

Tilburg University

Using Localised Quadratic Functions on an Irregular Grid for Pricing High-Dimensional American Options

Berridge, S.J.; Schumacher, J.M.

Publication date:
2004

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):

Berridge, S. J., & Schumacher, J. M. (2004). *Using Localised Quadratic Functions on an Irregular Grid for Pricing High-Dimensional American Options*. (CentER Discussion Paper; Vol. 2004-20). Finance.

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Center



Discussion Paper

No. 2004–20

**USING LOCALISED QUADRATIC FUNCTIONS ON AN
IRREGULAR GRID FOR PRICING HIGH-DIMENSIONAL
AMERICAN OPTIONS**

By S.J. Berridge, J.M. Schumacher

February 2004

ISSN 0924-7815

Using localised quadratic functions on an irregular grid for pricing high-dimensional American options

S.J. Berridge* and J.M. Schumacher

Department of Econometrics and Operations Research
and Center for Economic Research (CentER),
Tilburg University, Postbus 90153, 5000 LE Tilburg, The Netherlands.
{s.j.berridge, j.m.schumacher}@uvt.nl

February 6, 2004

Abstract

We propose a method for pricing high-dimensional American options on an irregular grid; the method involves using quadratic functions to approximate the local effect of the Black-Scholes operator. Once such an approximation is known, one can solve the pricing problem by time stepping in an explicit or implicit manner. We study stability of the method in two dimensions, and find that the grid structure is important in providing a stable approximation to the operator.

Keywords: American options, high-dimensional problems, free boundary problems, optimal stopping, variational inequalities, numerical methods, unstructured mesh, Markov chain approximation

MSC 2000: 35R35, 60G40, 65D15, 90C33

JEL Codes: C15, C61, C63

*Research support by the Dutch National Science Foundation (NWO)

1 Introduction

Recent literature tends to support the widely-held belief that pricing American options in a high-dimensional setting is a highly nontrivial task. Methods proposed by Carrière [5], Longstaff and Schwartz [9], Rogers [11] and Haugh and Kogan [7] give good results in many situations, but are sensitive to the choice of basis. Berridge and Schumacher [2, 3, 4]) and Bally and Pagès [1] propose methods which do not require the choice of a basis.

The difficulty with primal methods in a high-dimensional problem is determination of the expected continuation value. This is essentially an integral with respect to the risk neutral process and the optimal stopping rule. The latter poses the most difficulty since it is itself a high-dimensional function which must be approximated.

In classical finite difference methods the expected continuation value is determined through an approximation to the operator on a regular grid. In the irregular grid setting this approximation becomes more difficult, mainly due to instabilities in the approximation. The construction of stable approximations will be investigated in this paper through using a local polynomial representation of the value function.

Once a stable approximation is found, one has access to finite difference techniques for time stepping, including the explicit, Crank-Nicolson and implicit methods. In general, these θ -methods lead to linear complementarity problems (LCPs), which can be solved using the projected successive overrelaxation (PSOR) method introduced by Cryer [6] or by linear programming, for example.

It is of interest to note the connection between the methods presented in this paper and the literature on mesh-free methods (for an overview see Liu [8]). In particular the method of moving least squares provides an approach to partial differential equation (PDE) solution which is similar to that found in this paper in that an irregular grid is used on which to calculate local quadratic approximations. The essential difference is that our approach does not use weak forms, and thus does not require integration of basis functions. It seems the method we present is more sensitive to the structure of the grid, producing unstable approximations in some circumstances.

The remainder of the paper is organised as follows. In Section 2 we introduce the setting, the problem and a review of irregular grid methodology. Section 3 introduces the local quadratic approximation method; Sections 4 and 5 present results using this method for regular and irregular grids respectively. Finally Section 6 presents conclusions and ideas for future research.

2 The pricing problem

2.1 Formulation

We consider an American option at time t on d underlying assets with values $X(s) = (X_1(s), \dots, X_d(s))'$ at time $s \in [t, T]$, payoff $\psi(X(s), s)$ and expiry T . The assets follow the diffusion

$$dX_i(s) = \mu_i(X(s), s)ds + \sigma_i(X(s), s)dW(s) \quad (1)$$

for $i = 1, \dots, d$ and where $X(t)$ is known, $dW(s)$ are the increments of a standard d -dimensional Brownian motion and the μ_i and σ_i are measurable with respect to the filtration generated by the Brownian motion.

The price of an American option, giving the long party the right to receive the payoff $\psi(X(s), s)$ at any time $s \in [t, T]$, is given in the primal formulation as

$$v(X(t), t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_{X(t)}^{\mathbb{Q}} \left(e^{-r(\tau-t)} \psi(X(\tau)) \right) \quad (2)$$

where \mathcal{T} is the set of stopping times on $[t, T]$ with respect to the natural filtration, the expectation is taken with respect to the risk-neutral measure \mathbb{Q} , and the initial value is $X(t)$.

This can be reformulated as the following complementarity problem

$$\begin{cases} \frac{\partial v}{\partial t} + \mathcal{L}v \leq 0 \\ v - \psi \geq 0 \\ \left(\frac{\partial v}{\partial t} + \mathcal{L}v \right) (v - \psi) = 0 \end{cases} \quad (3)$$

for $(x, s) \in \mathbb{R}^d \times [t, T]$ with the terminal condition $v(\cdot, T) \equiv \psi(\cdot, T)$. Here \mathcal{L} is the Black-Scholes operator implied by the diffusion.

2.2 Discretisation

One way to discretise the complementarity formulation is to sample the state space. This is the approach taken in finite difference methods, however the traditional grid approach is not suitable for high-dimensional problems due to the curse of dimensionality. We thus consider an irregular sampling of the state space on which to approximate the problem.

Suppose now that we are given some sampling of the state space, $\mathcal{X} = (x_1, \dots, x_n) \subset \mathbb{R}^d$. We do not concentrate on the properties of the sampling, but we may assume it is a sequence of low discrepancy in the sense of Niederreiter [10] or low

distortion in the sense of Bally and Pagès [1]. These quantities may be best measured in terms of the terminal distribution of the process, which in our setting is a multivariate normal distribution.

