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A geometric characterisation of the compromise value

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Abstract

In this paper, we characterise the compromise value of a game as the barycentre of the edges of its core cover. For this, we introduce the τ^* value, which extends the adjusted proportional rule for bankruptcy situations and coincides with the compromise value on a large class of games.

1 Introduction

Most game-theoretic solution concepts that have been proposed in the literature are defined on the basis of or characterised by properties. These properties are usually formulated in terms of individual payoffs and reflect notions like monotonicity and rationality. For some values, there exist additional characterisations in terms of geometry. The best-known example is the Shapley value (Shapley (1953)), which is the barycentre of the extreme points of the Weber set (taking multiplicities into account).

For some classes of games, there exist nice geometric expressions for the *compromise* or τ *value* (Tijs (1981)). In particular, the compromise value is the barycentre of the core cover in big boss games (Muto et al. (1988)) and 1-convex games (Driessen (1988)).

In this paper, we extend the *APROP* rule for bankruptcy situations (Curiel et al. (1987)) to the whole class of compromise admissible (or quasi-balanced)

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games (cf. Tijs (1981)). This extended rule, which we call τ^*, turns out to be the barycentre of the *edges* of the core cover (taking multiplicities into account), which is our main result. Since this rule coincides with the compromise value if, after normalising such that each player's minimal right equals zero, each player's utopia payoff is at most the value of the grand coalition, our main result immediately provides a characterisation of the compromise value on this class of games.

This paper is organised as follows. In section 2, we extend the $APROP$ rule and define the barycentre ζ of the edges of the core cover. In section 3, we state our main result and give an overview of the proof, which consists of six main steps. Finally, in section 4, we prove our main result.

2 The τ^* value

A *transferable utility* or *TU game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to every coalition $S \subset N$ a payoff $v(S)$. By convention, $v(\emptyset) = 0$.

Following Tijs and Lipperts (1982), the *utopia vector* of a game (N, v), $M(v) \in \mathbb{R}^N$, is defined by

$$M_i(v) = v(N) - v(N \backslash \{i\})$$

for all $i \in N$. The *minimum right vector* $m_i(v) \in \mathbb{R}^N$ is defined by

$$m_i(v) = \max_{S \subset N, i \in S} \{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\}$$

for all $i \in N$.

The *core cover* of a game (N, v) consists of those allocations of $v(N)$ according to which every player receives at most his utopia payoff and at least his minimal right:

$$CC(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N),\ m(v) \leq x \leq M(v)\}.$$

A game is called *compromise admissible* if it has a nonempty core cover. We denote the class of compromise admissible games with player set N by CA^N. A *rule* on a subclass $A \subset CA^N$ is a function $f : A \to \mathbb{R}^N$ assigning to each $v \in A$ a payoff vector $f(v) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(v) = v(N)$.

The *compromise* or τ value (Tijs (1981)) is the rule on CA^N defined as the point on the line segment between $m(v)$ and $M(v)$ that is efficient with respect to $v(N)$:

$$\tau(v) = \lambda m(v) + (1-\lambda)M(v),$$

where $\lambda \in [0,1]$ is such that $\sum_{i\in N} \tau_i = v(N)$.

A *bankruptcy* situation is a triple (N, E, c), where $E \geq 0$ is the estate to be divided and $c \in \mathbb{R}^N_+$ with $\sum_{i\in N} c_i \geq E$ is the vector of claims. The corresponding *bankruptcy game* $(N, v_{E,c})$ is defined by $v_{E,c}(S) = \max\{E - \sum_{i\in N\setminus S} c_i, 0\}$ for all $S \subset N$. We denote the class of bankruptcy situations with player set N by BR^N. The class of corresponding games is a proper subclass of CA^N. A *bankruptcy rule* is a function $f : BR^N \to \mathbb{R}^N$ assigning to every bankruptcy situation $(N, E, c) \in BR^N$ a payoff vector $f(E, c) \in \mathbb{R}^N_+$ such that $\sum_{i\in N} f_i(E, c) = E$.

In the literature, many bankruptcy rules have been proposed. One interesting question is how these can be extended in a natural way to the whole class of compromise admissible games. In this paper, we consider the *proportional rule* and the *adjusted proportional rule* (cf. Curiel et al. (1987)). The proportional rule $PROP$ simply divides the estate proportional to the claims:

$$PROP_i(E, c) = \frac{c_i}{\sum_{j\in N} c_j} E$$

for all $(N, E, c) \in BR^N$ and $i \in N$. The adjusted proportional rule $APROP$ first gives each player $i \in N$ his minimal right $m_i(E, c) = \max\{E - \sum_{j\in N\setminus\{i\}} c_j, 0\}$ and the remainder is divided using the proportional rule, where each player's claim is truncated to the estate left:

$$APROP(E, c) = m(E, c) + PROP(E', c'),$$

where $E' = E - \sum_{i\in N} m_i(E, c)$ and for all $i \in N$, $c'_i = \min\{c_i - m_i(E, c), E'\}$.

The compromise value can be seen as an extension of the $PROP$ rule:

$$\tau(v) = m(v) + PROP(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)).$$

Note that it follows from the definition of compromise admissibility that the argument of $PROP$ is indeed a bankruptcy situation.

Similarly, we can extend the $APROP$ rule:

$$\tau^*(v) = m(v) + APROP(v(N) - \sum_{i\in N} m_i(v), M(v) - m(v)).$$

To simplify the expression for τ^*, we show that the minimum rights in the associated bankruptcy situation equal 0. Let $v \in CA^N$, $E = v(N) - \sum_{i\in N} m_i(v)$, $c = M(v) - m(v)$ and let $i \in N$. Then

$$
\begin{aligned}
E - \sum_{j \in N \setminus \{i\}} c_j &= v(N) - \sum_{i \in N} m_i(v) - \sum_{j \in N \setminus \{i\}} (M_j(v) - m_j(v)) \\
&= v(N) - \sum_{j \in N \setminus \{i\}} M_j(v) - m_i(v) \\
&\leq 0,
\end{aligned}
$$

since $m_i(v) \geq v(N) - \sum_{j \in N \setminus \{i\}} M_j(v)$. Hence, $m_i(E, c) = \max\{E - \sum_{j \in N \setminus \{i\}} c_j, 0\} = 0$. As a result, we have

$$
\tau^*(v) = m(v) + PROP(E', c') \tag{2.1}
$$

with $E' = v(N) - \sum_{i \in N} m_i(v)$ and $c'_i = \min\{M_i(v) - m_i(v), E'\}$ for all $i \in N$.

It follows that for a game $v \in CA^N$ with $M_i(v) - m_i(v) \leq v(N) - \sum_{j \in N} m_j(v)$ for all $i \in N$, τ^* coincides with the compromise value τ.

