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Compromise solutions based on bankruptcy

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Abstract

In this paper we introduce a new family of compromise solutions for the class of compromise admissible games. These solutions extend bankruptcy rules. In particular, we show that the compromise extension of the run-to-the-bank rule coincides with the barycentre of the core cover and characterise this rule by consistency. We also provide a characterisation of the TAL-family of rules.

Keywords: Bankruptcy rules, one point solutions, consistency.

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1 Introduction

The model of bankruptcy situations as introduced by O'Neill (1982) is a general framework for various kinds of simple allocation problems. In a bankruptcy problem, there is an estate to be divided and each player has a single claim on the estate. The total of the claims is larger than the estate available, so one has to find criteria on the basis of which the money is to be divided. In this context, many rules have been proposed to come to a fair allocation of the estate. For a recent overview of such rules, the reader is referred to Thomson (2003).

A bankruptcy situation can be seen as the most basic form of an allocation problem. As a consequence, many bankruptcy rules have a straightforward interpretation and appropriate properties of such rules are easily formulated. In a transferable utility game, the allocation problem is of a more complicated nature: instead of each player having a single claim, each coalition of players has a value which has to be taken into account. Our aim is to extend bankruptcy rules to the class of transferable utility games in such a way that both the interpretation and the appealing properties are maintained.

In this paper, we provide such an extension to the class of compromise admissible (or quasi-balanced) games (cf. Tijs and Lipperts (1982)). This approach has already been applied in Quant et al. (2003) and González Díaz et al. (2003) for two specific rules: the Talmud rule and the adjusted proportional rule, respectively. For compromise stable TU-games (ie, games for which the core coincides with the core cover), the compromise extension of the Talmud rule coincides with the nucleolus, while the APROP* rule is shown to coincide with the barycentre of the edges of the core cover. In the current paper, we look at the problem of extending bankruptcy rules from a more general viewpoint, using the concept of compromise extension of a bankruptcy rule.

An important concept in the bankruptcy literature is duality (cf. Aumann and Maschler (1985)). We use this notion to define for each rule a dual compromise extension and show that this coincides with the compromise extension of the dual rule.

In addition, we consider some more well-known specific cases: the run-to-the-bank-rule and the TAL-family of rules, which includes the constrained equal award rule, the Talmud rule and the constrained equal loss rule. We characterise the former by a consistency property and use a lexicographic construction to obtain a characterisation of each rule in the TAL-family.

This paper is organised as follows. In section 2 we present some basic definitions concerning transferable utility games and bankruptcy situations and define the concept of compromise extension. In section 3, we analyse the dual extension. Section 4 deals with the run-to-the-bank rule, while the TAL-family of rules is studied in section 5.

2   Bankruptcy and compromise extensions

A *transferable utility game* (in short *TU-game*) is a pair (N, v), where N denotes a finite set of players and $v : 2^N \to \mathbb{R}^N$ is a function assigning to each coalition $S \in 2^N$ a payoff $v(S)$. By convention $v(\emptyset) = 0$. The set of all TU-games with player set N is denoted by TU^N.

Let (N, v) be a TU-game. The *utopia demand* of a player $i \in N$, $M_i(v)$ is defined by

$$M_i(v) = v(N) - v(N \backslash \{i\}).$$

The *minimum right* of a player i, $m_i(v)$, is the minimum value this player can achieve by satisfying all other players in a coalition by giving them their utopia demands:

$$m_i(v) = \max_{S : i \in S} \left\{ v(S) - \sum_{j \in S \backslash \{i\}} M_j(v) \right\}.$$

The *core cover* of a game $v \in TU^N$, $CC(v)$, consists of all efficient allocation vectors, such that no player receives more than his utopia payoff or less than his minimum right:

$$CC(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \ m(v) \le x \le M(v) \right\}.$$

A game is called *compromise admissible* if it has a nonempty core cover. The class of all compromise admissible games with player set N is denoted by CA^N. From the definition of the core cover we immediately have that $v \in CA^N$ if and only if $m(v) \le M(v)$ and $\sum_{i \in N} m_i(v) \le v(N) \le \sum_{i \in N} M_i(v)$.

The core cover is a polytope with at most $|N|!$ extreme points. These so-called larginal vectors are introduced in Quant et al. (2003) and have been extensively studied in González Díaz et al. (2003).