Having taken such a sample we approximate the complementarity problem (3) by the new complementarity problem

$$\begin{cases} \frac{dv}{dt} + Av \leq 0 \\ v - \psi \geq 0 \\ \left(\frac{dv}{dt} + Av\right)'(v - \psi) = 0 \end{cases} \quad (4)$$

where v is an n -vector and the i th component of ψ is $\psi(x_i, t)$. The matrix A should approximate \mathcal{L} on our grid \mathcal{X} in that

$$(Av(t))_i \simeq (\mathcal{L}v)(x_i, t). \quad (5)$$

Let us write v as $v = (v^{(1)} \dots v^{(n)})'$ where each $v^{(i)} : [t, T] \rightarrow \mathbb{R}$.

2.3 Nearest neighbours

Just as in traditional finite difference methods, a great deal of efficiency can be gained by only considering local interactions. We thus use nearest neighbour sets on which to construct local approximations to \mathcal{L} .

The k th-nearest neighbour function $N_{k,\mathcal{X}} : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ for some set of points \mathcal{X} is then defined as

$$N_{k,\mathcal{X}}(i) \triangleq \{j : \|x - x_i\| \leq \|x_j - x_i\| \text{ for exactly } k \text{ different } x, x \in \mathcal{X}\}.$$

Note that $N_{1,\mathcal{X}}(i) = i$, that is x_i is the nearest neighbour of x_i in this definition. Further let $\mathcal{N}_i = \{N_{j,\mathcal{X}}(i)\}_{j=1,\dots,k}$ be the ordered set of the k nearest neighbours for each i . For brevity we denote the j th nearest neighbour of point x_i as $x_i^{(j)} \equiv N_{j,\mathcal{X}}(i)$.

In addition to considering other points as neighbours, we may also allow boundary points to be neighbours. Thus in some situations we use the extended nearest neighbour function $N_{k,\mathcal{X}} : \{1, \dots, n\} \rightarrow \{1, \dots, n + 2d\}$ where the $2d$ extra points are projections of x_i onto the boundaries.

3 Methodology

3.1 Approximating the differential operator

We consider now the construction of a direct approximation to A with respect to a grid \mathcal{X} . Let us form this approximation by assuming that v is approximately

locally quadratic about each grid point x_i . This is justified since we know that v is almost everywhere continuously differentiable, and convenient since we wish to approximate the effect of the second order operator \mathcal{L} .

We write \mathcal{L} as

$$\mathcal{L} = \alpha_{0,0} + \sum_{j=1}^d \alpha_{j,0} \frac{\partial}{\partial x_j} + \sum_{j=1}^d \sum_{k=1}^j \alpha_{j,k} \frac{\partial^2}{\partial x_j \partial x_k}$$

for some $\alpha_{j,k} \in \mathbb{R}$.

Now let us introduce the quadratic interpolant $\bar{v}^{(i)} : \mathbb{R}^d \rightarrow \mathbb{R}$ at grid point x_i as

$$\bar{v}^{(i)}(x) = a_{0,0}^{(i)} + \sum_{j=1}^d a_{j,0}^{(i)} x_j + \sum_{j=1}^d \sum_{k=1}^j a_{j,k}^{(i)} x_j x_k \quad (6)$$

where the $a_{j,k}^{(i)}$ are chosen so that $\bar{v}^{(i)}(x_i^{(j)})(s) = v^{(j)}(t)$ for all $s \in [t, T]$ whenever $j \in \mathcal{N}_i$. By finding this interpolant for each i we have an approximation for $v(x, t)$ in a neighbourhood of all our grid points. Letting $\eta = \frac{1}{2}(d^2 + 3d + 2)$ (the number of parameters in each $\bar{v}^{(i)}$), we can determine a unique quadratic interpolant for each i . Replacing the subscript i on x by the vector index, the coefficients of the i th interpolant are

$$\begin{aligned} & a^{(i)}(t) \\ & \equiv \left(a_{0,0}^{(i)}, a_{1,0}^{(i)}, \dots, a_{d,0}^{(i)}, a_{1,1}^{(i)}, a_{2,1}^{(i)}, \dots, a_{d,d}^{(i)} \right)' \\ & = \begin{pmatrix} 1 & x_1^{(1)} & \dots & x_d^{(1)} & \left(x_1^{(1)}\right)^2 & x_2^{(1)} x_1^{(1)} & \dots & \left(x_d^{(1)}\right)^2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_1^{(\eta)} & \dots & x_d^{(\eta)} & \left(x_1^{(\eta)}\right)^2 & x_2^{(\eta)} x_1^{(\eta)} & \dots & \left(x_d^{(\eta)}\right)^2 \end{pmatrix}^{-1} \begin{pmatrix} v_1^{(i)} \\ \vdots \\ v_\eta^{(i)} \end{pmatrix} \\ & \equiv \left(M^{(i)} \right)^{-1} v^{(i)}(t) \end{aligned} \quad (7)$$

assuming that the matrix $M^{(i)}$ is nonsingular.

