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Supermodular games and potential games

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Abstract

Potential games and supermodular games are attractive games, especially because under certain conditions they possess pure Nash equilibria. Subclasses of games with a potential are considered which are also strategically equivalent to supermodular games. The focus is on two-person zero-sum games and two-person Cournot games.

JEL classification: C72; C73

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1. Introduction

The aim of this paper is to investigate two interesting classes of games for which the existence of pure Nash equilibria is obtained under certain conditions, namely:

(i) the class of potential games (Monderer and Shapley, 1996);
(ii) the class of supermodular games (Topkis, 1998).

The question tackled here is whether there are games belonging to both classes. It turns out that two-person zero-sum supermodular games are potential games and conversely that two-person zero-sum potential games can be transformed in a canonical way into supermodular games. Also Cournot games are, under special conditions, members of both classes of games.

A connection between ordinal potential games (Monderer and Shapley, 1996) and supermodular games is also established for certain Cournot games.

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In Section 2 the definitions of potential games and of supermodular games are recalled together with some of their properties. In Section 3 the case of two-person zero-sum games is discussed and an example illustrating the connection between the two classes of games is given. Section 4 deals with Cournot duopoly competition and Cournot games. Section 5 contains some concluding remarks.

2. Preliminaries

Let $\langle A, B, K, L\rangle$ be a *two-person game* with strategy space A for player 1, strategy space B for player 2, and $K : A \times B \mapsto \mathbb{R}$, $L : A \times B \mapsto \mathbb{R}$ the pay-off function of players 1 and 2, respectively. If the players 1 and 2 choose $a \in A$ and $b \in B$, respectively, then player 1 obtains a pay-off $K(a, b)$ and player 2 obtains $L(a, b)$.

A *Nash equilibrium* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \leq K(\hat{a}, \hat{b})$ for each $a \in A$ and $L(\hat{a}, b) \leq L(\hat{a}, \hat{b})$ for each $b \in B$.

Such a game is called a *potential game* (Monderer and Shapley, 1996) if there is a *(potential) function* $P : A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b), \quad \text{for all } a_1, a_2 \in A \ \text{ and } \ \text{for each } b \in B,$$
$$L(a, b_1) - L(a, b_2) = P(a, b_1) - P(a, b_2), \quad \text{for each } a \in A \ \text{ and } \ \text{for all } b_1, b_2 \in B.$$

Clearly, elements of $\operatorname{argmax}(P)$ are Nash equilibria of the game.

The next lemma will be useful. It states that for a two-person potential game the pay-off function of player 1 (player 2) can be written as the sum of a potential and a function on the Cartesian product of the strategy spaces, which only depends on the strategy choice of player 2 (player 1). This is a known result (Slade, 1994; Facchini et al., 1997); an alternative proof is given here.

Lemma 1. *Let $\langle A, B, K, L\rangle$ be a potential game with potential P. Then there exist functions $f : A \mapsto \mathbb{R}$ and $g : B \mapsto \mathbb{R}$ such that*

$$K(a, b) = P(a, b) - 2g(b), \qquad L(a, b) = P(a, b) - 2f(a)$$

for each $a \in A$ and $b \in B$.

Proof. Take $a^* \in A$, $b^* \in B$ and define f and g as follows. For each $a \in A$ and $b \in B$, let

$$f(a) = \tfrac{1}{2}(P(a, b^*) - L(a, b^*)), \qquad g(b) = \tfrac{1}{2}(P(a^*, b) - K(a^*, b)).$$