An order on N is a bijective function $\sigma : \{1, \ldots, |N|\} \to N$. The player at position i in the order σ is denoted by $\sigma(i)$. The set of all orders on N is denoted by $\Pi(N)$. For $\sigma \in \Pi(N)$, the larginal $\ell^\sigma(v)$ is the efficient payoff

vector giving the first players in σ their utopia demands as long as it is still possible to satisfy the remaining players with their minimum rights.

Let $v \in CA^N$ and $\sigma \in \Pi(N)$. The *larginal vector* $\ell^\sigma(v)$ is defined by

$$
\ell^\sigma_{\sigma(i)}(v) = \begin{cases} M_{\sigma(i)}(v) & \text{if } \sum_{j=1}^{i} M_{\sigma(j)}(v) + \sum_{j=i+1}^{|N|} m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(i)}(v) & \text{if } \sum_{j=1}^{i-1} M_{\sigma(j)}(v) + \sum_{j=i}^{|N|} m_{\sigma(j)}(v) \geq v(N), \\ v(N) - \sum_{j=1}^{i-1} M_{\sigma(j)}(v) - \sum_{j=i+1}^{|N|} m_{\sigma(j)}(v) & \text{otherwise} \end{cases}
$$

for every $i \in \{1, \ldots, |N|\}$.

It is readily seen that the core cover equals the convex hull of all larginals:

$$
CC(v) = \text{conv}\{\ell^\sigma(v) \mid \sigma \in \Pi(N)\}.
$$

An allocation rule f on a subclass $A \subset CA^N$ is a function $f : A \to \mathbb{R}^N$ assigning to each game $v \in A$ a payoff vector $f(v)$ in $\mathbb{R}^N$. This paper introduces a new type of allocation rule on CA^N based on bankruptcy situations.

A *bankruptcy situation* is a triple (N, E, d), often abbreviated to (E, d). N is a set of players, $E \geq 0$ is the estate which has to be divided among the players and $d \in \mathbb{R}^N_+$ is a vector of claims, where for $i \in N$, d_i represents player i's claim on the estate. It is assumed that the estate is not large enough to satisfy all claims, so

$$
E \leq \sum_{i \in N} d_i.
$$

We denote the class of all bankruptcy situations with player set N by BR^N.

One can associate a bankruptcy game $v_{E,d}$ to a bankruptcy problem $(E, d) \in BR^N$. The value of a coalition S is determined by the amount of E that is not claimed by $N \backslash S$, so for all $S \subset N$,

$$
v_{E,d}(S) = \max\Big\{0, E - \sum_{i \in N \backslash S} d_i\Big\}.
$$

This class of games is a proper subset of the class of compromise admissible games.

A bankruptcy rule f is a function $f : BR^N \to \mathbb{R}^N_+$ assigning to each bankruptcy situation $(E, d) \in BR^N$ a payoff vector $f(E, d) \in \mathbb{R}^N_+$, such that $\sum_{i \in N} f_i(E, d) = E$ and $f_i(E, d) \leq d_i$ for all $i \in N$.

Many bankruptcy rules have been proposed in the literature, of which some are listed below.

- **Constrained equal award rule (CEA):**

$$CEA_i(E, d) = \min\{\alpha, d_i\},$$

  with α such that $\sum_{i \in N} \min\{\alpha, d_i\} = E$.

- **Constrained equal loss rule (CEL):**

$$CEL_i(E, d) = \max\{0, d_i - \beta\},$$

  with β such that $\sum_{i \in N} \max\{0, d_i - \beta\} = E$.

- **Proportional rule (PROP):**

$$PROP_i(E, d) = \frac{d_i}{\sum_{j \in N} d_j} \cdot E.$$

- **Talmud rule (TAL):**

$$TAL(E, d) = \begin{cases} CEA(E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j \geq 2E, \\ d - CEA\left(\sum_{j \in N} d_j - E, \frac{1}{2}d\right) & \text{if } \sum_{j \in N} d_j < 2E. \end{cases}$$

- **Run-to-the-bank rule (RTB):**

$$RTB(E, d) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r^{\sigma}(E, d),$$

  where for $\sigma \in \Pi(N), j \in \{1, \ldots, n\}$,

$$r^{\sigma}_{\sigma(j)}(E, d) = \max\left\{\min\{d_{\sigma(j)}, E - \sum_{k=1}^{j-1} d_{\sigma(k)}\}, 0\right\}.$$

- **Adjusted proportional rule (APROP):**

$$APROP(E, d) = m(E, d) + PROP(E', d'),$$

  where $m_i(E, d) = \max\{E - \sum_{j \in N \setminus \{i\}} d_j, 0\}$, $E' = E - \sum_{i \in N} m_i(E, d)$ and for all $i \in N$, $d'_i = \min\{d_i - m_i(E, d), E'\}$.

