $\gamma(v) = (132.5, 82.5, 65)$. Obviously, implementing the compensation rule based on a cooperative game approach and the two solution concepts will facilitate the reform to a great degree as c will also get a share from the surplus of the cooperation.

7.5 Concluding remarks: implications for the transition reform

As one may observe, the compensation problems exist in various sectors of the transition economy in China. The analysis of the effective compensation system for demolition and eviction can be applied to those sectors and the associated issues.

- Layoff and compensation

China's enterprise reform has resulted in large-scale layoff of workers from state and collective owned enterprises. In the 1980s there was virtually full employment in the urban sector. But since 1993, urban unemployment has been growing rapidly. By 1997, about 18.5 million workers had been laid off from state-owned enterprises and urban collective enterprises, raising the actual urban unemployment rate to as high as about 10 percent (ZGFB (1998)).

Although there is a so-called three-line security system for laid off workers in China (Li and Zax (2003)), it is very difficult for most of the laid off workers to be re-employed or find other jobs. One crucial reason is that those workers are eliminated through the restructuring of the SOEs and do not match for the new technology any more. Moreover, since the layoff benefit is relatively low, many of the laid off workers are living in poverty, which, as a result, generated large number of protests and social unrest.

While most existing studies on China's urban layoffs focus on the macroeconomic policy, the establishment of reemployment service and associated institutions, and improvement of the labor market (cf. Gu (1999); Yang (1999)), we would stress the issue of compensation for layoffs in the same spirit as the case of demolition and eviction.

According to the three-line security system for laid off workers, for the first three years after being laid off, the workers will still keep their employment relation with their firms and get their basic salary (funded by both the government and the firms);

after three years, if those workers do not find a job, they need to terminate their employment relation with their firms and will receive unemployment compensation (from government) for two years; thereafter, the workers will stop getting unemployment compensation and will receive income assistance from the government to maintain a minimum living standard.

We would say that the current compensation rule for layoffs might be suitable for the general cases of unemployment but it is not a fair rule for layoffs from SOEs.

The basic reason lies in the relationship between the workers and the SOEs. In the planned economy, once a person was employed by a state-owned enterprise, he got a permanent position. Now it is the transition of the whole economy to the market system that changed the situations of those SOEs. It is true that they have to lay some redundant workers off in order to survive in the market-oriented economy. However, the interests of those workers should be well protected by the society/government as it is not their responsibility for such a change. Consider a worker who specialized in a specific technique and was hired by a state-owned enterprise in the planned time. Now this enterprise is carrying out a reform to be adapt to the competitive market environment and will change into another profession. As a result, this worker becomes redundant and will be fired by the enterprise. Who should pay for his loss? The enterprise, exactly speaking, the government as the owner of the enterprise, because it breaches the contract with the worker. Therefore, such a compensation for layoffs should be different from the conventional views about the unemployment benefit, but is the compensation for the breach of contracts in nature. In addition, the rules, especially the specifications about unemployment compensation and income assistance are generic rules, which increases the unfairness of the compensation.

To formulate a practical compensation rule, the Shapley value and in particular, the consensus value will be very useful. Firstly, one can take the process as a cooperative game. The cooperation means that those workers agreed to be laid off from the enterprise, and consequently more profit will be generated by the enterprise. To justify the consensus value in this case, one may consider the quasi dummy property. Since a dummy player/worker in a state-owned enterprise is "created" by the transition process, e.g. one becomes a dummy player because the enterprise changed into a new profession, he should receive a share which is more than his individual value. More generally, as a fair compromise between individualism and collectivism, the consensus value can be well accepted by the layoffs and the SOEs in the transition period.

- Compensation facilitates transition

Compensation plays an important role in the privatization reforms of public utilities such as water, gas, and so on. Because of the budget constraint of the government and the increasing demand of the economy, and sometimes for the reason of providing a better service based on an advanced technology or management, privatizing the public utilities is introduced to the economy. Obviously, the losers need to be compensated. Specific compensation rules can be formulated in such sectors according to the above analysis.

In fact, compensation is not only intended for weak groups in the transition period, but also works for some strong groups. Compensating the interest-established groups to buy their concession or acceptance sometimes is a practical and necessary strategy to help in enacting a reform.

More generally, the approach of transition by compensation also provides solutions to ease social instabilities for the countries in Eastern Europe adopting the "big bang" strategies which are somehow an approach of transition by coercion.

- Compensation affects innovation

Concerning fairness, more compensation would be desirable. However, that may imply more protection to established groups, which may hinder innovation. For instance, the more compensation a worker receives, the less incentives he may have to accept a new job; and the more a compensator has to pay, the less reward for innovation remains. Hence, compensation may lower the incentive for innovation. Due to this tradeoff, it is not a simple task to find a fair as well as efficient compensation rule. One possible solution is the generalized consensus value. Depending on the political realities, the degree to which individualism (here maybe innovation or economic efficiency) or collectivism (here may be protection or fairness) is preferred in a society or by an authority may be adapted as the θ-dummy property (see chapter 2) suggests. The corresponding compensation rule follows.