Since P is a potential for the game $\langle A, B, K, L\rangle$, we have

$$K(a, b) - K(a^*, b) = P(a, b) - P(a^*, b) \quad \text{or} \quad K(a, b) - P(a, b) = -2g(b),$$

and also

$$L(a, b) - L(a, b^*) = P(a, b) - P(a, b^*) \quad \text{or} \quad L(a, b) - P(a, b) = -2f(a)$$

for all $a \in A$ and $b \in B$. □

The game $\langle A, B, K, L\rangle$ is called an *ordinal potential game* (Monderer and Shapley, 1996) if there is a (potential) function $P : A \times B \mapsto \mathbb{R}$ such that

$$\begin{aligned} &K(a_2, b) - K(a_1, b) > 0 \qquad \text{for all } a_1, a_2 \in A \text{ and for each } b \in B,\\ &\quad\Leftrightarrow P(a_2, b) - P(a_1, b) > 0,\\ &L(a, b_1) - L(a, b_2) > 0 \qquad \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.\\ &\quad\Leftrightarrow P(a, b_1) - P(a, b_2) > 0, \end{aligned}$$

We will use the following proposition.

Proposition 1 (Monderer and Shapley, 1996). *Let $\langle A, B, K, L\rangle$ be a two-person game. Let A, B be intervals of real numbers and K, L be twice continuously differentiable functions. Then $\langle A, B, K, L\rangle$ is a potential game if and only if*

$$\frac{\partial^2 K}{\partial a \partial b} = \frac{\partial^2 L}{\partial a \partial b}.$$

For more information on potential games see Voorneveld (1999) and Mallozzi et al. (2000).

Let us now recall some definitions related to supermodular games. A *partially ordered set* is a set X on which there is a binary relation $\preceq$ that is reflexive, antisymmetric and transitive. Let us consider a partially ordered set X and a subset X' of X. If $x' \in X$ and $x' \preceq x$ for each $x \in X'$, then x' is a *lower bound* for X'; if $x'' \in X$ and $x \preceq x''$ for each $x \in X'$, then x'' is an *upper bound* for X'. When the set of upper bounds of X' has a least element, then this least upper bound of X' is the *supremum* of X' in X; when the set of lower bounds of X' has a greatest element, then this greatest lower bound of X' is the *infimum* of X' in X.

If two elements x_1 and x_2 of a partially ordered set X have a supremum in X, it is called the *meet* of x_1 and x_2 and is denoted by $x_1 \wedge x_2$; if x_1 and x_2 have an infimum in X, it is called the *join* of x_1 and x_2 and is denoted by $x_1 \vee x_2$. A partially ordered set that contains the join and the meet of each pair of its elements is a *lattice*. If a subset X' of a lattice X contains the join and the meet (with respect to X) of each pair of elements of X', then X' is a *sublattice* of X. The real line $\mathbb{R}$ with the natural ordering denoted by $\leq$ is a lattice with $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ for $x, y \in \mathbb{R}$, and $\mathbb{R}^n$ $(n > 1)$ with the natural partial ordering denoted by $\leq$ is a lattice with $x \vee y = (x_1 \vee y_1, \dots, x_n \vee y_n)$ and $x \wedge y = (x_1 \wedge y_1, \dots, x_n \wedge y_n)$ for $x, y \in \mathbb{R}^n$. Any subset of $\mathbb{R}$ is a sublattice of $\mathbb{R}$, and a subset X of $\mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$ if it has the property that $x, y \in X$ imply that $(\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$ and $(\min\{x_1, y_1\}, \dots, \min\{x_n, y_n\})$ are in X.

The game $\langle A, B, K, L\rangle$ is called a *supermodular game* (Topkis, 1998) if the following three properties are satisfied:

(1) A is a sublattice of $\mathbb{R}^{m_1}$ and B is a sublattice of $\mathbb{R}^{m_2}$ for some $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$;
(2) K, L have increasing differences on $A \times B$, i.e. for all $(a_1, a_2) \in A^2$ and for all $(b_1, b_2) \in B^2$ such that $a_1 \geq a_2$ and $b_1 \geq b_2$,

$$K(a_1, b_1) - K(a_1, b_2) \geq K(a_2, b_1) - K(a_2, b_2),$$
$$L(a_1, b_1) - L(a_2, b_1) \geq L(a_1, b_2) - L(a_2, b_2);$$

(3) K is supermodular in the first coordinate and L is supermodular in the second coordinate, i.e. for each $b \in B$, for all $a_1, a_2 \in A$ we have

$$K(a_1, b) + K(a_2, b) \leq K(a_1 \vee a_2, b) + K(a_1 \wedge a_2, b)$$

and for each $a \in A$, for all $b_1, b_2 \in B$ we have

$$L(a, b_1) + L(a, b_2) \leq L(a, b_1 \vee b_2) + L(a, b_1 \wedge b_2).$$

We recall the following propositions.