Bankruptcy rules can be extended to allocation rules on the class of compromise admissible games in the following way.

Definition 2.1 *Let $v \in CA^N$ and let $f : BR^N \to \mathbb{R}^N$ be a bankruptcy rule. Then the* compromise extension *of f, $f^* : CA^N \to \mathbb{R}^N$ is defined by*

$$f^*(v) \;\; = \;\; m(v) + f\big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\big).$$

Note that because $v \in CA^N$, the bankruptcy situation to which f is applied is well-defined. Generally, if f is a bankruptcy rule and f^* is its compromise extension, then f^* will be efficient ($\sum_{i \in N} f_i^*(v) = v(N)$ for all $v \in CA^N$). Furthermore, for all bankruptcy rules mentioned earlier f^* is relatively invariant with respect to strategic equivalence (for $v, \hat{v} \in CA^N$ with $\hat{v} = kv + a, k > 0, a \in \mathbb{R}^N$, we have $f^*(\hat{v}) = kf^*(v) + a$). To prove this, it suffices to show that it satisfies homogeneity, ie, for all $k > 0$ and all $(E, d) \in BR^N$ it holds that $f(kE, kd) = kf(E, d)$.

It is immediately clear that the compromise value (or τ value) introduced by Tijs (1981) equals PROP*, since τ is the efficient convex combination of the vectors $M(v)$ and $m(v)$. The TAL* rule is considered in Quant et al. (2003), while González Díaz et al. (2003) study the APROP* rule. Note that for a game $v \in CA^N$ with $m(v) = 0$ and $M_i(v) \leq v(N)$ for all $i \in N$, we have $APROP^*(v) = PROP^*(v)$.

In the following example general compromise extensions are illustrated.

Example 2.1 *Let $v \in CA^N$ with $N = \{1, 2, 3\}$ be the game defined by*

S	1	2	3	12	13	23	N
$v(S)$	0	0	0	3	2	4	6

Then $M(v) = (2, 4, 3)$ and $m(v) = (0, 1, 0)$. The larginals are given in the table below.

σ	$\ell^\sigma(v)$
(123)	(2, 4, 0)
(132)	(2, 1, 3)
(213)	(2, 4, 0)
(231)	(0, 4, 2)
(312)	(2, 1, 3)
(321)	(0, 3, 3)

The RTB solution equals*

$$
\begin{aligned}
RTB^*(v) &= (0,1,0) + RTB(5,(2,3,3)) \\
&= (0,1,0) + \frac{1}{6}(8,11,11) \\
&= (\frac{8}{6}, \frac{17}{6}, \frac{11}{6})
\end{aligned}
$$

Note that for this game v, the RTB solution coincides with the average of the larginals. It is proved later that this is true in general. Since this game only has three players, the core coincides with the core cover and $n(v) = TAL^*(v) = (1,3,2)$. Furthermore $\tau(v) = (\frac{10}{8}, \frac{23}{8}, \frac{15}{8})$, $\phi(v) = (\frac{3}{2}, \frac{5}{2}, 2)$, $CEL^*(v) = (1,3,2)$ and $CEA^*(v) = (\frac{5}{3}, \frac{8}{3}, \frac{5}{3})$.*

The following example shows that a compromise solution which is symmetric, is not necessarily a core element, although the game (N, v) is convex.

Example 2.2 *Let $v \in CA^N$ with $N = \{1,2,3,4,5\}$ be the game defined by*

S	$\lvert S\rvert = 1$	12	other $\lvert S\rvert = 2$	123	124	125	other $\lvert S\rvert = 3$	$\lvert S\rvert = 4$	N
$v(S)$	0	6	0	6	6	6	2	8	14

In this game, players 1 and 2 are symmetric, as are players 3, 4 and 5. The payoff of a coalition S depends on the size of S and on whether $\{1,2\}$ is a part of S. The utopia vector equals $M(v) = (6,6,6,6,6)$ and the minimum right vector equals $m(v) = (0,0,0,0,0)$. Consider a symmetric bankruptcy rule f and its compromise extension f^*. Because all players are symmetric with respect to the value of $m(v)$ and $M(v)$ it holds that $f^*(v) = (\frac{14}{5}, \frac{14}{5}, \frac{14}{5}, \frac{14}{5}, \frac{14}{5})$. Since $f_1^*(v) + f_2^*(v) = \frac{28}{5} < 6$, $f^*(v) \notin C(v)$.