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Samenvatting

Coöperatie, Competitie en Transitie

Coöperatie speelt een fundamentele rol in maatschappelijke ontwikkelingen. Dit maakt het bestuderen van coöperatie tot een zinvolle bezigheid. Het feit dat economische agenten zich competitief en coöperatief kunnen gedragen, wordt weerspiegeld in modellen van competitief en coöperatief gedrag binnen de speltheorie, een moderne tak van de wiskunde. De coöperatieve speltheorie betreft onderwerpen als coalitie formatie tussen spelers, rationele onderhandelingen over gemeenschappelijke acties en oplossingsmethoden voor de verdeling van baten uit samenwerking. Vooral dit laatste onderwerp is essentieel voor de samenwerking zelf. De reden is simpel: zonder een goed onderbouwde verdelingsregel zal samenwerking niet van de grond komen. Coöperatieve speltheorie bevat veel verschillende modellen en oplossingsconcepten. Het meest bekend is het model van spelen met overdraagbaar nut, geïntroduceerd door von Neumann en Morgenstern (1944). Men kan een dergelijk spel beschouwen als een surplus verdelingsprobleem waarin een geldbedrag (zoals de netto winst van een groep spelers) onder de spelers verdeeld wordt zonder rekening te houden met de mogelijkheid dat spelers verschillende waarden kunnen toekennen aan een zelfde geldbedrag.

Compensatie is een bijzonder aspect van een surplus verdelingsprobleem dat aan agenten de mogelijkheid biedt om hun eigen directe belangen voor een moment te vergeten om een hogere opbrengst voor de coalitie als geheel te verkrijgen. Zo opgevat kan een compensatieprobleem bestudeerd worden binnen het kader van de coperatieve speltheorie. In de werkelijkheid vormt het verschaffen van een redelijk bedrag aan compensatie aan spelers een praktische strategie om de noodzakelijke middelen te verkrijgen om efficintie te bereiken voor de gehele groep. Compensatie is echter nauwelijks expliciet bestudeerd in de literatuur en soms volledig genegeerd. In de klassieke benadering van externe effecten in de literatuur, bijvoorbeeld, wordt de aandacht alleen op efficintie gericht, zonder de mogelijkheid te bieden om bepaalde agenten die te lijden hebben onder een uitkomst te kunnen compenseren.

Men kan dus coperatie makkelijker maken door compensatie te bieden aan bepaalde spelers. Dit idee vindt natuurlijk vaak toepassing in de praktijk. In de politieke economie vormt het concept "transitie door compensatie" een belangrijke methode. Immers, men kan een transitie proces beschouwen als een soort samenwerking van de hele maatschappij, waarbij bepaalde agenten compensatie ontvangen om hun medewerking te verkrijgen bij het realiseren van hervormingen en daardoor sociale stabiliteit te handhaven.

In het eerste deel van het proefschrift (de hoofdstukken 2 en 3) worden verschillende oplossingsconcepten van coöperatieve spelen behandeld. In hoofdstuk 2 wordt een nieuw oplossingsconcept geïntroduceerd, de consensuswaarde. Het is gebaseerd op een paper van Ju, Borm en Ruys (2004). De consensuswaarde is de unieke verdeelregel die wordt gekarakteriseerd door vier eigenschappen: efficiëntie, symmetrie, de quasi-dummy eigenschap en additiviteit. Middels de transfer eigenschap wordt een tweede karakterisering gegeven. Een recursieve formule voor deze waarde volgt uit de formulering van een adequaat gereduceerd spel. De consensuswaarde blijkt het gemiddelde van de Shapley-waarde en de egalitaire oplossing. Bij wijze van voorbeeld wordt de consensuswaarde toegepast op het probleem van fusies in netwerkindustrien.

In hoofdstuk 3 wordt de consensuswaarde gegeneraliseerd tot de klasse van spelen in partitiefunctievorm. Deze waarde wordt gekarakteriseerd als een unieke verdeelregel met de eigenschappen: efficiëntie, complete symmetrie, de quasi-nulspeler eigenschap en additiviteit. Een tweede karakterisering volgt met behulp van de transfer eigenschap. Verder wordt aangetoond dat deze waarde onder bepaalde voorwaarden voldoet aan de eigenschap individuele rationaliteit en een mooi uitgebalanceerde oplossing biedt in de afweging tussen coalitionele effecten en externe effecten. Ook hier kan een recursieve formule worden afgeleid door een herformulering via gereduceerde spelen. Verdere generaliseringmogelijkheden worden bediscussieerd. Tenslotte wordt deze waarde toegepast op twee voorbeelden: op oligopolie spelen in partitiefunctievorm en op incentive problemen bij free-rider situaties.