Proposition 2 (Topkis, 1998). *Let f be a differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ if and only if $(\partial f/\partial x_i)$ is increasing in x_j for all distinct i and j and all x.*

Proposition 3 (Topkis, 1998). *Let f be a twice differentiable function on $\mathbb{R}^n$, then f has increasing differences on $\mathbb{R}^n$ if and only if $\partial^2 f/\partial x_i \partial x_j \geq 0$, for all distinct i and j.*

The following two examples show that the classes of potential games and supermodular games do not coincide. So the study of special subclasses becomes interesting.

Example 1. Let $A = B = [0, 1]$ and $K(a, b) = 2ab$, $L(a, b) = a + b$ for all $a, b \in [0, 1]$. Then the game $\langle A, B, K, L \rangle$ is a supermodular game because A and B are sublattices of $\mathbb{R}$, K, L have increasing differences on $[0, 1] \times [0, 1]$, and K is supermodular in the first coordinate and L in the second coordinate. This game is not an exact potential game because the condition in Proposition 1 is not satisfied since $(\partial^2 K/\partial a \partial b) = 2 \neq \partial^2 L/\partial a \partial b = 0$. Let us remark that the game is an ordinal potential game with potential function P given by $P(a, b) = a + b$ for all $a, b \in [0, 1]$.

On the other hand there are games that are exact potential games and not supermodular games.

Example 2. Let $A = B = [0, 1]$ and $K(a, b) = a^2 - 2a(b - (1/2))^2 + b$, $L(a, b) = -2a(b - (1/2))^2$ for all $a, b \in [0, 1]$. Then the game $\langle A, B, K, L \rangle$ is a potential game with potential function P given by $P(a, b) = a^2 - 2a(b - (1/2))^2$ for all $a, b \in [0, 1]$ but it is not a supermodular game in view of Proposition 3 because $(\partial^2 K/\partial a \partial b) = -4(b - (1/2)) < 0$ if $b > (1/2)$.

3. Zero-sum potential games and supermodular games

A two-person game of the form $\langle A, B, K, -K \rangle$ is called a *zero-sum game*. Such a game will be denoted by $\langle A, B, K \rangle$. In a zero-sum game one player pays the other. A *saddle point* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \leq K(\hat{a}, \hat{b}) \leq K(\hat{a}, b)$ for each $a \in A$ and $b \in B$. We denote by $S(A, B, K)$ the set of all saddle points of $\langle A, B, K \rangle$. Note that $\langle A, B, K \rangle$ is a potential game if there is a (potential) function $P : A \times B \mapsto \mathbb{R}$

such that

$$K(a_2, b) - K(a_1, b) \quad \text{for all } a_1, a_2 \in A \quad \text{and} \quad \text{for each } b \in B,$$
$$= P(a_2, b) - P(a_1, b),$$
$$-K(a, b_1) + K(a, b_2) \quad \text{for each } a \in A \quad \text{and} \quad \text{for all } b_1, b_2 \in B.$$
$$= P(a, b_1) - P(a, b_2),$$

Clearly, elements of argmax(P) are saddle-points of the game. Also the converse turns out to hold as we see in Remark 2.

Following theorem will be useful.