However, if (N, v) is a compromise stable game, then all compromise solutions are core elements.

3 Duality

In section 2, we define the compromise extension of a bankruptcy rule f by

$$
f^*(v) = m(v) + f\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).
$$

Another way to extend a bankruptcy rule to an allocation rule on CA^N is to take a dual approach. Instead of first giving each player his minimum right

and then dividing what is left, one could first give each player his utopia demand and take back the excess amount using f. This dual extension of a rule $f : BR^N \to \mathbb{R}^N$ is defined by

$$f^\star = M(v) - f\Big(\sum_{i \in N} M_i(v) - v(N), M(v) - m(v)\Big).$$

The *dual* of a bankruptcy rule f (cf. Aumann and Maschler (1985)), $\bar{f}$ is defined by

$$\bar{f}(E, d) = d - f(\sum_{i \in N} d_i - E, d)$$

and a rule is called *self-dual* if $f = \bar{f}$. Of the rules presented in the previous section, only $PROP$, TAL and RTB are self-dual.

As is stated in the following proposition, first taking the dual of f and then extending it yields the same rule as taking the dual extension of f.

Proposition 3.1 *Let $f : BR^N \to \mathbb{R}^N$ be a bankruptcy rule. Then $\bar{f}^*(v) = f^\star(v)$ for all $v \in CA^N$.*

Proof: Let $v \in CA^N$. Then

$$
\begin{aligned}
\bar{f}^*(v) &= m(v) + \bar{f}\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right) \\
&= m(v) + M(v) - m(v) - \\
&\quad f\left(\sum_{i \in N} M_i(v) - \sum_{i \in N} m_i(v) - (v(N) - \sum_{i \in N} m_i(v)), M(v) - m(v)\right) \\
&= M(v) - f\left(\sum_{i \in N} M_i(v) - v(N), M(v) - m(v)\right) \\
&= f^\star(v).
\end{aligned}
$$

$\square$

As a corollary, we obtain that if f is self-dual, then $f^\star = f^*$.

4   Run-to-the-bank rule

In this section we consider the compromise extension of the run-to-the-bank rule. We provide an interpretation in terms of larginals and a characterisation based on a consistency property.

RTB* is similar to the Shapley value in the sense that it is the average of all larginals (rather than marginals)[1]. This is shown in the following theorem.

Theorem 4.1 *Let* $v \in CA^N$, *then* $RTB^*(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(v)$.

Proof: Consider the game w defined by $w(S) = v(S) - \sum_{i \in S} m_i(v)$ for all $S \subset N$. Then $w \in CA^N$ and $\ell^\sigma(w) = \ell^\sigma(v) - m(v)$ for all $\sigma \in \Pi(N)$, $m(w) = 0$ and $M(w) = M(v) - m(v)$. Next, it is readily seen that $\ell^\sigma(w) = r^\sigma(w(N), M(w)))$ for all $\sigma \in \Pi(N)$ and hence,

$$
\begin{aligned}
RTB^*(w) &= RTB(w(N), M(w)) \\
&= \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r^\sigma(w(N), M(w)) \\
&= \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(w).
\end{aligned}
$$

Because RTB* is relative invariant with respect to strategic equivalence, we have

$$
\begin{aligned}
RTB^*(v) &= m(v) + RTB^*(w) \\
&= m(v) + \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(w) \\
&= m(v) + \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} [\ell^\sigma(v) - m(v)] \\
&= \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} \ell^\sigma(v).
\end{aligned}
$$

$\square$

Theorem 4.1 implies that if (N, v) is a compromise stable game, then $RTB^*(v)$ is the barycentre of $C(v)$. For example, in big boss games (cf. Muto, Nakayama, Potters, and Tijs (1988)) the RTB^* solution is the barycentre of the core and equals the nucleolus and the compromise value.