Now let us consider the effect of the operator \mathcal{L} on $\bar{v}^{(i)}$. First note that

$$\begin{aligned} \frac{\partial \bar{v}^{(i)}}{\partial x_j} &= a_{j,0}^{(i)} + \sum_{k=1}^d a_{j,k}^{(i)} (1 + \delta_{j,k}) x_k^{(i)} \\ \frac{\partial^2 \bar{v}^{(i)}}{\partial x_j \partial x_k} &= a_{j,k}^{(i)} (1 + \delta_{j,k}) \end{aligned}$$

where $\delta_{j,k}$ is the Kronecker delta function. Hence the effect is

$$\begin{aligned}
(\mathcal{L}\bar{v}^{(i)})(x^{(i)}) &= \alpha_{0,0}\bar{v}^{(i)} + \sum_{j=1}^d \alpha_{j,0} \frac{\partial \bar{v}^{(i)}}{\partial x_j} + \sum_{j=1}^d \sum_{k=1}^j \alpha_{j,k} \frac{\partial^2 \bar{v}^{(i)}}{\partial x_j \partial x_k} \\
&= \alpha_{0,0} \left\{ a_{0,0}^{(i)} + \sum_{j=1}^d a_{j,0}^{(i)} x_j^{(i)} + \sum_{j=1}^d \sum_{k=1}^j a_{j,k}^{(i)} x_j^{(i)} x_k^{(i)} \right\} \\
&+ \sum_{j=1}^d \alpha_{j,0} \left\{ a_{j,0}^{(i)} + \sum_{k=1}^d a_{j,k}^{(i)} (1 + \delta_{j,k}) x_k^{(i)} \right\} \\
&+ \sum_{j=1}^d \sum_{k=1}^j \alpha_{j,k} \left\{ a_{j,k}^{(i)} (1 + \delta_{j,k}) \right\} \\
&= a_{0,0}^{(i)} \{ \alpha_{0,0} \} + \sum_{j=1}^d a_{j,0}^{(i)} \{ \alpha_{0,0} x_j^{(i)} + \alpha_{j,0} \} \\
&+ \sum_{j=1}^d \sum_{k=1}^j a_{j,k}^{(i)} \left\{ \alpha_{0,0} x_j^{(i)} x_k^{(i)} + (1 + \delta_{j,k}) \alpha_{j,k} + (\alpha_{k,0} x_j^{(i)} + \alpha_{j,0} x_k^{(i)}) \right\}
\end{aligned}$$

or in matrix form

$$\begin{aligned}
(\mathcal{L}\bar{v}^{(i)})(x^{(i)}) &= \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} + \alpha_{0,0} x_1^{(i)} \\ \vdots \\ \alpha_{d,0} + \alpha_{0,0} x_d^{(i)} \\ \alpha_{0,0} (x_1^{(i)})^2 + 2\alpha_{1,0} x_1^{(i)} + 2\alpha_{1,1} \\ (\alpha_{0,0} x_2^{(i)} x_1^{(i)} + \alpha_{2,1}) + (\alpha_{1,0} x_2^{(i)} + \alpha_{2,0} x_1^{(i)}) \\ \vdots \\ \alpha_{0,0} (x_d^{(i)})^2 + 2\alpha_{d,d} + 2\alpha_{d,0} x_d^{(i)} \end{pmatrix}' \begin{pmatrix} a_{0,0}^{(i)} \\ a_{1,0}^{(i)} \\ \vdots \\ a_{d,0}^{(i)} \\ a_{1,1}^{(i)} \\ a_{2,1}^{(i)} \\ \vdots \\ a_{d,d}^{(i)} \end{pmatrix} \\
&= \beta(\alpha, x^{(i)})' a^{(i)}(t)
\end{aligned}$$

and substituting for a and evaluating at $x^{(i)}$ we have

$$\begin{aligned}
(\mathcal{L}\bar{v}^{(i)})(x^{(i)}) &= \beta(\alpha, x^{(i)})' (M^{(i)})^{-1} v^{(i)}(t) \\
&= A^{(i)} v^{(i)}(t)
\end{aligned} \tag{8}$$

where $A^{(i)}$ is a row vector of length η which we can think of as containing the elements of the i th row of some matrix A which defines the constrained system of ordinary differential equations (4).

In order to unify these n equations into the form of Equation 4, introduce an operator $\mathcal{S} : \mathbb{R}^\eta \rightarrow \mathbb{R}^n$ which stretches and rearranges the η -vector $v^{(i)}$ so that it becomes an n -vector with the entries placed in positions corresponding to the nearest neighbours of $x^{(i)}$.

$$\mathcal{S}_{\mathcal{X}}(x, i) \triangleq (0, \dots, 0, x_{j_1}, 0, \dots, 0, x_{j_2}, 0, \dots, 0, x_{j_\eta}, 0, \dots, 0)$$

where x_{j_k} is in the $N_{k, \mathcal{X}}(i)$ th position (in particular x_i is in the i th position). Now let

$$A \triangleq \begin{pmatrix} \mathcal{S}_{\mathcal{X}}(A^{(1)}, 1) \\ \vdots \\ \mathcal{S}_{\mathcal{X}}(A^{(n)}, n) \end{pmatrix}. \quad (9)$$

3.2 Weighted least squares

As a simple extension to the above one may consider a least squares regression using $\xi > \eta$ nearest neighbours. As a further extension one may consider weighting the points in the regression according to their distance from x_i .

Letting $M^{(i)}$ be the matrix in (7), but now with ξ rows. Furthermore let Λ denote a diagonal weighting matrix, yet to be specified. The least squares criterion then gives

$$a^{(i)}(t) = \left(M^{(i)'} \Lambda^2 M^{(i)} \right)^{-1} M^{(i)'} \Lambda v^{(i)}(t). \quad (10)$$

Now, as in (8), we have the approximation

$$A^{(i)} \triangleq \beta(\alpha, x^{(i)})' \left(M^{(i)'} \Lambda^2 M^{(i)} \right)^{-1} M^{(i)'} \Lambda \quad (11)$$

and we define A from the $A^{(i)}$ as in (9).

3.3 Time stepping

We can now discretise time using a θ -method. Let $t_k = \frac{kT}{K}$ for some δt , $k \in \{0, \dots, K\}$ and $\theta \in [0, 1]$, which can be thought of as the implicitness.

In the unconstrained case we form the finite difference equation

$$\frac{v_{k+1} - v_k}{\delta t} + (1 - \theta)Av_{k+1} + \theta Av_k = 0$$

where $v_k \equiv v(t_k)$. Then the set of equations we have to solve at each time step is

$$(I + \delta t(1 - \theta)A) v_{k+1} = (I - \delta t\theta A) v_k$$

where the initial conditions are given by $v_K = \psi(x, T)$.