The extended rule τ^* turns out to be a kind of barycentre of the core cover, which is the main result of our paper. To define this barycentre rule ζ, we need to introduce some more concepts. A *permutation* on N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(p)$ denotes the player at position p, and consequently, $\sigma^{-1}(i)$ denotes the position of player i. The set of all permutations on N is denoted by $\Pi(N)$. $\sigma^{i,j}$ denotes the permutation obtained from σ by switching players i and j. Two permutations σ and $\sigma^{\sigma(p),\sigma(p+1)}$ are called *permutation neighbours*. The set of permutation neighbours of σ is denoted by $\Pi^\sigma(N)$.

The core cover is a polytope whose extreme points are called *larginal vectors* or *larginals*. The larginal $\ell^\sigma \in \mathbb{R}^N$ corresponding to order $\sigma \in \Pi(N)$ (cf. Quant et al. (2003)) is defined by

$$
\ell^\sigma_{\sigma(p)}(v) = \begin{cases} M_{\sigma(p)}(v) \text{ if } \sum_{k=1}^{p} M_{\sigma(k)}(v) + \sum_{k=p+1}^{n} m_{\sigma(k)}(v) \leq v(N), \\ m_{\sigma(p)}(v) \text{ if } \sum_{k=1}^{p-1} M_{\sigma(k)}(v) + \sum_{k=p}^{n} m_{\sigma(k)}(v) > v(N), \\ v(N) - \sum_{k=1}^{p-1} M_{\sigma(k)}(v) - \sum_{k=1}^{n} m_{\sigma(k)}(v) \text{ otherwise} \end{cases}
$$

for all $p \in \{1, \ldots, n\}$.

Note that two permutations that are neighbours yield larginals which are adjacent extreme points of the core cover (possibly coinciding), which we also call permutation neighbours.

We define the ζ rule as a weighted average of the larginal vectors:[1]

[1] In the degenerate case where $M = m$, the core cover consists of a single point, which we define to be ζ. Otherwise, there are at least two different larginals and the denominator is positive.

$$\zeta(v) = \frac{\sum_{\sigma\in\Pi(N)} w^\sigma(v)\ell^\sigma(v)}{\sum_{\sigma\in\Pi(N)} w^\sigma(v)}, \tag{2.2}$$

where

$$w^\sigma(v) = \frac{1}{\sqrt{2}} \sum_{\tau\in\Pi^\sigma(N)} d(\ell^\sigma(v), \ell^\tau(v))$$

equals the sum of the Euclidean distances between $\ell^\sigma(v)$ and all its permutation neighbours, divided by the common factor $\sqrt{2}$. The ζ value can be viewed as the barycentre of the edges of the core cover, taking the multiplicities into account.

To simplify the proofs later on, we first show that both τ^* and ζ satisfy the properties (SEQ) and (RTRUNC). Two games v and $\hat{v}$ are called strategically equivalent if there exists a real number $k > 0$ and a vector $a \in \mathbb{R}^N$ such that for all $S \subset N$,

$$\hat{v}(S) = kv(S) + a(S), \tag{2.3}$$

with $a(S) = \sum_{i\in S} a_i$. A function $f : CA^N \to \mathbb{R}^N$ is *relatively invariant wrt strategic equivalence* (SEQ) if for all $v, \hat{v} \in CA^N$ such that (2.3) holds for some $k > 0, a \in \mathbb{R}^N$, we have

$$f(\hat{v}) = kf(v) + a.$$

It is well-known that the utopia vector M and the minimum right vector m both satisfy (SEQ).

Proposition 2.1 *The τ^* rule and the ζ rule satisfy (SEQ).*

Proof: The proof for τ^* is straightforward and therefore omitted.
It readily follows from (SEQ) of m and M that ℓ^σ also satisfies (SEQ) for all $\sigma \in \Pi(N)$. Let $v, \hat{v} \in CA^N$ and let $k > 0, a \in \mathbb{R}^N$ such that (2.3) holds and let $\sigma \in \Pi(N)$. Then

$$\begin{aligned} w^\sigma(\hat{v}) &= \frac{1}{\sqrt{2}} \sum_{\tau\in\Pi^\sigma(N)} d(\ell^\sigma(\hat{v}), \ell^\tau(\hat{v})) \\ &= \frac{1}{\sqrt{2}} \sum_{\tau\in\Pi^\sigma(N)} d(k\ell^\sigma(v) + a, k\ell^\tau(v) + a) \\ &= k\frac{1}{\sqrt{2}} \sum_{\tau\in\Pi^\sigma(N)} d(\ell^\sigma(v), \ell^\tau(v)) \\ &= kw^\sigma(v) \end{aligned}$$

Hence,

$$\begin{aligned}\zeta(\hat{v}) &= \frac{\sum_{\sigma\in\Pi(N)} w^\sigma(\hat{v})\ell^\sigma(\hat{v})}{\sum_{\sigma\in\Pi(N)} w^\sigma(\hat{v})} \\ &= \frac{k\sum_{\sigma\in\Pi(N)} w^\sigma(v)[k\ell^\sigma(v)+a]}{k\sum_{\sigma\in\Pi(N)} w^\sigma(v)} \\ &= k\zeta(v)+a.\end{aligned}$$

And so, ζ satisfies (SEQ). □

A rule $f : CA^N \to \mathbb{R}^N$ satisfies the *restricted truncation property* (RTRUNC) if for all $v \in CA^N$ with $m(v) = 0$ it holds that for all $\hat{v} \in CA^N$ with $\hat{v}(N) = v(N)$, $m(\hat{v}) = 0$ and $M_i(\hat{v}) = \min\{M_i(v), v(N)\}$ we have $f(\hat{v}) = f(v)$. The idea behind (RTRUNC) is that if a player's utopia value (or, in bankruptcy terms, his claim) is higher than the value of the grand coalition (the estate), his payoff according to f should not by influenced by truncating this claim.

Proposition 2.2 *The τ^* rule and the ζ rule satisfy (RTRUNC).*

Proof: Let $v \in CA^N$ with $m(v) = 0$. Then (2.1) reduces to

$$\tau^*(v) = PROP(v(N), (\min\{M_i(v), v(N)\})_{i\in N}).$$

From this it immediately follows that τ^* satisfies (RTRUNC).
For the ζ rule, it suffices to note that truncating the utopia vector has no influence on the larginal vectors. □

3 Main result

In this section, we present our main result: equality between τ^* and ζ on CA^N. After dealing with some simple cases, we present a six step outline of the proof, which we will give in the next section.

Theorem 3.1 *Let $v \in CA^N$. Then*

$$\tau^*(v) = \zeta(v).$$

As a result of Proposition 2.1, it suffices to show equality for every game $v \in CA^N$ with $m(v) = 0$. Next, we can use Proposition 2.2 and conclude that we have to show that for all $v \in CA^N$ with $m(v) = 0$ and $M_i(v) \leq v(N)$[2] for all $i \in N$ we have[3]

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

In case there are only two players, equality between τ^* and ζ follows from $M_1(v) = M_2(v) = v(N)$.