The RTB rule is the unique bankruptcy rule satisfying consistency. A bankruptcy rule f satisfies *consistency* (cf. O'Neill (1982)) if

$$
f_i(E, d) = \frac{1}{|N|} \left(\min\{d_i, E\} + \sum_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, E - \min\{d_j, E\}, d_{N \setminus \{j\}}) \right)
$$

---

[1]Note however that larginals do not satisfy additivity, since the vector m is not additive.

for all $(N, E, d) \in BR^N$ and all $i \in N$.

Note that in the definition of consistency, for each player $j \in N\backslash\{i\}$, the corresponding situation without j is again a well-defined bankruptcy situation. To extend this property to our framework of compromise admissible games, we have to consider a subclass, which is closed with respect to "sending one player away with his claim". The class $\mathcal{A}^N \subset CA^N$ consists of all TU-games $v \in CA^N$ such that for all $S \in 2^N$,

(i) $v(S) \geq 0$,

(ii) $v(S) + \sum_{k \in N\backslash S} m_k(v) \leq v(N)$.

We denote $\mathcal{A} = \bigcup_N \mathcal{A}^N$.

Property 4.1 (Consistency) *An allocation rule f on $\mathcal{A}$ satisfies consistency if for all N, for all $v \in \mathcal{A}^N$ and all $i \in N$ we have*

$$f_i(v) \;\;=\;\; \frac{1}{|N|} \min\Big\{ M_i(v), v(N) - \sum_{j \in N\backslash\{i\}} m_j(v) \Big\} +$$
$$\frac{1}{|N|} \sum_{j \in N\backslash\{i\}} \big(m_i(v) + f_i(v_j) \big),$$

where the game $v_j \in TU^{N\backslash\{j\}}$ is defined by $v_j(S) = \max\{ v(S \cup \{j\} - \sum_{k \in S} m_k(v) - M_j(v), 0 \}$ for all $S \subset N\backslash\{j\}$, $j \in N\backslash\{i\}$.

The game v_j in the previous property can be interpreted as if each player gets his minimal right and player j leaves the group and takes $M_j(v)$ if possible. The game v_j is again an element of $\mathcal{A}$, as is shown in the following lemma.

Lemma 4.1 *Let $v \in \mathcal{A}^N$, then $v_j \in \mathcal{A}^{N\backslash\{j\}}$ for all $j \in N$.*

Proof: Let $j \in N$. Then it is immediately clear from the definition of $v_j(S)$ that $v_j(S) \geq 0$ for all $S \subset N\backslash\{j\}$. Note that it follows from condition (ii) applied to (N, v) that for all $S \subset N\backslash\{j\}$ we have

$$v(S \cup \{j\}) - \sum_{i \in S} m_i(v) - M_j(v) \;\;\leq\;\; v(N) - \sum_{i \in N\backslash\{j\}} m_i(v) - M_j(v),$$

from which it easily follows that

$$v_j(S) \leq v_j(N\backslash\{j\}). \tag{1}$$

We assume that $v_j(N\backslash\{j\}) > 0$, since if $v_j(N\backslash\{j\}) = 0$, then $v_j(S) = 0$ for all $S \subset N\backslash\{j\}$ and $v_j \in \mathcal{A}^{N\backslash\{j\}}$ trivially. It remains to prove that $v_j \in CA^{N\backslash\{j\}}$ and that v_j satisfies condition (ii).

First we calculate $M(v_j)$ and $m(v_j)$. Let $i \in N$. If $v_j(N\backslash\{i,j\}) = 0$, then $M_i(v_j) = v_j(N\backslash\{j\})$. Otherwise,

$$
\begin{aligned}
M_i(v_j) &= v_j(N\backslash\{j\}) - v_j(N\backslash\{i,j\}) \\
&= v(N) - v(N\backslash\{i\}) - m_i(v) \\
&= M_i(v) - m_i(v).
\end{aligned}
$$

Combining the two cases, we obtain

$$
M_i(v_j) = \min\left\{M_i(v) - m_i(v), v_j(N\backslash\{j\})\right\}.
$$

We next show that $m_i(v_j) = 0$. To do so, we prove that for each $S \subset N\backslash\{j\}$ and $i \in S$ we have

$$
\rho_i^S \leq 0, \tag{2}
$$

where $\rho_i^S = v_j(S) - \sum_{k\in S\backslash\{i\}} M_k(v_j)$. Since $m_i(v_j) \geq v_j(\{i\}) \geq 0$ this proves that $m_i(v_j) = 0$.