Theorem 1. *Let $\langle A, B, K\rangle$ be a two-person zero-sum game. Then the following assertions are equivalent*:

(1_{i}) *$\langle A, B, K\rangle$ is a potential game*;
(1_{ii}) *there exists a pair of functions (f, g) with $f : A \mapsto \mathbb{R}$ and $g : B \mapsto \mathbb{R}$ such that $K(a, b) = f(a) - g(b)$ for all $a \in A$, $b \in B$ (separation property).*

Proof. That (1_{ii}) implies (1_{i}) follows by taking the potential P defined by

$$P(a, b) = f(a) + g(b) \text{ for all } a \in A \text{ and } b \in B.$$

Conversely, suppose (1_{i}). Then by Lemma 1, there are functions $f : A \mapsto \mathbb{R}$ and $g : B \mapsto \mathbb{R}$ such that for each $a \in A$ and $b \in B$

$$K(a, b) = P(a, b) - 2g(b), \qquad -K(a, b) = P(a, b) - 2f(a).$$

So $K(a, b) = f(a) - g(b)$ for all $(a, b) \in A \times B$. □

Remark 1. This theorem is also proved in Potters et al. (1999), in an alternative way. In that paper it was also observed that for 2×2-subgames of a two-person zero-sum potential game the "diagonal property" holds. This is

$$K(a_1, b_1) + K(a_2, b_2) = K(a_1, b_2) + K(a_2, b_1)$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. This property follows easily from (1_{ii}) in Theorem 1. Conversely, it was proved in Potters et al. (1999) that the diagonal property for two-person zero-sum games implies also that the game is a potential game.

Remark 2. A pair (f, g) as in (1_{ii}) of Theorem 1 is called a *separating pair* for the potential game $\langle A, B, K\rangle$. For a potential P of this game we have $P(a, b) = c + f(a) + g(b)$ for each $a \in A$, $b \in B$ and some $c \in \mathbb{R}$. Clearly, $(\hat{a}, \hat{b})$ is a saddle point of $\langle A, B, K\rangle$ if and only if $\hat{a} \in \text{argmax}_{a\in A} f(a)$, $\hat{b} \in \text{argmax}_{b\in B} g(b)$ if and only if $(\hat{a}, \hat{b}) \in \text{argmax}(P)$.

Theorem 1 gives us the possibility to connect a two-person zero-sum potential game with a related game where the strategy spaces are ordered subsets of $\mathbb{R}$ and the pay-off function satisfies monotonicity conditions.

Given $\langle A, B, K\rangle$ with potential function P and separating pair (f, g) such that $P(a, b) = f(a) + g(b)$ for all $a \in A, b \in B$, define $\langle \bar{A}, \bar{B}, \bar{K}\rangle$ as follows. Take $\bar{A} = f(A)$, $\bar{B} = g(B)$ and for $(\bar{a}, \bar{b}) \in \bar{A} \times \bar{B}$ let $\bar{K}(\bar{a}, \bar{b}) = \bar{a} - \bar{b}$.

So we use the real-valued functions $f : A \mapsto \mathbb{R}$ and $g : B \mapsto \mathbb{R}$ to find a game $\langle \bar{A}, \bar{B}, \bar{K}\rangle$ with strategy spaces in $\mathbb{R}$, which is strategically equivalent to $\langle A, B, K\rangle$ because

$$K(a, b) = \bar{K}(f(a), g(b)) \quad \text{for all } (a, b) \in A \times B,$$

$$\bar{K}(c, d) = K(a, b) \quad \text{for all } a \in f^{-1}(c),\ b \in g^{-1}(d).$$

From this follows

$$(a, b) \in S(A, B, K) \Rightarrow (f(a), g(b)) \in S(\bar{A}, \bar{B}, \bar{K}),$$

$$(c, d) \in S(\bar{A}, \bar{B}, \bar{K}) \Rightarrow (a, b) \in S(A, B, K) \quad \text{for all } a \in f^{-1}(c),\ b \in g^{-1}(d).$$

The strategy space $\bar{A}$ can be smaller than A because two strategies a_1 and a_2 in A which are equivalent in the sense that

$$K(a_1, b) = K(a_2, b) \quad \text{for all } b \in B$$

are mapped into the same point $f(a_1) = f(a_2) \in \bar{A}$.