In the constrained case, we obtain the complementarity problem

$$\begin{cases} (I + (1 - \theta)A\delta t) v_{k+1} - (I - \theta A\delta t) v_k \leq 0 \\ v_k - \psi \geq 0 \\ ((I + (1 - \theta)A\delta t) v_{k+1} - (I - \theta A\delta t) v_k)' (v_k - \psi) = 0 \end{cases} \quad (12)$$

which can be solved using Cryer's PSOR [6] or linear programming, for example.

3.4 Stability

The stability of the time stepping algorithm depends crucially on the eigenvalues of the matrix A . In particular, the stability of the time stepping method in the unconstrained case requires that real eigenvalues of A must be nonpositive. The mapping of eigenvalues from the matrix A to the time stepping matrix is shown in Figure 3.1.

Algebraically, stability can be guaranteed through the diagonal dominance condition

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad (13)$$

where $a_{ii} \leq 0$ for all i . This condition ensures the real parts of the eigenvalues of A are negative, a direct consequence of the Gershgorin disc theorem. Since row sums are zero, this also implies that off-diagonal entries must be nonnegative. The condition (13) is not necessary for stability however.

4 Application to regular grid

Before applying the discretisation method in its generality, we would first like to investigate its behaviour on regular grids. In particular it is of interest to compare the irregular grid method to standard finite difference methods. In the following analysis we consider only internal grid elements, and not boundary elements.

4.1 One dimension

The standard finite difference method on a regular grid approximates the first and second derivatives as

$$\frac{\partial v}{\partial x} \simeq \frac{v_{i+1} - v_{i-1}}{2\delta x}, \quad \frac{\partial^2 v}{\partial x^2} \simeq \frac{v_{i+1} - 2v_i + v_{i-1}}{\delta x^2} \quad (14)$$

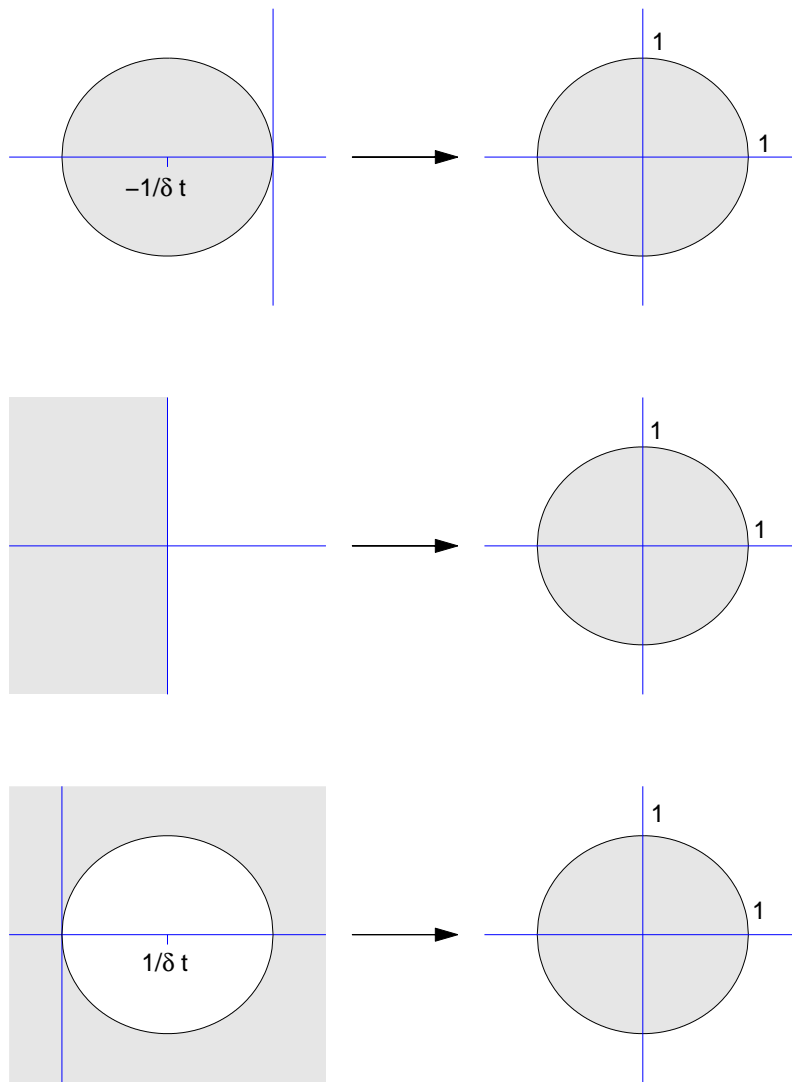


Figure 3.1: Mapping of eigenvalues from generator matrix A to time stepping matrix M in the explicit, Crank-Nicolson and implicit schemes respectively. The shaded areas correspond to stable eigenvalues.

thus leading to the standard finite difference matrix A which has nonzero components in row i of A_i where

$$A_i = \left(\frac{\alpha_{11}}{\delta^2} - \frac{\alpha_{10}}{2\delta}, \alpha_{00} - 2\frac{\alpha_{11}}{\delta^2}, \frac{\alpha_{11}}{\delta^2} + \frac{\alpha_{10}}{2\delta} \right). \quad (15)$$

This implies the approximation

$$(\mathcal{L}v)(x_i) \simeq A_i \begin{pmatrix} v_{i-1} \\ v_i \\ v_{i+1} \end{pmatrix}. \quad (16)$$