Let $S \subset N\backslash\{j\}, i \in S$. If $v_j(S) = 0$, then (2) follows from the fact that $M_k(v_j) \geq 0$ for all $k \in N\backslash\{j\}$. Assume that $v_j(S) > 0$. We consider two cases.

Case 1: $M_k(v_j) = M_k(v) - m_k(v)$ for all $k \in S\backslash\{i\}$. Then

$$
\begin{aligned}
\rho_i^S &= v_j(S) - \sum_{k\in S\backslash\{i\}} \left(M_k(v) - m_k(v)\right) \\
&= v(S\cup\{j\}) - \sum_{k\in S} m_k(v) - M_j(v) - \sum_{k\in S\backslash\{i\}} \left(M_k(v) - m_k(v)\right) \\
&= v(S\cup\{j\}) - \sum_{k\in S\cup\{j\}\backslash\{i\}} M_k(v) - m_i(v) \\
&\leq 0,
\end{aligned}
$$

where the inequality follows from the definition of $m_i(v)$.

Case 2: There exists a $k \in S\backslash\{i\}$ with $M_k(v_j) = v_j(N\backslash\{j\})$. Then

$$
\rho_i^S = v_j(S) - v_j(N\backslash\{j\}) - \sum_{\ell\in S\backslash\{i,k\}} M_\ell(v_j) \leq 0,
$$

because of (1) and $M_\ell(v_j) \geq 0$ for all $\ell \in N\backslash\{j\}$.

Hence, $\rho_i^S \leq 0$ for all $S \subset N\backslash\{j\}, i \in S$ and $m(v_j) = 0$. Condition (ii) then directly follows from (1).

It remains to prove that $v_j \in CA^{N\backslash\{j\}}$. We already have $m(v_j) = 0 \leq M(v_j)$ and $\sum_{i\in N\backslash\{j\}} m_i(v_j) = 0 \leq v_j(N\backslash\{j\})$. Furthermore, for $k \in N\backslash\{j\}$ we have

$$
\begin{aligned}
v_j(N\backslash\{j\}) - \sum_{i\in N\backslash\{j\}} M_i(v_j) &= v_j(N\backslash\{j\}) - \sum_{i\in N\backslash\{j,k\}} M_i(v_j) - M_k(v_j) \\
&= \rho_k^{N\backslash\{j\}} - M_k(v_j) \\
&\leq 0.
\end{aligned}
$$

Hence condition (ii) is satisfied as well and $v_j \in \mathcal{A}^{N\backslash\{j\}}$. $\qquad\square$

The following theorem characterises RTB*.

Theorem 4.2 *RTB* is the unique rule on $\mathcal{A}$ satisfying consistency.*

Proof: Let f be an allocation rule on $\mathcal{A}$ satisfying consistency. Consistency uniquely determines the outcome of f for all one-player games and, by induction, every game in $\mathcal{A}$. Therefore, there can only be one rule satisfying consistency on $\mathcal{A}$. Hence, it suffices to show that RTB* satisfies consistency on $\mathcal{A}$.

Let $v \in \mathcal{A}^N$. From consistency of RTB it follows that for all $i \in N$ we have

$$
\begin{aligned}
RTB_i^*(v) &= m_i(v) + RTB_i\big(N, v(N) - \sum_{j\in N} m_j(v), M(v) - m(v)\big) \\
&= m_i(v) + \frac{1}{|N|}\Big[\min\{M_i(v) - m_i(v), v(N) - \sum_{j\in N} m_j(v)\} + \\
&\quad + \sum_{j\in N\backslash\{i\}} RTB_i(N\backslash\{j\}, E_j, d_{-j})\Big]
\end{aligned}
$$

with $E_j = v(N) - \sum_{j\in N} m_j(v) - \min\{M_j(v) - m_j(v), v(N) - \sum_{j\in N} m_j(v)\}$ and $d = M_{-j} - m_{-j} = (M_k(v) - m_k(v))_{k\in N\backslash\{j\}}$. Note that by construction $E_j = v_j(N\backslash\{j\})$. Then,

$$
\begin{aligned}
RTB_i^*(v) &= \frac{1}{|N|}\min\big\{M_i(v), v(N) - \sum_{j\in N\backslash\{i\}} m_j(v)\big\} + \\
&\quad + \frac{1}{|N|}\sum_{j\in N\backslash\{i\}} \Big(m_i(v) + RTB_i\big(N\backslash\{j\}, v_j(N\backslash\{j\}), M_{-j} - m_{-j}\big)\Big).
\end{aligned}
$$