Relations between $\langle A, B, K\rangle$ and $\langle \bar{A}, \bar{B}, \bar{K}\rangle$ are described in the following theoerm.

Theorem 2. *Let $\langle A, B, K\rangle$ a game with potential P and let $\langle \bar{A}, \bar{B}, \bar{K}\rangle$ be as above. Then*

(2_{i}) *$\langle \bar{A}, \bar{B}, \bar{K}\rangle$ is a potential game with potential $\bar{P} : \bar{A} \times \bar{B} \mapsto \mathbb{R}$ such that $\bar{P}(\bar{a}, \bar{b}) = \bar{a} + \bar{b}$ for all $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$;*
(2_{ii}) $\max(\bar{A}) \times \max(\bar{B}) = \operatorname{argmax}(\bar{P}) = S(\bar{A}, \bar{B}, \bar{K})$;
(2_{iii}) $(a, b) \in S(A, B, K) \Leftrightarrow f(a) = \max(\bar{A}),\ g(b) = \max(\bar{B})$.

Note that $S(\bar{A}, \bar{B}, \bar{K})$ has cardinality 0 *or* 1.

Example 3. Consider the matrix game

	L	R	E
T	8	13	13
M	5	10	10
F	10	15	15

corresponding to the two-person zero-sum game $\langle A, B, K\rangle$ where $A = \{T, M, F\}$, $B = \{L, R, E\}$ and $K(T, L) = 8$, $K(T, R) = K(T, E) = 13$, $K(M, L) = 5$, $K(M, R) = K(M, E) = K(F, L) = 10$, $K(F, R) = K(F, E) = 15$. If we take $f : \{T, M, F\} \mapsto \mathbb{R}$ and $g : \{L, R, E\} \mapsto \mathbb{R}$ as follows: $f(T) = 5$, $f(M) = 2$, $f(F) = 7$, $g(L) = -3$, $g(R) = g(E) = -8$, then $K(a, b) = f(a) - g(b)$ for all $a \in A, b \in B$ and $P : A \times B \mapsto \mathbb{R}$ with $P(a, b) = f(a) + g(b)$ for all $a \in A, b \in B$ is a potential for this matrix game.

Transforming this game to $\langle \bar{A}, \bar{B}, \bar{K} \rangle$ with the aid of (f, g) results in $\bar{A} = \{2, 5, 7\}$, $\bar{B} = \{-8, -3\}$ and $\bar{K}(\bar{a}, \bar{b}) = \bar{a} - \bar{b}$ or the "monotonic" matrix game

	−8	−3
2	10	5
5	13	8
7	15	10

with the unique saddle point in $(7, -3)$ corresponding to maximum 4 of the potential $\bar{P}$ which can be written in matrix form as follows:

	−8	−3
2	−6	−1
5	−3	−2
7	−1	4

Note that $7 = \max(\bar{A})$, $-3 = \max(\bar{B})$.

Remark 3. If $\max(\bar{A})$ (or $\max(\bar{B})$) does not exists, then there are no saddle points. If K is bounded, then there are ε-saddle points for each $\varepsilon > 0$ corresponding to points (a', b') with $P(a', b') \geq \sup(P(a, b) - \varepsilon)$.

Theorem 3. *The game $\langle \bar{A}, \bar{B}, \bar{K} \rangle$ with $\bar{K}(a, b) = a - b$ for each $a \in \bar{A}$ and $b \in \bar{B}$ is a supermodular game.*

Proof. The subsets $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$. For each $b \in \bar{B}$ the function $a \mapsto \bar{K}(a, b)$ is supermodular on A and also $b \mapsto -\bar{K}(a, b)$ is supermodular on B for each $a \in A$. We have finished the proof if we show that for each $a_1, a_2 \in A$, $b_1, b_2 \in B$ the functions

$$a \mapsto \bar{K}(a, b_1) - \bar{K}(a, b_2) \quad (a \in \bar{A}), \qquad b \mapsto -\bar{K}(a_1, b) + \bar{K}(a_2, b) \quad (b \in \bar{B})$$

are monotonic. This is true because these functions are in fact constant:

$$\bar{K}(a, b_1) - \bar{K}(a, b_2) = -b_1 + b_2, \qquad -\bar{K}(a_1, b) + \bar{K}(a_2, b) = -a_1 + a_2. \qquad \square$$

We have seen in Theorem 3 that two-person zero-sum potential games can be embedded in the family of supermodular games. The converse is treated in following theorem.