If $M_i(v) = 0$ for a player $i \in N$, then we have $\tau_i^*(v) = \zeta_i(v) = 0$. Furthermore, for each $\sigma \in \Pi(N)$, the payoff to the players in $N \backslash \{i\}$ according to $\ell^\sigma(v)$ equals their payoff in the situation without player i[4] according to the larginal corresponding to the restricted permutation $\sigma_{N\backslash\{i\}} \in \Pi(N \backslash \{i\})$, defined by $\sigma^{-1}_{N\backslash\{i\}}(h) < \sigma^{-1}_{N\backslash\{i\}}(j) \Leftrightarrow \sigma^{-1}(h) < \sigma^{-1}(j)$ for all $h, j \in N \backslash \{i\}$. It is readily verified that also the total weight of each larginal (taking multiplicities into account) is the same in the game with and without player i. Hence, we can omit player i from the game and establish equality between τ^* and ζ for the remaining players.[5]

We establish equality between τ^* and ζ by combining the permutations in the numerator and denominator in (2.2) into so-called *chains*. In the denominator, these chains allow us to combine terms in such a way that the total weight can be expressed as a simple function of $M(v)$. In the numerator, we construct an iterative procedure to find an expression for the weighted larginals, in which the chains allow us to keep track of changes that occur from one iteration to the next.

The proof of Theorem 3.1 consists of six steps:

1. We first find an expression for the weight of each permutation. This is done by introducing the concept of *pivot* and classifying each permutation in terms of its pivot and its neighbours' pivots.

2. Using the concept of pivot, we introduce *chains*, which constitute a partition

[2]Note that the condition $M_i(v) \leq v(N)$ is necessary and sufficient to have $M_i(v) = \max_{\sigma \in \Pi(N)} \ell_i^\sigma(v)$. Only in this case, the utopia vector can be reconstructed from the core cover.

[3]The denominator is zero if and only if $M(v) = 0 (= m(v))$. In this degenerate case equality between τ^* and ζ is trivial and we therefore assume $M(v) \gneqq 0$.

[4]Ie, the situation with player set $N \backslash \{i\}$, utopia vector $M_{N\backslash\{i\}}(v)$ and the same amount $v(N)$ to be distributed.

[5]Geometrically, the core cover, which lies in the hyperplane $M_i(v) = 0$, is projected onto a space which is one dimension lower.

of the set of all permutations. The results of the previous step are then used to compute the total weight of each chain.

3. We define a family of auxiliary functions f^{ij} and g^{ij}, which are used to show that each player "belongs" to the same number of chains. As a result, we use our expression of the previous step to compute the total of all the weights, ie, the denominator in (2.2).

4. In the numerator, we partition the set of chains on the basis of the first player in each permutation. Within each part, we compute the total weighted payoff to all the players. For the first player, this total weighted payoff can easily be computed.

5. The expression for the payoffs to the other players is proved using an iterative argument, varying the utopia vector while keeping $v(N)$ constant. We start with a utopia vector for which our expression is trivial and lower this vector step by step until we reach $M(v)$. In each step of the iteration, (generically) only two chains change and using this, we show that the total weighted payoff to each player who is not first does not change as function of the utopia vector.

6. Combining the previous three steps, we derive an expression for ζ and show that this equals τ^*.

4 Proof of main result

Throughout this section, let $v \in CA^N$ be such that $|N| \geq 3$, $m(v) = 0$, $M(v) > 0$, and $v(N) \geq M_i(v)$ for all $i \in N$. For Theorem 3.1 is suffices to show that for this v we have

$$PROP(v(N), M(v)) = \frac{\sum_{\sigma \in \Pi(N)} w^\sigma(v) \ell^\sigma(v)}{\sum_{\sigma \in \Pi(N)} w^\sigma(v)}.$$

Since v is fixed for the rest of the section, we will suppress it as argument and write M rather than $M(v)$, etc. The weight $w^\sigma(v)$ will be denoted by $w(\sigma)$.

Step 1: pivots

Let $\sigma \in \Pi(N)$. Player $\sigma(p)$ with $p \geq 2$ is called the *pivot* in ℓ^σ if $\ell^\sigma_{\sigma(p-1)} = M_{\sigma(p-1)}$, $\ell^\sigma_{\sigma(p)} > 0$ and $\ell^\sigma_{\sigma(p+1)} = 0$. The pivot of a larginal is the player who gets a lower

amount according this larginal if the amount $v(N)$ is decreased slightly. In the boundary case where $M_{\sigma(1)} = v(N)$, $v(N)$ cannot be decreased without violating the condition $M_{\sigma(1)} \leq v(N)$; in this case, player $\sigma(2)$ is defined to be the pivot, being the player who gets a higher amount if $v(N)$ is increased slightly. Note that $m = 0$ implies that $\sum_{j \in N \setminus \{i\}} M_j \geq v(N)$ and hence, player $\sigma(n)$ can never be the pivot.

In the following example, we introduce a game which we will use throughout the paper to illustrate the various concepts.

Example 4.1 Consider the game (N, v) with $N = \{1, \ldots, 5\}$, $v(N) = 10$ and $M = (5, 7, 1, 3, 4)$. For this game, we have $\tau^* = \zeta = \frac{1}{2}M$. Take σ_1 to be the identity permutation. Then

$$\ell^{\sigma_1} = (5, 5, 0, 0, 0)$$

and player 2 is the pivot. $\triangleleft$

For a permutation $\sigma \in \Pi(N)$, we define p_σ to be the position at which the pivot[6] is located. We define $\sigma^L = \sigma^{\sigma(p_\sigma - 1), \sigma(p_\sigma)}$ to be the *left neighbour* of σ and $\sigma^R = \sigma^{\sigma(p_\sigma), \sigma(p_\sigma + 1)}$ to be the right neighbour of σ. It follows from the definition of pivot that the left and right neighbours of ℓ^σ are the only two permutation neighbours that can give rise to a larginal different from ℓ^σ.

Recall that the weight of ℓ^σ, $w(\sigma)$, equals the sum of the (Euclidean) distances between ℓ^σ and all its permutation neighbours. In line with the previous paragraph, we only have to take the left and right neighbours into account. So,

$$w(\sigma) = \frac{1}{\sqrt{2}} \left[d(\ell^\sigma, \ell^{\sigma^L}) + d(\ell^\sigma, \ell^{\sigma^R}) \right].$$

We classify the larginals into four categories, depending on the pivot in the left and right neighbours. Let $\sigma = (\ldots, h, i, j, \ldots)$ be a permutation with pivot i. Then the four types are given in the following table:

Type	Pivot in σ^L	Pivot in σ	Pivot in σ^R
PPP	i	i	i
$-PP$	h	i	i
$PP-$	i	i	j
$-P-$	h	i	j

[6] As with *neighbours*, we use the term *pivot* as property of a permutation as well as the corresponding larginal.