Since $(M_{-j} - m_{-j})_k \geq \min\{v_j(N\backslash\{j\}), M_{-j} - m_{-j})\} = M_k(v_j)$ for all $k \in N\backslash\{j\}$, the truncation property[2] gives

$$
\begin{aligned}
RTB_i^*(v) &= \frac{1}{|N|} \min\Big\{M_i(v), v(N) - \sum_{j \in N\backslash\{i\}} m_j(v)\Big\} + \\
&\quad + \frac{1}{|N|} \sum_{j \in N\backslash\{i\}} \Big(m_i(v) + RTB_i\big(N\backslash\{j\}, v_j(N\backslash\{j\}), M(v_j)\big)\Big) \\
&= \frac{1}{|N|} \min\Big\{M_i(v), v(N) - \sum_{j \in N\backslash\{i\}} m_j(v)\Big\} + \\
&\quad + \frac{1}{|N|} \sum_{j \in N\backslash\{i\}} \big(m_i(v) + RTB_i^*(v_j)\big),
\end{aligned}
$$

where the last equality holds because $m(v_j) = 0$ for all $j \in N$. Hence, RTB^* satisfies consistency. $\square$

5   TAL-family rules

In this section we consider the compromise extension of the so-called TAL-family of rules (cf. Moreno-Ternero and Villar (2002)). This family of rules is parametrised by $\theta \in [0, 1]$ and defined by

$$
F_\theta(E, d) = \begin{cases} CEA(E, \theta d) & \text{if } E \leq \theta D, \\ \theta d + CEL\left(E - \theta D, (1-\theta)d\right) & \text{if } E > \theta D. \end{cases}
$$

Obviously, if $\theta = 1$, then F_θ coincides with CEA and if $\theta = 0$, then F_θ coincides with CEL. If $\theta = \frac{1}{2}$, then $\theta d + CEL(E - \theta D, (1-\theta)d) = d - CEA(D - E, \theta d)$ and F_θ coincides with TAL.

We are going to characterise F_θ^* on the class $CA_0^N = \{v \in CA^N \mid m(v) = 0\}$. For $v \in CA_0^N$, F_θ^* is given by

$$
F_\theta^*(v) = \begin{cases} CEA\left(v(N), \theta M(v)\right) & \text{if } v(N) \leq \theta \sum_{i \in N} M_i(v), \\ \theta M(v) + CEL\big(v(N) - \\ \quad \theta \sum_{i \in N} M_i(v), (1-\theta)M(v)\big) & \text{if } v(N) > \theta \sum_{i \in N} M_i(v). \end{cases}
$$

Since $F_\theta = \widetilde{F_{1-\theta}}$ for all $\theta \in [0, 1]$, we have that the compromise extension of CEA coincides with the dual compromise extension of CEL, ie,

---

[2] A bankruptcy rule f satisfies the truncation property if for all $(E, d) \in BR^N$ it holds that $f(E, d) = f(E, d')$, with $d_i' = \min\{E, d_i\}$ for all $i \in N$. The RTB rule satisfies the truncation property.

$CEA^* = CEL^\star$ and vice versa, $CEA^\star = CEL^*$.

Our characterisation is inspired by the definition of the nucleolus (cf. Schmeidler (1969)). The nucleolus of a game is the unique point in the imputation set for which it holds that the so-called excesses (which measure the dissatisfaction of each coalition with the proposed allocation) are as small and as equal as possible. Using this idea, we provide a characterisation of F_θ^* by replacing the excess function by the appropriate auxiliary function (depending on θ) and by considering elements of the core cover rather than the imputation set.

The phrase "as small and as equal as possible" in the previous paragraph is formalised by lexicographic optimisation. Let $x, y \in \mathbb{R}^t$. Then we say that x is lexicographically smaller than y, or $x \leq_L y$, if $x = y$ or if there exists an $s \in \{1, \ldots, t\}$ such that $x_k = y_k$ for all $k \in \{1, \ldots, s-1\}$ and $x_s < y_s$. For a finite set A, we denote $x \leq_L^* y$ with $x, y \in \mathbb{R}^A$, if $x' \leq_L y'$ where x' (y') is the vector in $\mathbb{R}^{|A|}$ containing the elements of x (y) in increasing order.