Theorem 4. *Let $\langle A, B, K \rangle$ be a two-person zero-sum game with $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, which is supermodular. Then $\langle A, B, K \rangle$ is a potential game.*

Proof. The supermodularity implies that for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $a_1 < a_2, b_1 < b_2$ we have

$$K(a_2, b_2) - K(a_2, b_1) \geq K(a_1, b_2) - K(a_1, b_1),$$
$$-K(a_2, b_2) + K(a_1, b_2) \geq -K(a_2, b_1) + K(a_1, b_1).$$

From these two inequalities follows the diagonal property. Then, according to Remark 1, $\langle A, B, K\rangle$ is a potential game. □

Example 4. Let $\langle A, B, K, L\rangle$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(i, j) = i + j$ for all $i \in A$ and $j \in B$, and $L(1, 1) = 4$, $L(1, 2) = 7$, $L(2, 1) = 5$ and $L(2, 2) = 9$. Then this game is a supermodular game but not a potential game.

Example 5. Let $\langle A, B, K, L\rangle$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(1, 1) = 3, K(1, 2) = 1, K(2, 1) = 5, K(2, 2) = 2$ and $L(1, 1) = 3, L(1, 2) = 8, L(2, 1) = 6, L(2, 2) = 10$. Then the game is a potential game but not a supermodular game.

Remark 4. A subclass of general two-person potential games can be embedded into the class of supermodular games in a similar way as we embedded two-person zero-sum potential games. These are games of the form $\langle A, B, K, L\rangle$ with separable pay-off functions, i.e. K and L can be written in the form

$$K(a, b) = f(a) + g(b), \qquad L(a, b) = h(a) + k(b)$$

for all $a \in A, b \in B$, and where f and h are real-valued functions on A such that f is injective, and g and k are real-valued functions on B such that k is injective. A potential P is then given by $P(a, b) = f(a) + k(b)$ for each $a \in A$ and $b \in B$. A strategically equivalent supermodular game is the game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ where $\bar{A} = f(A)$, $\bar{B} = k(B)$ and where for all $c \in \bar{A}, d \in \bar{B}$:

$$\bar{K}(c, d) = K(f^{-1}(c), k^{-1}(d)), \qquad \bar{L}(c, d) = L(f^{-1}(c), k^{-1}(d)).$$

4. Cournot games

Consider Cournot's model of duopoly where the demand arises from a competitive market of a single homogeneous commodity.

Suppose that firm i, $i = 1, 2$, can supply the single homogeneous product in any non negative bounded quantity $q_i \in [0, q_i^0]$ with production cost $c_i(q_i)$. The price of the single homogeneous commodity is given by the inverse demand function $Q(q_1, q_2)$ which is assumed to be twice continuously differentiable function. We suppose that firm i's cost $c_i(q_i)$, $i = 1, 2$, is differentiable.

Given the output level selected by the other firm, the objective of firm i is to maximize its profit

$$\Pi_i(q_1, q_2) = q_i Q(q_1, q_2) - c_i(q_i)$$

by the choice of its output q_i, where $q_i Q(q_1, q_2)$ expresses the revenue (return) of firm i. We assume that the marginal revenue of firm i (i.e. $Q(q_1, q_2) + q_i(\partial Q(q_1, q_2)/\partial q_i)$) is decreasing with respect to $q_j (j \neq i)$.