Consider now the irregular grid approximation introduced in the previous section. Since we are using a regular grid, let us choose our x to be multiples of some δ and let $x_i = i\delta$ for all i . Making use of three nearest neighbours, we then have

$$M^{(i)} = \begin{pmatrix} 1 & (i-1)\delta & (i-1)^2\delta^2 \\ 1 & i\delta & i^2\delta^2 \\ 1 & (i+1)\delta & (i+1)^2\delta^2 \end{pmatrix} \quad (17)$$

and thus our approximation to \mathcal{L} at x_i is

$$\begin{aligned} A^{(i)} &= \beta(\alpha, x^{(i)})' (M^{(i)})^{-1} \\ &= \begin{pmatrix} \alpha_{00} \\ \alpha_{10} + \alpha_{00}i\delta \\ \alpha_{00}i^2\delta^2 + 2\alpha_{10}i\delta + 2\alpha_{11} \end{pmatrix}' \begin{pmatrix} \frac{1}{2}i(i+1) & 1-i^2 & \frac{1}{2}i(i-1) \\ -\frac{1}{2\delta}(2i+1) & \frac{1}{2\delta}i & \frac{1}{2\delta}(1-2i) \\ \frac{1}{2\delta^2} & -\frac{1}{\delta^2} & \frac{1}{2\delta^2} \end{pmatrix} \\ &= \left(\frac{\alpha_{11}}{\delta^2} - \frac{\alpha_{10}}{2\delta}, \alpha_{00} - 2\frac{\alpha_{11}}{\delta^2}, \frac{\alpha_{11}}{\delta^2} + \frac{\alpha_{10}}{2\delta} \right). \end{aligned}$$

Hence we see that for a regular grid in one dimension the method of the previous section is equivalent to the standard finite difference method.

4.2 Two dimensions

We now compare the irregular grid method to the standard finite difference method on a regular grid in two dimensions. For the finite difference scheme, in addition to the derivative approximations given above for one dimension we introduce the standard cross-derivative approximation (see for example Wilmott [12])

$$\frac{\partial^2 v}{\partial x_1 \partial x_2} \simeq \frac{v_{i+1,j+1} - v_{i-1,j+1} - v_{i+1,j-1} + v_{i-1,j-1}}{4\delta x^2}. \quad (18)$$

7	5	9
2	1	3
6	4	8

Table 4.1: Assignment of indices of v to neighbours.

In this case the i th row of the approximation matrix is

$$A_i = \left(\alpha_{00} - \frac{2(\alpha_{11} + \alpha_{22})}{\delta^2}, -\frac{\alpha_{10}}{2\delta} + \frac{\alpha_{11}}{\delta^2}, \frac{\alpha_{10}}{2\delta} + \frac{\alpha_{11}}{\delta^2}, \dots \right. \\ \left. -\frac{\alpha_{20}}{2\delta} + \frac{\alpha_{22}}{\delta^2}, \frac{\alpha_{20}}{2\delta} + \frac{\alpha_{22}}{\delta^2}, \frac{\alpha_{12}}{4\delta^2}, -\frac{\alpha_{12}}{4\delta^2}, -\frac{\alpha_{12}}{4\delta^2}, \frac{\alpha_{12}}{4\delta^2} \right)$$

implying the approximation

$$(\mathcal{L}v)(x_i) \simeq A_i \begin{pmatrix} v_{i,j} \\ v_{i-1,j} \\ v_{i+1,j} \\ v_{i,j-1} \\ v_{i,j+1} \\ v_{i-1,j-1} \\ v_{i-1,j+1} \\ v_{i+1,j-1} \\ v_{i+1,j+1} \end{pmatrix}. \quad (19)$$

Pictorially, we present in Table 4.1 the way in which entries in the vector v correspond to the points around $v_{i,j}$.

Nine neighbours In order to compare the finite difference scheme to the irregular grid method, we consider nine nearest neighbours in the approximation. We thus use the least squares approach and denote $x_{i,j} = (i\delta, j\delta)$. The resulting ap-

proximation is

$$A^{(i)} = \begin{pmatrix} \frac{5}{9}\alpha_{00} - \frac{2(\alpha_{11} + \alpha_{22})}{3\delta^2} \\ \frac{2}{9}\alpha_{00} + \frac{2\alpha_{11} - \alpha_{10}\delta - 4\alpha_{22}}{6\delta^2} \\ \frac{2}{9}\alpha_{00} + \frac{2\alpha_{11} + \alpha_{10}\delta - 4\alpha_{22}}{6\delta^2} \\ \frac{2}{9}\alpha_{00} + \frac{2\alpha_{22} - \alpha_{20}\delta - 4\alpha_{11}}{6\delta^2} \\ \frac{2}{9}\alpha_{00} + \frac{2\alpha_{22} + \alpha_{20}\delta - 4\alpha_{11}}{6\delta^2} \\ -\frac{1}{9}\alpha_{00} + \frac{4(\alpha_{11} + \alpha_{22}) + 2(-\alpha_{20} - \alpha_{10})\delta + 3\alpha_{12}}{12\delta^2} \\ -\frac{1}{9}\alpha_{00} + \frac{4(\alpha_{11} + \alpha_{22}) + 2(\alpha_{20} - \alpha_{10})\delta - 3\alpha_{12}}{12\delta^2} \\ -\frac{1}{9}\alpha_{00} + \frac{4(\alpha_{11} + \alpha_{22}) + 2(-\alpha_{20} + \alpha_{10})\delta - 3\alpha_{12}}{12\delta^2} \\ -\frac{1}{9}\alpha_{00} + \frac{4(\alpha_{11} + \alpha_{22}) + 2(\alpha_{20} + \alpha_{10})\delta + 3\alpha_{12}}{12\delta^2} \end{pmatrix}. \quad (20)$$

One can view this approximation as a weighted finite difference method, with the weightings being given in Tables 4.2–4.7. Note that the only case in which the weights are the same is for α_{12} ; in all other cases the weights for the nine neighbour irregular grid scheme have less concentrated weights than in the finite difference method.

0	0	0
0	1	0
0	0	0

$\frac{1}{9}$	-1	2	-1
	2	5	2
	-1	2	-1

Table 4.2: Weights for α_{00} , for finite difference and irregular grid respectively.

Six neighbours Alternatively if we use six nearest neighbours, which is the minimum required to find a quadratic interpolant, we must make a choice between the diagonally located neighbours. Focusing on the points $x_{i,j}, x_{i,j-1}, x_{i,j+1}, x_{i-1,j}, x_{i+1,j}$

$\frac{1}{2\delta}$	0	0	0
	-1	0	1
	0	0	0

$\frac{1}{6\delta}$	-1	0	1
	-1	0	1
	-1	0	1

Table 4.3: Weights for α_{10} , for finite difference and irregular grid respectively.