For $\theta \in [0, 1]$ and $v \in CA_0^N$, we define the function $g_\theta^v : CC(v) \to \mathbb{R}^N$ by

$$(g_\theta^v(x))_i = \begin{cases} \min\{x_i, \theta M_i(v)\} & \text{if } v(N) \leq \theta \sum_{i \in N} M_i(v), \\ \min\{(M_i(v) - x_i), \theta M_i(v)\} & \text{if } v(N) > \theta \sum_{i \in N} M_i(v) \end{cases} \quad (3)$$

for all $x \in CC(v), i \in N$.

Theorem 5.1 *Let $v \in CA_0^N$. Then $g_\theta^v(F_\theta^*(v)) \geq_L^* g_\theta^v(x)$ for all $x \in CC(v)$.*

Proof: Throughout this proof we say that for $x, y \in CC(v)$, x is *larger* than y if $g_\theta^v(x) \geq_L^* g_\theta^v(y)$ and we use the term *maximal* accordingly. We distinguish between two cases:

1. $v(N) \leq \theta \sum_{i \in N} M_i(v)$: First note that if $x \in CC(v)$ is such that $x_i > \theta M_i(v)$ for some $i \in N$, then because of the minimum expression in (3), there always exists another point in $CC(v)$ which is strictly larger. If $\frac{v(N)}{|N|} \leq \theta M_i(v)$ for all $i \in N$, then the point $(\frac{v(N)}{|N|}, \ldots, \frac{v(N)}{|N|}) \in CC(v)$ is obviously maximal, and equals $CEA(v(N), \theta M(v))$. Otherwise, the maximum is located at the boundary of $CC(v)$, where some of the players with the lowest claims receive their full claim and the others receive an equal amount. A similar construction as in the proof of Theorem 1 in Moreno-Ternero and Villar (2002) can be used to show that the resulting allocation equals $CEA(v(N), \theta M(v))$.

2. $v(N) > \theta \sum_{i \in N} M_i(v)$: In this case, as a result of the minimum operator, we can immediately conclude that a point $x \in CC(v)$ cannot be

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maximal if $x_i < \theta M_i(v)$ for some $i \in N$. The remainder of the proof is analogous to the first case.

From these two cases we conclude that the point $F_\theta^*(v)$ is maximal. $\quad\square$

A similar argument as in Maschler et al. (1992) can be used to show that the lexicographic maximum is unique and hence, Theorem 5.1 provides a characterisation of F_θ^* on CA_0^N.

References

Aumann, R.J. and M. Maschler (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, **36**, 195–213.

González Díaz, J., P. Borm, R. Hendrickx, and M. Quant (2003). A characterization of the τ-value. CentER DP 2003-88, Tilburg University, Tilburg, The Netherlands.

Maschler, M., J. Potters, and S. Tijs (1992). The general nucleolus and the reduced game property. *International Journal of Game Theory*, **21**, 85–106.

Moreno-Ternero, J. and A. Villar (2002). The TAL-family of rules for bankruptcy problems. Mimeo.

Muto, S., M. Nakayama, J. Potters, and S.H. Tijs (1988). On big boss games. *Economic Studies Quarterly*, **39**, 303–321.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, **2**, 345–371.

Quant, M., P. Borm, H. Reijnierse, and S. van Velzen (2003). Compromise stable TU-games. CentER DP 2003-55, Tilburg University, Tilburg, The Netherlands.

Schmeidler, D. (1969). The nucleolus of a characteristic function game. *SIAM Journal on Applied Mathematics*, **17**, 1163–1170.

Thomson, W. (2003). Axiomatic and game theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, **45**, 249–297.

Tijs, S.H. (1981). Bounds for the core and the τ-value. In O. Moeschlin and D. Pallaschke (Eds.), *Game Theory and Mathematical Economics*, pp. 123–132. North-Holland, Amsterdam.

Tijs, S.H. and F.A.S. Lipperts (1982). The hypercube and the core cover of n-person cooperative games. *Cahiers du Centre d'Études de Researche Opérationelle*, **24**, 27–37.