A *Cournot game* is a game of the form $\langle A, B, K, L\rangle$ where $A = [0, q_1^0]$, $B = [0, q_2^0]$ and

$$K(a, b) = aQ(a, b) - c_1(a), \qquad L(a, b) = bQ(a, b) - c_2(b)$$

for all $a \in A$ and $b \in B$. If the inverse demand function Q is linear in $a + b$, then the corresponding Cournot duopoly game is also called a *quasi-Cournot game*.

Now we put $\bar{a} = a$ and $\bar{b} = -b$ for each $a \in A$ and $b \in B$ and consider the game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ where $\bar{A} = A$, $\bar{B} = -B = [-q_2^0, 0]$ and

$$\bar{K}(\bar{a}, \bar{b}) = K(\bar{a}, -\bar{b}), \qquad \bar{L}(\bar{a}, \bar{b}) = L(\bar{a}, -\bar{b})$$

for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. So

$$\bar{K}(\bar{a}, \bar{b}) = \bar{a}Q(\bar{a}, -\bar{b}) - c_1(\bar{a}), \qquad \bar{L}(\bar{a}, \bar{b}) = -\bar{b}Q(\bar{a}, -\bar{b}) - c_2(-\bar{b}).$$

The game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ is strategically equivalent to $\langle A, B, K, L\rangle$ because $\bar{K}(\bar{a}, \bar{b}) = K(a, b)$ and $\bar{L}(\bar{a}, \bar{b}) = L(a, b)$ for all $a \in A, b \in B$. We will denote by $\mathrm{NE}(A, B, K, L)$ the set of all Nash equilibria of the game $\langle A, B, K, L\rangle$. Note that $(a, b) \in \mathrm{NE}(A, B, K, L)$ if and only if $(a, -b) \in \mathrm{NE}(\bar{A}, \bar{B}, \bar{K}, \bar{L})$. Moreover, if $\langle A, B, K, L\rangle$ is a Cournot potential game with potential function P, then the game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ as above is also a potential game with potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(\bar{a}, -\bar{b})$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$.

Theorem 5. *Let $\langle A, B, K, L\rangle$ be a Cournot game and consider $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ as above. Then*

(5_{i}) *if the cost functions c_i are of the form $c_i(q_i) = cq_i$ for $i = 1, 2$, then $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ is an ordinal potential game and also a supermodular game*;

(5_{ii}) *if the inverse demand function Q is linear in the aggregate output level, given by $Q(a, b) = \alpha - \beta(a + b)$, $\alpha, \beta > 0$ (i.e. $\langle A, B, K, L\rangle$ is a quasi-Cournot game), then $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ is a potential game and also a supermodular game.*

Proof.

(5_{i}) The Cournot duopoly with cost functions c_i, $i = 1, 2$ is an ordinal potential game with potential function P given by $P(a, b) = ab[Q(a, b) - c]$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ is also an ordinal potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(a, b)$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. Moreover, $\bar{K}(\bar{a}, \bar{b}) = \bar{a}[Q(\bar{a}, -\bar{b}) - c]$ and $\bar{L}(\bar{a}, \bar{b}) = -\bar{b}[Q(\bar{a}, -\bar{b}) - c]$ satisfy the increasing differences property because by Proposition 3 we have

$$\frac{\partial^2 \bar{K}}{\partial \bar{a}\partial \bar{b}} = -\frac{\partial}{\partial b}\left[Q + a\frac{\partial Q}{\partial a}\right] \geq 0$$

and

$$\frac{\partial^2 \bar{L}}{\partial \bar{a}\partial \bar{b}} = -\frac{\partial}{\partial a}\left[Q + b\frac{\partial Q}{\partial b}\right] \geq 0,$$

since we assumed that the marginal revenue is decreasing. Moreover, the transformed strategy spaces $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate and $\bar{L}$ is supermodular in the second coordinate. Then the Cournot game is a supermodular game.