$\frac{1}{2\delta}$	0	1	0
	0	0	0
	0	-1	0

$\frac{1}{6\delta}$	1	1	1
	0	0	0
	-1	-1	-1

Table 4.4: Weights for α_{20} , for finite difference and irregular grid respectively.

and $x_{i+1,j+1}$ we find that

$$A^{(i)} = \begin{pmatrix} \alpha_{00} - \frac{2(\alpha_{11} + \alpha_{22}) - \alpha_{12}}{\delta^2} \\ -\frac{\alpha_{10}}{2\delta} + \frac{\alpha_{11} - \alpha_{12}}{\delta^2} \\ \frac{\alpha_{10}}{2\delta} + \frac{\alpha_{11}}{\delta^2} \\ -\frac{\alpha_{20}}{2\delta} + \frac{\alpha_{22} - \alpha_{12}}{\delta^2} \\ \frac{\alpha_{20}}{2\delta} + \frac{\alpha_{22}}{\delta^2} \\ \frac{\alpha_{12}}{\delta^2} \end{pmatrix}. \quad (21)$$

We thus can find no equivalence between the irregular grid method and the standard finite difference method on a two dimensional regular grid, in contrast to the one dimensional case. One can however see the irregular grid method as a modified finite difference scheme in which different weights are used in the finite difference approximations.

We shall see in the Section 5 that the local quadratic approximation method leads to a stable scheme when six neighbours are used as in (21), but an unstable scheme when nine neighbours are used as in (20).

$\frac{1}{\delta^2}$	0	0	0
	1	-2	1
	0	0	0

$\frac{1}{3\delta^2}$	1	-2	1
	1	-2	1
	1	-2	1

Table 4.5: Weights for α_{11} , for finite difference and irregular grid respectively.

$\frac{1}{\delta^2}$	0	1	0
	0	-2	0
	0	1	0

$\frac{1}{3\delta^2}$	1	1	1
	-2	-2	-2
	1	1	1

Table 4.6: Weights for α_{22} , for finite difference and irregular grid respectively.

5 Experimental results

In order to solve the complementarity problem related to the American option pricing problem, we must find a stable and convergent method for time stepping. As outlined in Section 3.4, a necessary condition for obtaining a stable time stepping matrix is that the real eigenvalues of A are nonpositive.

Having ascertained that A will lead to a stable time stepping scheme, it also remains to check the convergence conditions for the LCP solution method.

5.1 Grids on the unit cube in \mathbb{R}^2

We first consider grids on the region $\Omega = [0, 1]^2 \subset \mathbb{R}^2$. This is a natural place to start since we know that finite difference schemes using regular grids lead to stable A matrices in this case. Boundaries are allowed to be neighbours in this setting.

We present in Figures 5.1–5.7 point sets of size approximately 500 and the eigenvalues of the corresponding A matrix obtained using the irregular grid method (plotted in the complex plane).

In the case of the Sobol’ grid one can in practise observe which points are causing instability by examining the eigenvector corresponding to the eigenvalues with positive real part. The instability can often be resolved by changing the neighbour configuration, in particular so that neighbours are well-distributed about the point. No systematic method was found to perform this stabilisation however.

Note that the least squares scheme using nine neighbours was not stable, despite the fact that it uses the same points as in the regular finite difference scheme. It was also found that none of the grids considered above leads to a stable A when

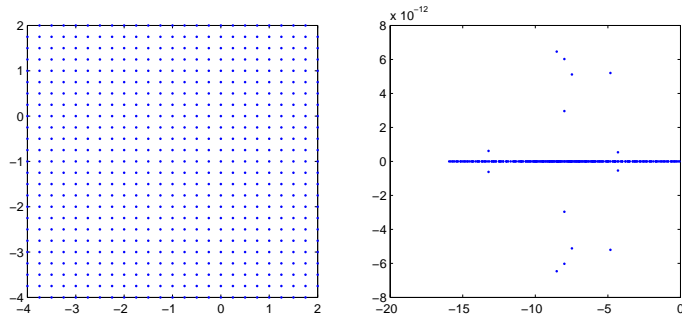


Figure 5.1: Points and eigenvalues of A for regular grid with 529 interior points and using 6 neighbours - stable. Note that the vertical scale in the eigenvalue plot is close to zero.

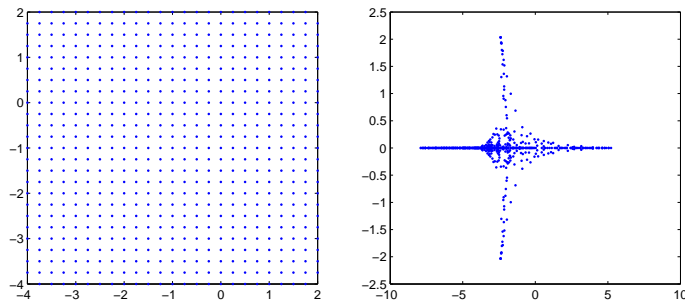


Figure 5.2: Points and eigenvalues of A for regular grid with 529 interior points and using 9 neighbours - unstable.

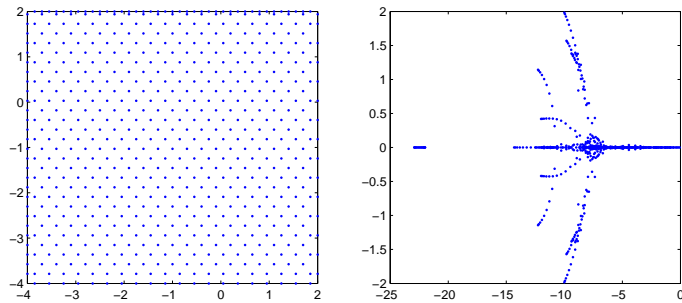


Figure 5.3: Points and eigenvalues of A for triangular grid with 546 interior points and using 6 neighbours - stable.