We can now determine the weight of each larginal, depending on its type. Take $\sigma \in \Pi(N)$ to be the identity permutation and assume that ℓ^σ is of type $PP-$ and has pivot i. Then

$$\begin{aligned}
\ell^\sigma &= (M_1, \ldots, M_{i-2}, M_{i-1}, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0), \\
\ell^{\sigma^L} &= (M_1, \ldots, M_{i-2}, 0, v(N) - \sum_{j=1}^{i-2} M_j, 0, \ldots, 0), \\
\ell^{\sigma^R} &= (M_1, \ldots, M_{i-2}, M_{i-1}, 0, v(N) - \sum_{j=1}^{i-1} M_j, 0, \ldots, 0).
\end{aligned}$$

So,

$$\begin{aligned}
d(\ell^\sigma, \ell^{\sigma^L}) &= \sqrt{2M_{i-1}^2} = \sqrt{2}M_{i-1}, \\
d(\ell^\sigma, \ell^{\sigma^R}) &= \sqrt{2(v(N) - \sum_{j=1}^{i-1} M_j)^2} = \sqrt{2}(v(N) - \sum_{j=1}^{i-1} M_j), \\
w(\sigma) &= (v(N) - \sum_{j=1}^{i-2} M_j).
\end{aligned}$$

Doing these calculations for all types and arbitrary $\sigma \in \Pi(N)$, we obtain the following weights:

Type	$w(\sigma)$
PPP	$M_{\sigma(p_\sigma-1)} + M_{\sigma(p_\sigma+1)}$
$-PP$	$\sum_{k=1}^{p_\sigma+1} M_{\sigma(k)} - v(N)$
$PP-$	$v(N) - \sum_{k=1}^{p_\sigma-2} M_{\sigma(k)}$
$-P-$	$M_{\sigma(p_\sigma)}$

Example 4.2 With σ_1 the identity permutation, we have (the player with $\hat{}$ is the pivot):

$$\begin{array}{ll}
\sigma_1 = (1, \hat{2}, 3, 4, 5) & \ell^{\sigma_1} = (5, 5, 0, 0, 0) \\
\sigma_1^L = (2, \hat{1}, 3, 4, 5) & \ell^{\sigma_1^L} = (3, 7, 0, 0, 0) \\
\sigma_1^R = (1, 3, \hat{2}, 4, 5) & \ell^{\sigma_1^R} = (5, 4, 1, 0, 0)
\end{array}$$

So, ℓ^{σ_1} is of type $-PP$. The weight of σ_1 equals

$$\begin{aligned}
w(\sigma_1) &= \frac{1}{\sqrt{2}} \left[d(\sigma_1, \sigma_1^L) + d(\sigma_1, \sigma_1^R) \right] \\
&= 2 + 1 \\
&= 3.
\end{aligned}$$

Indeed, we have that $w(\sigma_1) = \sum_{k=1}^{p_{\sigma_1}+1} M_{\sigma_1(k)} - v(N) = M_1 + M_2 + M_3 - v(N) = 5 + 7 + 1 - 10 = 3$, as the table shows. $\triangleleft$

Step 2: chains

A *chain* of length q and with pivot i is a set of q permutations $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ such that

- $(\sigma_m)^R = \sigma_{m+1}$ for all $m \in \{1, \ldots, q-1\}$,
- i is pivot in σ_m for all $m \in \{1, \ldots, q\}$,
- i is not pivot in σ_1^L and σ_q^R.

If $q = 1$, then it follows from the definitions of the four types that σ_1 is of type $-P-$. If $q > 1$, then σ_1 is of type $-PP$, σ_m is of type PPP for all $m \in \{2, \ldots, q-1\}$, and σ_q is of type $PP-$. Observe that the set of all chains, which we denote by $\mathcal{C}$, constitutes a partition of the set of permutations $\Pi(N)$.

Denoting by σ^* the permutation on the $n-1$ players obtained from σ by removing the pivot, we characterise the chains in the following lemma.

Lemma 4.1 *$\sigma_1 \in \Pi(N)$ and $\sigma_2 \in \Pi(N)$ are in the same chain if and only if $\sigma_1^* = \sigma_2^*$.*

Given the weights of the larginal vectors, depending on the type, we can easily compute the weight of a chain Γ, which is simply defined as the total weight of its elements, ie, $w(\Gamma) = \sum_{\sigma \in \Gamma} w(\sigma)$.

Lemma 4.2 *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$. Then*

$$w(\Gamma) = \sum_{k=p_{\sigma_1}}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)}.$$

Proof: Denoting $p = p_{\sigma_1}$, we have (for $q \geq 5$; for smaller chains the proof is similar):

$$
\begin{array}{ccccccccc}
w(\sigma_1) & = & \sum_{k=1}^{p-1} M_{\sigma_1(k)} & - & v(N) & + & M_{\sigma_1(p)} & + & M_{\sigma_1(p+1)} \\
w(\sigma_2) & = & & & & + & M_{\sigma_1(p+1)} & + & M_{\sigma_1(p+2)} \\
w(\sigma_3) & = & & & & + & M_{\sigma_1(p+2)} & + & M_{\sigma_1(p+3)} \\
\vdots & = & & & & & \vdots & & \vdots \\
w(\sigma_{q-1}) & = & & & & + & M_{\sigma_1(p+q-2)} & + & M_{\sigma_1(p+q-1)} \\
w(\sigma_q) & = & -\sum_{k=1}^{p-1} M_{\sigma_1(k)} & + & v(N) & - & \sum_{k=p+1}^{p+q-2} M_{\sigma_1(k)} & & + \\
\hline
w(\Gamma) & = & & & & & \sum_{k=p}^{p+q-1} M_{\sigma_1(k)} & &
\end{array}
$$

□

We say that player $i \in N$ *belongs* to chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\}$ if $i \in \{\sigma_1(p_{\sigma_1}), \ldots, \sigma_1(p_{\sigma_1} + q - 1)\}$, ie, if his position is not constant throughout the chain. Note that if a player does belong to a chain, his utopia payoff contributes to its weight. We define $C(i)$ to be the set of chains to which i belongs. By $P(i) \subset C(i)$ we denote the set of chains in which i is pivot and by $\bar{P}(i) = C(i) \backslash P(i)$ its complement. For each $\Lambda \in \bar{P}(i)$, we denote the permutation in Λ in which i is immediately before the pivot by λ_{bi} and the permutation in which i is immediately after the pivot by λ_{ai}.