(5_{ii}) The quasi-Cournot competition is a potential game with potential function P given by $P(a, b) = \alpha(a + b) - \beta(a^2 + b^2) - \beta ab - c_1(a) - c_2(b)$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $\langle \bar{A}, \bar{B}, \bar{K}, \bar{L}\rangle$ is also a potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a}, \bar{b}) = P(a, b)$ for all $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$. Moreover, $\bar{K}(\bar{a}, \bar{b}) = \bar{a}[\alpha - \beta(\bar{a} - \bar{b})] - c_1(\bar{a})$ and $\bar{L}(\bar{a}, \bar{b}) = -\bar{b}[\alpha - \beta(\bar{a} - \bar{b})] - c_2(-\bar{b})$ satisfy the increasing differences property because by Proposition 3 we have

$$\frac{\partial^2 \bar{K}}{\partial \bar{a}\partial \bar{b}} = \frac{\partial^2 \bar{L}}{\partial \bar{a}\partial \bar{b}} = \beta > 0.$$

As in the previous case, $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate, $\bar{L}$ is supermodular in the second coordinate and the quasi-Cournot game is a supermodular game. □

5. Concluding remarks

Let us first summarize the main results we obtained:

(i) a supermodular two-person zero-sum game is a potential game (Theorem 4). Conversely, if a two-person zero-sum game is a potential game then it is strategically equivalent to a supermodular game (Theorems 2 and 3), which is monotonic and has at most one saddle point; the set of pure saddle points of a two-person zero-sum potential game turns out to coincide with the potential maximizers (Remark 2);

(ii) two subclasses of Cournot games are described, which are strategically equivalent to supermodular games and which are simultaneously (ordinal or exact) potential games (Theorem 5).

In Remark 4 we discussed a subclass of general two-person potential games which can be embedded in the class of supermodular games. This result holds for a similar subclass of general n-person strategic games with separable pay-off functions.

A game of the form $\langle A_1, \ldots, A_n, K_1, \ldots, K_n\rangle$ where $K_i(a_i, a_{-i}) = f_i(a_i) + g_i(a_{-i})$ for all $a_i \in A_i$ and $a_{-i} \in \Pi_{j \in \mathrm{N}-\{i\}} A_j$ is a potential game and it is strategically equivalent to a supermodular game if $f_1, \ldots, f_n$ are injective functions. A potential is given by

$$P(a) = \sum_{i=1}^{n} f_i(a_i)$$

and the (strategically equivalent) supermodular game is defined as follows:

- for each $i \in N = \{1, \dots, n\}$, $\bar{A}_i = f_i(A_i)$;
- for all $b_1 \in \bar{A}_1, \dots, b_n \in \bar{A}_n$ and all $i \in N$

$$\bar{K}_i(b_1, \dots, b_n) = K_i(f_1^{-1}(b_1), \dots, f_n^{-1}(b_n)).$$

Also duopoly results in Section 4 can be extended to multimarket oligopoly (Topkis, 1998). It is interesting to find other economic situations leading to strategic games which are potential games and also supermodular games.

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References

Facchini, G., Van Megen, F., Borm, P., Tijs, S., 1997. Congestion models and weighted Bayesian potential games. Theory and Decision 42, 193–206.
Mallozzi, L., Tijs, S., Voorneveld, M., 2000. Infinite hierarchical potential games. Journal of Optimization Theory and Applications 107, 287–296.
Monderer, P., Shapley, L., 1996. Potential games. Games and Economic Behavior 14, 124–143.
Potters, J., Raghavan, T.E.S., Tijs, S., 1999. Pure equilibrium strategies for stochastic games via potential functions. Department of Mathematics, University of Nijmegen, The Netherlands, Report No. 9910.
Slade, M.E., 1994. What does an oligopoly maximize? Journal of Industrial Economics 42, 45–61.
Topkis, D., 1998. Supermodularity and Complementarity. Princeton University Press, Princeton, NJ.
Voorneveld, M., 1999. Potential games and interactive decisions with multicriteria, Ph.D. Thesis, No. 61, CentER Tilburg University.