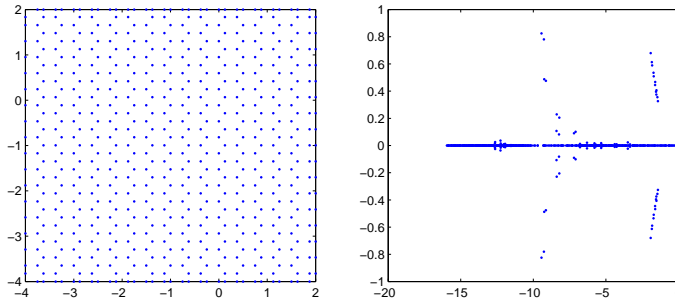


Figure 5.4: Points and eigenvalues of A for hexagonal grid with 512 interior points and using 6 neighbours - stable.

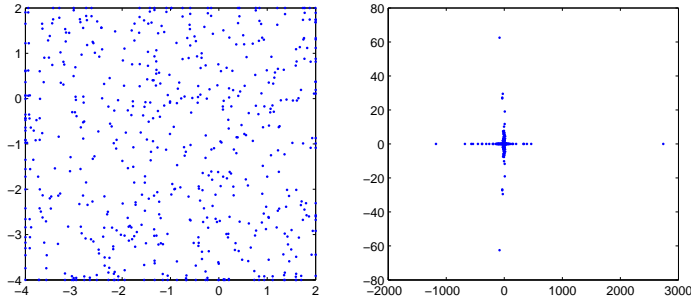


Figure 5.5: Points and eigenvalues of A for uniform pseudo-random grid with 500 interior points and using 6 neighbours - unstable.

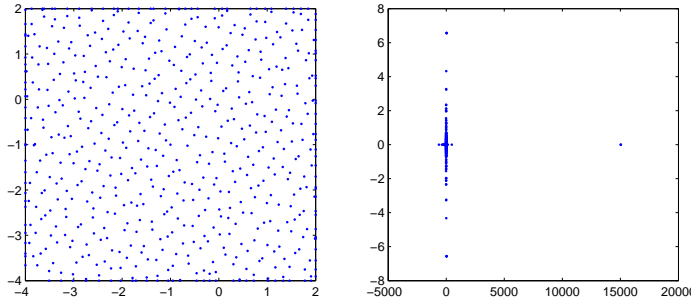


Figure 5.6: Points and eigenvalues of A for Sobol' grid with 500 interior points and using 6 neighbours - unstable.

$\frac{1}{4\delta^2}$	-1	0	1
	0	0	0
	1	0	-1

$\frac{1}{4\delta^2}$	-1	0	1
	0	0	0
	1	0	-1

Table 4.7: Weights for α_{12} , for finite difference and irregular grid respectively.

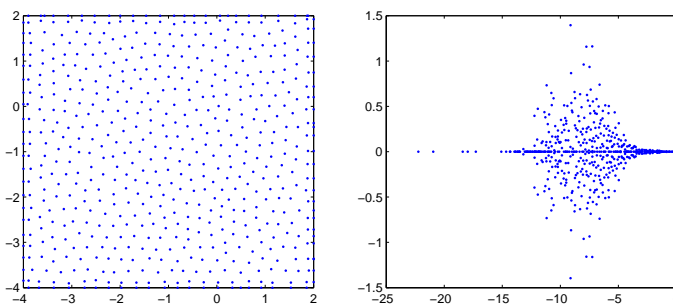


Figure 5.7: Points and eigenvalues of A for low distortion grid with 500 interior points and using 6 neighbours - stable.

considering local least squares fits over 7 neighbours.

5.2 Normally distributed grids in \mathbb{R}^2

Grids constructed on the unit cube are not optimal for the application under consideration. The main reason for this is that they are not representative of the regions of space that are likely to be visited by the stochastic process introduced in (1).

A further problem with using a grid on the unit cube is that neither Dirichlet nor Neumann boundary conditions are known. Approximate conditions may be specified, thus adding an extra source of error to the computed solution.

A natural grid choice for our problem would be one that is related to the process, and in this case a normally distributed grid seems appropriate. This also alleviates the second problem mentioned, in that a normally distributed grid covers \mathbb{R}^d asymptotically, and so boundary conditions have a vanishing effect on the solution.

Given the previous results of grids on the unit cube, we choose to focus on low distortion grids in the following. We generate a low distortion grid with respect to the standard normal density in \mathbb{R}^2 , and apply the irregular grid method to it to obtain the matrix A . We then examine the eigenvalues of A .

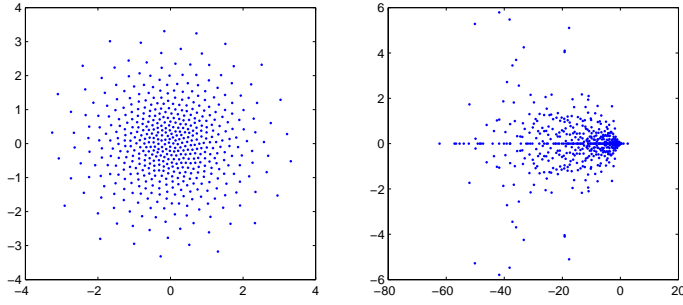


Figure 5.8: Points and eigenvalues of A for low distortion normal grid with 500 points and using $r = \infty$ and 6 neighbours - unstable, $\max(\Re(\lambda)) = 2.54$.