Example 4.3 Since player 2 is not the pivot in σ_1^L, σ_1 is the first permutation of a chain. This chain Γ consists of σ_1, $\sigma_2 = \sigma_1^R$ and $\sigma_3 = \sigma_2^R$, all having player 2 as pivot. In line of Lemma 4.1, we have $\sigma_1^* = \sigma_2^* = \sigma_3^* = (1, 3, 4, 5)$. Players 2, 3 and 4 belong to Γ and $w(\Gamma) = M_2 + M_3 + M_4 = 11$. ◁

Step 3: denominator

In this step, we derive an expression for the denominator in ζ. We do this by showing that each player belongs to the same number of chains, ie,

$$|C(i)| = |C(j)| \tag{4.1}$$

for all $i, j \in N$. If $M_i = M_j$, then this is trivial, so throughout this step, let $i, j \in N$ be such that $M_i > M_j$. We prove only one part of (4.1):

$$|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|. \tag{4.2}$$

The proof of the reverse inequality goes along similar lines, as will be indicated later on.

An immediate consequence of Lemma 4.4 is that it follows from $M_i > M_j$ that $|P(i)| \geq |P(j)|$ and $|\bar{P}(j)| \geq |\bar{P}(i)|$. We establish (4.2) in Proposition 4.5 by partnering all the chains in $P(j)$ to some of the chains in $P(i)$ and partnering all the chains $\bar{P}(i)$ to some of the chains in $\bar{P}(j)$. We then show that for every chain in $\bar{P}(j)$ which has no partner in $\bar{P}(i)$, we can find a chain in $P(i)$ which has no partner in $P(j)$.

To partner the various chains, we define two functions. First, we define f^{ij}:

$$\begin{array}{rcl} P(j) & \stackrel{f^{ij}}{\rightarrow} & P(i) \\ \Delta & \mapsto & f(\Delta) = \Lambda \end{array}$$

where $\Delta = \{\delta_1, \ldots, \delta_q\}$ and Λ is the chain to which $\delta_1^{i,j}$ belongs. Note that the function f^{ij} is well-defined: since $M_i > M_j$, player i is indeed the pivot in $\delta_1^{i,j}$ and hence, in Λ.

Similarly, we define the function g^{ij}:

$$\begin{array}{rcl} \bar{P}(i) & \stackrel{g^{ij}}{\rightarrow} & \bar{P}(j) \\ \Lambda & \mapsto & g(\Lambda) = \Delta \end{array}$$

where for all $\Lambda \in \bar{P}(i)$, Δ is the chain containing $\lambda_{bi}^{i,j}$.[7]

In the following lemma, we show that g^{ij} is well-defined, ie, that the chain Δ thus constructed is indeed an element of the range of g^{ij}, $\bar{P}(j)$.

Lemma 4.3 *The function g^{ij} is well-defined.*

Proof: Denote the pivot player in λ_{bi} (and hence, λ_{ai}) by h. Observe that as a result of $M_i > M_j$, player h cannot coincide with j. Distinguish between the following two cases:

- i is before j in λ_{bi}:

$$\begin{array}{ll} \lambda_{ai} = (\ldots, \hat{h}, i, \ldots, j, \ldots) & \lambda_{ai}^{i,j} = (\ldots, \hat{h}, j, \ldots, i, \ldots) \\ \lambda_{bi} = (\ldots, i, \hat{h}, \ldots, j, \ldots) & \lambda_{bi}^{i,j} = (\ldots, j, \hat{h}, \ldots, i, \ldots) \end{array}$$

[7] By $\lambda_{bi}^{i,j}$ we mean $(\lambda_{bi})^{i,j}$, ie, the permutation which is obtained by switching i and j in the permutation in Λ where i is immediately before the pivot.

Since h is pivot in λ_{ai}, it immediately follows that h is also pivot in $\lambda_{ai}^{i,j}$. Player j cannot be the pivot in $\lambda_{bi}^{i,j}$, because i is before the pivot in λ_{bi} and $M_i > M_j$. Combining this with the fact that h is pivot in $\lambda_{ai}^{i,j}$, h is also pivot in $\lambda_{bi}^{i,j}$. But then $\lambda_{ai}^{i,j}$ belongs to the same chain Δ as $\lambda_{bi}^{i,j}$. From this, $\Delta \in C(j)$, and because j is not the pivot in Δ, $\Delta \in \bar{P}(j)$.

- j is before i in λ_{bi}:

$$\begin{array}{ll} \lambda_{ai} = (\ldots, j, \ldots, \hat{h}, i, \ldots) & \lambda_{ai}^{i,j} = (\ldots, i, \ldots, \hat{h}, j, \ldots) \\ \lambda_{bi} = (\ldots, j, \ldots, i, \hat{h}, \ldots) & \lambda_{bi}^{i,j} = (\ldots, i, \ldots, j, \hat{h}, \ldots) \end{array}$$

Since h is pivot in λ_{bi}, we immediately have that h is pivot in $\lambda_{bi}^{i,j}$. Because of this, the pivot in $\lambda_{ai}^{i,j}$ cannot be before h. It can also not be after h, because h is pivot in λ_{ai} and $M_i > M_j$. By the same argument as in the first case, $\Delta \in \bar{P}(j)$.

From these two cases, we conclude that g^{ij} is well-defined. □

For our partnering argument to hold, we need that the functions f^{ij} and g^{ij} are injective. This is shown in the following lemma.

Lemma 4.4 *The functions f^{ij} and g^{ij} are injective.*

Proof: To see that f^{ij} is injective, let $\Delta, \tilde{\Delta} \in P(j)$ be such that $f^{ij}(\Delta) = f^{ij}(\tilde{\Delta})$. By construction, i is pivot in both $f^{ij}(\Delta)$ and $f^{ij}(\tilde{\Delta})$, so i is pivot in both $\delta_1^{i,j}$ and $\tilde{\delta}_1^{i,j}$. Since by assumption these permutations are in the same chain, by Lemma 4.1 we have $(\delta_1^{i,j})^* = (\tilde{\delta}_1^{i,j})^*$. But since j is pivot in both δ_1 and $\tilde{\delta}_1$, it follows that $\delta_1^* = \tilde{\delta}_1^*$. So, δ_1 and $\tilde{\delta}_1$ are in the same chain and $\Delta = \tilde{\Delta}$.
For injectivity of g^{ij}, let $\Lambda, \tilde{\Lambda} \in \bar{P}(i)$ be such that $g^{ij}(\Lambda) = g^{ij}(\tilde{\Lambda})$. Then $\lambda_{bi}^{i,j}$ and $\tilde{\lambda}_{bi}^{i,j}$ are in the same chain. By the same arguments as used before, j is just before the pivot in both permutations and hence, $\lambda_{bi}^{i,j} = \tilde{\lambda}_{bi}^{i,j}$. From this, we conclude $\lambda_{bi} = \tilde{\lambda}_{bi}$ and $\Lambda = \tilde{\Lambda}$. □

From Lemma 4.4, we conclude

$$|P(j)| \leq |P(i)|$$

and

$|\bar{P}(i)| \leq |\bar{P}(j)|$.