Het tweede deel is gewijd aan het thema compensatie en introduceert daarnaast nieuwe samenwerkingsmodellen. Hoofdstuk 4 ontwikkelt een algemeen kader voor compensatie en coperatie: de zogenaamde project-allocatie situaties. Verder worden toepassingen van de consensuswaarde op problemen van verliescompensatie en winstdeling onderzocht. Hoofdstuk 5 ontwerpt een nieuw model om externaliteiten te bestuderen met het daarbij horende compensatieprobleem. Er worden diverse specifieke oplossingen voorgesteld.

Door het introduceren van de begrippen “project” en “aandelen” in hoofdstuk 4, zoals in een paper van Ju, Ruys en Borm (2004), kan een nieuwe klasse van economische omgevingen worden geanalyseerd. Deze klasse, de zogenaamde project-allocatie situaties, is relevant als de maatschappij als geheel profijt kan hebben van samenwerking tussen agenten. Voorwaarde hierbij is echter dat agenten bereid zijn tot reallocatie van de aandelen die zij bezitten. Voor dit doel wordt een corresponderend project-allocatie spel geconstrueerd en een verwant stelsel van spelen die het onderliggende coöperatieve proces expliciet modelleren. Ook in het kader van de project-allocatie situaties blijkt de consensuswaarde uitstekend te voldoen.

In afwijking van de klassieke literatuur, waaronder Pigou (1920), Coase (1960), Arrow (1970) en meer recente auteurs, zie Varian (1994), die het externaliteiten-probleem uitsluitend met efficiëntie verbinden, wordt in hoofdstuk 5 ook het compensatieprobleem in dit verband aan de orde gesteld. Daarbij worden normatieve compensatieregels gegeven in de context van externaliteiten. Door rekening te houden met zogenaamde “stand-alone” situaties van agenten kan een nieuw type spel ontworpen worden, de “oerspelen” (primeval games). Deze oerspelen kunnen eigenschappen van inter-individuele externaliteiten goed beschrijven en maken het mogelijk de verwante compensatieproblemen goed te analyseren. In dit hoofdstuk worden verschillende oplossingsconcepten gentroduceerd die als benchmark kunnen dienen voor het oplossing van zulke problemen. Eerst wordt de Shapley waarde voor dit kader gegeneraliseerd en wordt een aangepaste Shapley waarde verkregen. Door een bilateraal standpunt in te nemen inzake de gevolgen van externaliteiten wordt de consensuswaarde voor oerspelen gedefinieerd. Voor beide oplossingsconcepten worden karakteriseringen gegeven. Verder suggereert dit hoofdstuk een meer context-specifiek oplossingsconcept, de oerwaarde. Een analyse van enkele eigenschappen van deze waarde volgt. Tenslotte worden mogelijke verbanden tussen dit nieuwe kader en de klassieke coöperatieve spelen bediscussieerd.

Het derde deel is een toepassing van de eerste twee delen op compensatieproblemen in transitie-economieën. Hoofdstuk 6 analyseert de geleidelijke transitie hervorming in China. Geconstateerd wordt dat China eigenlijk een “experimentele” methode gehanteerd heeft voor het transitieproces. Het samenstel van hervormingen in de banksector in China wordt als pars pro toto voor de gehele economische hervorming in China beschouwd. Deze hervorming reflecteert de basis ideeën van China’s transitiemodel: gradueel in proces, experimenteel in methode, pragmatisch in houding en evolutionair van aard. Door in het kort terug te zien op deze hervormingen en de verschillende maatregelen te analyseren, verschaft dit hoofdstuk een venster waardoor we in de unieke filosofie van China's hervormingen kunnen kijken, welke samengevat wordt door de "transitie door experimenteren" benadering.

Hoofdstuk 7 is gewijd aan beleidsimplicaties van de theoretische analyses in de voorgaande hoofdstukken. In dit hoofdstuk wordt voorgesteld van nu af aan een alternatieve methode voor het transitieproces in China te hanteren, namelijk de "transitie met compensatie". In het algemeen kan gesteld worden dat de overgang van een economie van een toestand naar een andere nauwelijks geïmplementeerd kan worden zonder adequate compensaties van die partijen welke verliezen lijden, privileges moeten opgeven, verworven rechten moeten opgeven of andere belangen hebben bij de status-quo. Beargumenteerd wordt dat de "transitie door compensatie" methode bij uitstek geschikt is om van nu af aan op het Chinese transitie proces toegepast te worden. Deze methode is gebaseerd op oplossingsbegrippen voor coperatieve spelen en le-vert eerlijke compensatieregels op. De analyse concentreert zich op een case study omtrent afbraak en uitwijzing in China, maar de implicaties ervan strekken zich over de hele economie uit.