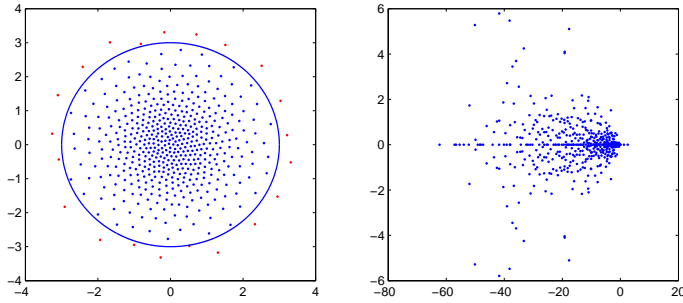


Figure 5.9: Points and eigenvalues of A for low distortion normal grid with 500 points and using $r = 3.0$ and 6 neighbours - unstable, $\max(\Re(\lambda)) = 2.51$.

Since there are no natural boundary points when dealing with normally distributed grids, we choose a radius outside which all points are considered to be boundary points. In particular we consider the radii $r = \infty, 3.0, 2.5$ and 2.0 . In general one expects the eigenvalues to be different for different choices of r , and in particular that A should only be stable for smaller choices of r .

The results are presented in Figures 5.8 – 5.11. The maximum real parts of the eigenvalues are given in the captions. In this case the transition to stability occurs when r is between 3.0 and 2.5.

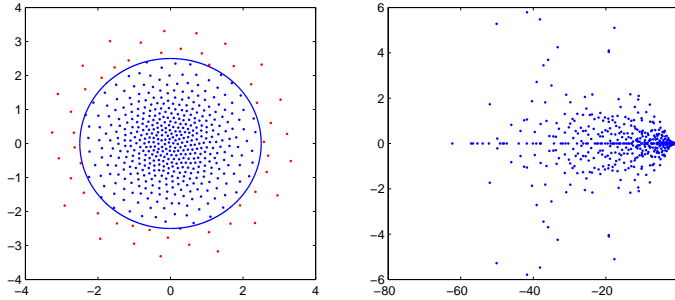


Figure 5.10: Points and eigenvalues of A for low distortion normal grid with 500 points and using $r = 2.5$ and 6 neighbours - stable, $\max(\Re(\lambda)) = 0$.

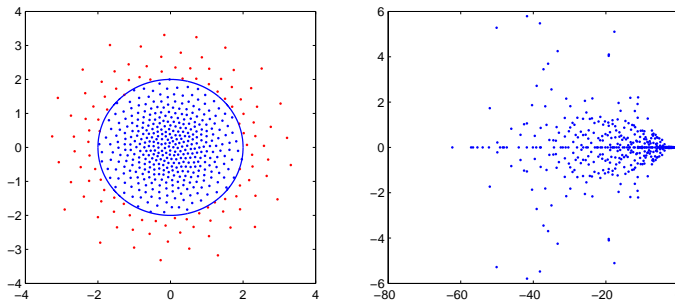


Figure 5.11: Points and eigenvalues of A for low distortion normal grid with 500 points and using $r = 2.0$ and 6 neighbours - stable, $\max(\Re(\lambda)) = 0$.

6 Conclusions

We presented a method for approximating a differential operator on an irregular grid. The method uses local polynomial interpolants to construct derivative approximations. We analysed the stability of the operator approximation using different grid and boundary configurations.

In one dimension, we showed that the method is equivalent to the standard finite difference method.

Our main finding in two dimensions was that grids with a regular local structure are more likely to lead to stable approximations. Thus square, triangular and hexagonal grids lead to stable approximations, but pseudo-random and quasi-random grids did not. Low distortion grids as used in Bally and Pagès [1] were also found to lead to stable approximations. We were able to induce stability in the case of a low discrepancy grid by altering neighbour configurations so that the neighbours were more uniformly distributed in an angular sense. For a low distortion grid adapted to the normal distribution, we found that the approximations constructed were unstable when the boundary radius was too large, but stable for smaller radii.

Summarised, this study indicates that instabilities in the approximation are a consequence of the local roughness of points in the grid, and of boundary effects.

The hurdle in extending this work to higher dimensions is stability, in particular more research is required either in the direction of determining sufficient conditions for stability on arbitrary grids or towards modifications of the approximation method.

The literature on mesh-free methods provides one solution in the form of the moving least squares method, where one attempts to integrate a particular interpolant, as opposed to working with derivatives of the interpolants. The moving least squares method seems to be less susceptible to instabilities than the present method.

References

- [1] Vlad Bally and Gilles Pagès. A quantization algorithm for solving multidimensional discrete-time optimal stopping problems. *Bernoulli*, 6:1003–1049, 2003.
- [2] S.J. Berridge and J.M. Schumacher. An irregular grid approach for pricing high-dimensional American options. *CentER Discussion Paper*, 99, 2002.
- [3] S.J. Berridge and J.M. Schumacher. An irregular grid method for solving high-dimensional problems in finance. In Sloot et al., editor, *International*

Conference on Computational Science (2), pages 510–519. Springer, Berlin, 2002.

- [4] S.J. Berridge and J.M. Schumacher. An irregular grid method for high-dimensional free-boundary problems in finance. Forthcoming, *Future Generation Computer Systems: Special Issue on Numerical Methods*, 2003.
- [5] Jacques Carrière. Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance: Mathematics and Economics*, 19:19–30, 1996.
- [6] Colin W. Cryer. The solution of a quadratic programming problem using systematic overrelaxation. *SIAM Journal of Control*, 9:385–392, 1971.
- [7] Martin B. Haugh and Leonid Kogan. Pricing American options: a duality approach. *Working Paper, The Wharton School, University of Pennsylvania*, 2001.
- [8] G.R. Liu. *Mesh Free Methods: Moving Beyond the Finite Element Method*. CRC Press, Boca Raton, 2002.
- [9] Francis A. Longstaff and Eduardo S. Schwartz. Valuing American options by simulation: a simple least squares approach. *Review of Financial Studies*, 14:113–147, 2001.
- [10] Harald Niederreiter. *Random Number Generation and Quasi-Monte Carlo Methods*. SIAM, 1992.
- [11] L.C.G. Rogers. Monte Carlo valuation of American options. *Mathematical Finance*, 12:271–286, 2002.
- [12] Paul Wilmott. *Derivatives: The Theory and Practice of Financial Engineering*. John Wiley & Sons, 1998.