With these inequalities, we can now apply our partnering argument to prove that each player belongs to the same number of chains.

Proposition 4.5 *Let $i, j \in N$. Then $|C(i)| = |C(j)|$.*

Proof: If $M_i = M_j$, then the statement is trivial. Hence, assume without loss of generality that $M_i > M_j$.
We only show (4.2). Let $\Delta \in \bar{P}(j)$ be such that there exists no $\Lambda \in \bar{P}(i)$ with $g^{ij}(\Lambda) = \Delta$. Denote the pivot in Δ by h and distinguish between the following three cases:

- $h \neq i$ and i is after j in δ_{bj}:

$$\begin{array}{ll} \delta_{aj} = (\ldots, \hat{h}, j, \ldots, i, \ldots) & \delta_{aj}^{i,j} = (\ldots, \hat{h}, i, \ldots, j, \ldots) \\ \delta_{bj} = (\ldots, j, \hat{h}, \ldots, i, \ldots) & \delta_{bj}^{i,j} = (\ldots, \hat{i}, h, \ldots, j, \ldots) \end{array}$$

  Of course, h is also the pivot in $\delta_{aj}^{i,j}$. If h were the pivot in $\delta_{bj}^{i,j}$, then $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$ would belong to the same chain $\Lambda \in \bar{P}(i)$. But then $g^{ij}(\Lambda) = \Delta$, which is impossible by assumption. Since $M_i > M_j$, player i must be the pivot in $\delta_{bj}^{i,j}$. The chain to which $\delta_{bj}^{i,j}$ belongs cannot be an image under f^{ij}, since it is obtained by switching i and j in a permutation in which j is not the pivot. Furthermore, two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing $\delta_{bj}^{i,j}$ and $\tilde{\delta}_{bj}^{i,j}$, because both permutations are of type $PP-$ or $-P-$ and there can be only one such permutation in a chain.

- $h \neq i$ and i is before j in δ_{bj}:

$$\begin{array}{ll} \delta_{aj} = (\ldots, i, \ldots, \hat{h}, j, \ldots) & \delta_{aj}^{i,j} = (\ldots, j, \ldots, h, \hat{i}, \ldots) \\ \delta_{bj} = (\ldots, i, \ldots, j, \hat{h}, \ldots) & \delta_{bj}^{i,j} = (\ldots, j, \ldots, i, \hat{h}, \ldots) \end{array}$$

  Again, it easily follows that h is pivot in $\delta_{bj}^{i,j}$ and by the same argument as in the first case, i must be pivot in $\delta_{aj}^{i,j}$. Also, the chain to which $\delta_{aj}^{i,j}$ belongs cannot be an image under f^{ij} and two different starting chains $\Delta, \tilde{\Delta} \in \bar{P}(j)$ cannot give rise to one single chain containing $\delta_{aj}^{i,j}$ and $\tilde{\delta}_{aj}^{i,j}$, because both permutations

are of type $-PP$ or $-P-$. Moreover, the chains constructed in this second case, containing $\delta_{aj}^{i,j}$, must differ from the chains constructed in the first case, containing $\delta_{bj}^{i,j}$, as a result of the relative positions of h and j.

- $h = i$:

$$\begin{array}{ll} \delta_{aj} = (\ldots, \hat{i}, j, \ldots) & \delta_{aj}^{i,j} = (\ldots, j, \hat{i}, \ldots) \\ \delta_{bj} = (\ldots, j, \hat{i}, \ldots) & \delta_{bj}^{i,j} = (\ldots, \hat{i}, j, \ldots) \end{array}$$

  Obviously, i is pivot in both $\delta_{aj}^{i,j}$ and $\delta_{bj}^{i,j}$. So, these two permutations belong to the same chain $\Lambda \in P(i)$. Again Λ cannot be an image under f^{ij}, and since $\Lambda = \Delta$, different starting chains give rise to different Λ's. Finally, since j belongs to the "new" chains constructed in this case, they must differ from the chains in the first two cases.

Combining the three cases, for every element of $\bar{P}(j)$ that is not an image under g^{ij} of any chain in $\bar{P}(i)$, we have found a different element of $P(i)$ that is not an image under f^{ij} of any chain in $P(j)$. Together with Lemma 4.4, $|P(j)| + |\bar{P}(j)| \leq |P(i)| + |\bar{P}(i)|$.
Similarly, by taking $\Lambda \in P(i)$ such that there exists no $\Delta \in P(j)$ with $\Lambda = f^{ij}(\Delta)$, one can prove the reverse inequality of (4.2). Combining the two inequalities, we obtain $|C(i)| = |C(j)|$. $\square$

Using the previous proposition, we can compute the total weight of all larginals.

Proposition 4.6 $\sum_{\sigma \in \Pi(N)} w(\sigma) = (n-1)! \sum_{i \in N} M_i$.

Proof: Since each of the n players belongs to the same number of chains and there are $n!$ permutations making up the chains, each player belongs to $\frac{n!}{n} = (n-1)!$ chains. But then the statement immediately follows from Lemma 4.2. $\square$

Step 4: numerator, first player

Now we turn our attention to the numerator of ζ. For this, we partition the set of chains into subsets with the same starting player:

$$\mathcal{C}_k = \{\{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C} \mid \sigma_1(1) = k\}.$$

Note that since player k is by definition never the pivot in σ_1, he is also the first player in $\sigma_2, \ldots, \sigma_q$. It is easily verified that $\{\mathcal{C}_k\}_{k \in N}$ is indeed a partition of $\mathcal{C}$.

For a chain $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}$, we define L^Γ to be the weighted sum of its corresponding larginals:

$$L^\Gamma = \sum_{k=1}^{q} w(\sigma_k) \ell^{\sigma_k}.$$

We compute the numerator in (2.2) by combining the permutations that belong to the same $\mathcal{C}_k$, $k \in N$. We derive an expression for $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$ for each player $i \in N$. In this step, we consider the special case where $i = k$, while in the next step we compute the payoff to the other players.

Lemma 4.7 *For all $i \in N$, $\sum_{\Gamma \in \mathcal{C}_i} L_i^\Gamma = (n-2)! M_i \sum_{j \in N \setminus \{i\}} M_j$.*

Proof: In a similar way as in Proposition 4.5, we can show that $|\mathcal{C}_i \cap C(j)| = |\mathcal{C}_i \cap C(k)|$ for all $j, k \in N \setminus \{i\}$. Analogous to Proposition 4.6, we then have $\sum_{\sigma \in \Pi(N): \sigma(1) = i} w(\sigma) = (n-2)! \sum_{j \in N \setminus \{i\}} M_j$. Since player i always gets M_i at the first position, the statement follows. □

Step 5: numerator, other players

In this step, we finish the expression for the numerator in ζ by computing $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma$ for all $i \in N, i \neq k$. First, in a similar way as in Lemma 4.2, one can compute the total weighted larginal for each chain, as is done in the next lemma.

Lemma 4.8 *Let $\Gamma = \{\sigma_1, \ldots, \sigma_q\} \in P(i)$. Then for $j = \sigma(s)$ we have*

$$L_j^\Gamma = \begin{cases} w(\Gamma) M_j & \text{if } s < p_{\sigma_1}, \\ (v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)}) M_j & \text{if } j = i, \\ (v(N) - \sum_{k=1, k \neq p_{\sigma_1}}^{s-1} M_{\sigma_1(k)} + \sum_{k=s+1}^{p_{\sigma_1}+q-1} M_{\sigma_1(k)}) M_j & \text{if } \Gamma \in \bar{P}(j), \\ 0 & \text{if } s > p_{\sigma_1} + q - 1. \end{cases}$$

Example 4.4 Of course, $L_1^\Gamma = w(\Gamma) M_1 = 11 \cdot 5 = 55$ and $L_5^\Gamma = 0$. For player 2, the pivot, we have

$$
\begin{aligned}
L_2^\Gamma &= w(\sigma_1)(v(N) - M_1) + w(\sigma_2)(v(N) - M_1 - M_3) \\
&\quad + w(\sigma_3)(v(N) - M_1 - M_3 - M_4) \\
&= 3 \cdot (10 - 5) + 4 \cdot (10 - 5 - 1) + 4 \cdot (10 - 5 - 1 - 3) \\
&= 35.
\end{aligned}
$$

Indeed, this equals $(v(N) - \sum_{k=1}^{p_{\sigma_1}-1} M_{\sigma_1(k)})M_2 = (10-5) \cdot 2$, as stated in Lemma 4.8.

For player 3, which belongs to Γ but is not the pivot, we have

$$
\begin{aligned}
L_3^\Gamma &= w(\sigma_1) \cdot 0 + w(\sigma_2)M_3 + w(\sigma_3)M_3 \\
&= 0 + 4 \cdot 1 + 4 \cdot 1 \\
&= 8,
\end{aligned}
$$

which equals the expression in Lemma 4.8. For player 4, the computation is similar. $\triangleleft$

Lemma 4.9 *For all $i, k \in N, i \neq k$, we have $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma = (n-2)!(v(N) - M_k)M_i$.*

Proof: We prove the assertion using an iterative procedure, varying the utopia payoffs while keeping $v(N)$ constant. We denote the utopia vector in iteration t by M^t and throughout the procedure, this vector satisfies all our assumptions. We first show that the statement holds for $M^1 = (v(N), \ldots, v(N)) \geq M$. Then we iteratively reduce the components of the utopia vector one by one until we, after finitely many steps, end up in M. For every M^t, we show that for the corresponding (induced) set of chains, the total weighted payoff to i is as stated, *as function of the utopia vector.*

Step 1

Take $M^1 = (v(N), \ldots, v(N))$. Then all chains consist of one permutation, in which the second player is the pivot. Player i gets 0 if he is after the pivot and $v(N) - M_k^1$ if he is the pivot. There are $(n-2)!$ chains in which the latter occurs, each having weight M_i^1. Hence, $\sum_{\Gamma \in \mathcal{C}_k} L_i^\Gamma = (n-2)!(v(N) - M_k^1)M_i^1$.

Step t

Suppose that the statement holds for utopia vector M^{t-1}. If $M^{t-1} = M$, then we

are finished. Otherwise, there exists a $j \in N$ such that $M_j^{t-1} > M_j$. We now reduce j's utopia payoff until one of the chains changes, or until M_j is reached.
A chain changes if in one of its permutations, the pivot changes. Obviously, this can only happen if player j is before the pivot. Because in the first permutation of each chain the gap between what the pivot gets and his utopia payoff is smallest, this permutation is the first to change. Denoting this gap corresponding to $\sigma \in \Pi(N)$ by $\gamma(\sigma)$, ie,

$$\gamma(\sigma) = M_{\sigma(p_\sigma)}^{t-1} - (v(N) - \sum_{k=1}^{p_\sigma - 1} M_{\sigma(k)}^{t-1}),$$

the first chain changes when j's utopia payoff is decreased by

$$\gamma = \min\{\gamma(\sigma_1) \,|\, \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}_k, \sigma_1^{-1}(j) \leq p_{\sigma_1}\}. \tag{4.3}$$

Assume *for the moment* that the corresponding argmin is unique and denote its first permutation by $\hat{\sigma}$.
If $\gamma \geq M_j^{t-1} - M_j$, then decreasing j's utopia payoff from M_j^{t-1} to M_j does not result in any change in the chains. In this case, the statement holds for M_j^t defined by $M_j^t = M_j, M_h^t = M_h^{t-1}$ for all $h \in N \backslash \{j\}$. Proceed to step $t+1$.
Otherwise, define the second-highest gap $\tilde{\gamma}$ by

$$\tilde{\gamma} = \min\{\gamma(\sigma_1) \,|\, \{\sigma_1, \ldots, \sigma_q\} \in \mathcal{C}_k, \sigma_1^{-1}(j) \leq p_{\sigma_1}, \gamma(\sigma_1) > \gamma(\hat{\sigma})\}$$

and take $M_j^t = M_j^{t-1} - (\gamma + \varepsilon)$ for some $\varepsilon \in (0, \tilde{\gamma} - \gamma)$ and $M_h^t = M_h^{t-1}$ for all $h \in N \backslash \{j\}$. We show that the statement holds for this new utopia vector.
As mentioned before, $\hat{\sigma}$ is the first in a chain, say $\Gamma \in \mathcal{C}_k$. So, $\hat{\sigma}$ must be either of type $-P-$ or $-PP$. Define $s = \hat{\sigma}^{-1}(i)$ and distinguish between the two cases:

- $\hat{\sigma}$ is of type $-P-$:
  $\hat{\sigma}^R$ belongs to another chain, say $\Delta \in \mathcal{C}_k$ with length q. Note that the players $\hat{\sigma}(p_{\hat{\sigma}} - q + 1), \ldots, \hat{\sigma}(p_{\hat{\sigma}} - 1)$ and $\hat{\sigma}(p_{\hat{\sigma}} + 1)$ belong to Δ. When the pivot changes in $\hat{\sigma}$, this permutation joins Δ, as type $PP-$, forming chain $\Delta \cup \{\hat{\sigma}\}$. Hence, chain Γ disappears and the length of Δ is increased by one, while the other chains are not affected. So, it suffices to show that $L_i^{\Gamma,t-1} + L_i^{\Delta,t-1}$ as function of M^{t-1} equals $L_i^{\Delta \cup \{\hat{\sigma}\},t}$ as function of M^t. Using Lemma 4.8, we have:

  - $1 < s < p_{\hat{\sigma}} - q + 1$:

$$
\begin{aligned}
L_i^{\Gamma,t-1} &= M_{\hat{\sigma}(p_{\hat{\sigma}})}^{t-1} M_i^{t-1} \ (i \text{ is before } \Gamma), \\
L_i^{\Delta,t-1} &= (M_{\hat{\sigma}(p_{\hat{\sigma}}+1)}^{t-1} + \sum_{k=p_{\hat{\sigma}}-q+1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(k)}^{t-1}) M_i^{t-1} \ (i \text{ is before } \Delta), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (\sum_{k=p_{\hat{\sigma}}-q+1}^{p_{\hat{\sigma}}+1} M_{\hat{\sigma}(k)}^{t}) M_i^{t} \ (i \text{ is before } \Delta\cup\{\hat{\sigma}\}).
\end{aligned}
$$

– $s = p_{\hat{\sigma}}$:

$$
\begin{aligned}
L_i^{\Gamma,t-1} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(k)}^{t-1}) M_i^{t-1} \ (\Gamma \in P(i)), \\
L_i^{\Delta,t-1} &= 0 \ (i \text{ is after } \Delta), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(k)}^{t}) M_i^{t} \ (i \text{ is last in } \Delta\cup\{\hat{\sigma}\}).
\end{aligned}
$$

– $p_{\hat{\sigma}} - q + 1 \leq s < p_{\hat{\sigma}}$:

$$
\begin{aligned}
L_i^{\Gamma,t-1} &= M_{\hat{\sigma}(p_{\hat{\sigma}})}^{t-1} M_i^{t-1} \ (i \text{ is before } \Gamma), \\
L_i^{\Delta,t-1} &= (v(N) - \sum_{k=1}^{s-1} M_{\hat{\sigma}(k)}^{t-1} + \sum_{k=s+1}^{p_{\hat{\sigma}}-1} M_{\hat{\sigma}(k)}^{t-1}) M_i^{t-1} \ (\Delta \in \bar{P}(i)), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{s-1} M_{\hat{\sigma}(k)}^{t} + \sum_{k=s+1}^{p_{\hat{\sigma}}} M_{\hat{\sigma}(k)}^{t}) M_i^{t} \ (\Delta\cup\{\hat{\sigma}\} \in \bar{P}(i)).
\end{aligned}
$$

– $s = p_{\hat{\sigma}} + 1$:

$$
\begin{aligned}
L_i^{\Gamma,t-1} &= 0 \ (i \text{ is after } \Gamma), \\
L_i^{\Delta,t-1} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-q} M_{\hat{\sigma}(k)}^{t-1}) M_i^{t-1}, \ (\Delta \in P(i)), \\
L_i^{\Delta\cup\{\hat{\sigma}\},t} &= (v(N) - \sum_{k=1}^{p_{\hat{\sigma}}-q} M_{\hat{\sigma}(k)}^{t}) M_i^{t} \ (\Delta\cup\{\hat{\sigma}\} \in P(i)).
\end{aligned}
$$

– $s > p_{\hat{\sigma}} + 1$:

$$L_i^{\Gamma,t-1} = L_i^{\Delta,t-1} = L_i^{\Delta\cup\{\hat{\sigma}\},t} = 0 \text{ (i is after all three chains).}$$

It is readily checked that in all cases, $L_i^{\Gamma,t-1} + L_i^{\Delta,t-1}$ as function of M^{t-1} equals $L_i^{\Delta\cup\{\hat{\sigma}\},t}$ as function of M^t.

- $\hat{\sigma}$ is $-PP$:
  $\hat{\sigma}^R$ belongs to the same chain as $\hat{\sigma}$. When the pivot changes in $\hat{\sigma}$, this permutation will form a new chain of length one. In the same manner as in the previous case, we can show that the total weighted payoff to i as function of the utopia vector in these two chains remains the same.

So, from these two cases, we conclude that the statement holds for the new set of chains induced by the (lower) utopia vector M^t. Proceed to step $t+1$.

We assumed that the minimal gap in (4.3) is obtained for a unique permutation, $\hat{\sigma}$. Suppose now that there exists another permutation, $\tilde{\sigma}$, with this minimal gap. Since both $\hat{\sigma}$ and $\tilde{\sigma}$ are of type $-P-$ or $-PP$, they must belong to different chains Γ and $\tilde{\Gamma}$. Also the two corresponding "neighbouring" chains Δ and $\tilde{\Delta}$ are different, and different from Γ and $\tilde{\Gamma}$. Hence, we can consider the analysis in step t for $\hat{\sigma}$ and $\tilde{\sigma}$ separately to prove the statement.
Finally, our procedure stops after finitely many steps, because in all the changes, the pivot concerned moves towards the back of a permutation. □

Step 6: final

In this final step, we combine our previous results to prove the main theorem.

Proof of Theorem 3.1: Let $i \in N$. Then applying Lemmas 4.7 and 4.9 yields

$$\begin{aligned}\sum_{\sigma\in\Pi(N)} w(\sigma)\ell_i^\sigma &= \sum_{\Gamma\in\mathcal{C}} L_i^\Gamma \\ &= \sum_{j\in N\setminus\{i\}} \sum_{\Gamma\in\mathcal{C}_k} L_i^\Gamma + \sum_{\Gamma\in\mathcal{C}_i} L_i^\Gamma\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k \in N \setminus \{i\}} (n-2)!(v(N) - M_k)M_i + (n-2)!M_i \sum_{k \in N \setminus \{i\}} M_k \\
&= (n-1)!v(N)M_i.
\end{aligned}
$$

Then, using Proposition 4.6, we have

$$
\begin{aligned}
\zeta_i &= \frac{\sum_{\sigma \in \Pi(N)} w(\sigma)\ell_i^\sigma}{\sum_{\sigma \in \Pi(N)} w(\sigma)} \\
&= \frac{(n-1)!v(N)M_i}{(n-1)! \sum_{j \in N} M_j} \\
&= \frac{v(N)}{\sum_{j \in N} M_j} M_i.
\end{aligned}
$$

Hence, $\tau^* = \zeta$. □

As stated in section 2, for the class of compromise admissible games in which, after normalising such that the minimal rights vector equals zero, each player's utopia payoff is at most the value of the grand coalition, the τ^* value coincides with the compromise value. As a result, Theorem 3.1 gives a geometric characterisation of the latter on this class of games.

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