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Published in:
Journal of Mathematical Analysis and Applications

Publication date:
2003

Citation for published version (APA):
Brânzei, R., Morgan, J., Scalzo, V., & Tijs, S. H. (2003). Approximate fixed point theorems in Banach spaces with applications in game theory. *Journal of Mathematical Analysis and Applications*, *285*(2), 619-628.

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J. Math. Anal. Appl. 285 (2003) 619–628

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Approximate fixed point theorems in Banach spaces with applications in game theory ☆

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Received 28 March 2002

Submitted by J.A. Filar

Abstract

In this paper some new approximate fixed point theorems for multifunctions in Banach spaces are presented and a method is developed indicating how to use approximate fixed point theorems in proving the existence of approximate Nash equilibria for non-cooperative games.

Keywords: Approximate fixed point; Approximate Nash equilibrium; Banach space; Closed multifunction; Upper semicontinuous multifunction

1. Introduction

In this paper we are interested in multifunctions $F: X \rightarrow\rightarrow X$ which possess (fixed points or) approximate fixed points. Fixed point theorems deal with sufficient conditions on X and F guaranteeing that there exists a fixed point, that is, an $\hat{x} \in X$ with $\hat{x} \in F(\hat{x})$. There are many fixed point theorems known on topological spaces (Brouwer [5], Kaku-

☆ Supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilita e loro Applicazioni (GNAMPA) and Universita di Napoli Federico II.

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0022-247X/$ – see front matter
doi:10.1016/S0022-247X(03)00450-5

tani [8], Banach [3], etc.) which have proved to be useful in many applied fields such as game theory, mathematical economics and the theory of quasi-variational inequalities (cf. Baiocchi and Capelo [2]). If X is a metric space, approximate fixed point theorems are interesting. Such theorems deal with sufficient conditions on X and F guaranteeing that, for each $\varepsilon > 0$, there is an ε-fixed point, i.e., an $x^* \in X$ with $d(x^*, F(x^*)) \leqslant \varepsilon$, where $d(x^*, F(x^*)) = \inf\{d(x^*, z) \mid z \in F(x^*)\}$. In Tijs et al. [22], approximate fixed point theorems in the spirit of Brouwer, Kakutani and Banach were derived. In the first two theorems, in finite dimensional spaces, the compactness conditions used in the above quoted theorems have been replaced by boundedness conditions. In the third one, the completeness of the metric space (used in Banach's contraction theorem) has been dropped.

In this paper we will present some new approximate fixed point theorems for multifunctions defined on Banach spaces. Weak and strong topologies play here a role and both bounded and unbounded regions are considered.

The outline of the paper is as follows. In Section 2, we present some approximate fixed point theorems for closed or upper semicontinuous (with respect to the weak or strong topologies) multifunctions on bounded, totally bounded convex regions or on unbounded convex regions, respectively. Here the notion of tame multifunction plays a crucial role. Section 3 gives an outline of how to use approximate fixed point theorems to guarantee that non-cooperative games have approximate Nash equilibria, and Section 4 concludes with some remarks.

2. New approximate fixed point theorems

In this section, V will be a real Banach space and for $F : X \to\to X$ with $X \subseteq V$, the set $\{x \in V \mid d(x, F(x)) = \inf_{y \in F(x)} \|y - x\| \leqslant \varepsilon\}$ of the ε-fixed points of the multifunction F on X is denoted by $FIX^{\varepsilon}(F)$.

The assumptions of closedness and boundedness for a set of a reflexive real Banach space is an usual and classical assumption in many theoretical and applied problems. In light of the Alaoglu theorem, a closed and bounded set is sequentially compact and in these cases we have to deal with weak convergence.

Thus, first, we present two theorems where the weak topology plays a role.

Theorem 2.1. *Let V be a reflexive real Banach space and let X be a bounded and convex subset of V with non-empty interior. Assume that $F : X \to\to X$ is a weakly closed multifunction (that is, a multifunction closed with respect to the weak topology) such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Suppose without loss of generality that $0 \in \operatorname{int} X$. Let $\alpha = \sup\{\|x\| \mid x \in X\}$. Take $\varepsilon > 0$ and $0 < \delta < 1$ such that $\delta\alpha \leqslant \varepsilon$. Let Y be the weakly compact and convex subset of X defined by $Y = (1 - \delta)\overline{X}$, where $\overline{X}$ is the closure of X. Define the multifunction $G : Y \to\to Y$ by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$. Then G is a weakly closed multifunction with non-empty, convex and weakly compact values. But, with respect to the weak topology, V is an Hausdorff locally convex topological vector space, so, in view of Glicksberg's theorem [7], G has at least one fixed point on Y. So there is an $x^* \in Y$ such that $x^* \in G(x^*) = (1 - \delta) \times$

$F(x^*)$. Then there is a $z \in F(x^*)$ such that $x^* = (1-\delta)z$, so $\|z - x^*\| = \delta\|z\| \leqslant \delta\alpha \leqslant \varepsilon$. Hence x^* is an ε-fixed point of F. □

Theorem 2.2. *Let V be a reflexive and separable real Banach space and let X be a bounded and convex subset of V with non-empty interior. Assume that $F : X \to\to X$ is a weakly upper semicontinuous multifunction (that is, a multifunction upper semicontinuous with respect to the weak topology) such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. As in the proof of Theorem 2.1, we assume that $0 \in \operatorname{int} X$ and $\alpha = \sup\{\|x\| \mid x \in X\}$. Take $\varepsilon > 0$, $0 < \delta < 1$ such that $\delta\alpha \leqslant \varepsilon/2$ and $Y = (1-\delta)\overline{X}$. Define the multifunction $G : Y \to\to Y$ by $G(x) = (1-\delta)\overline{F(x)}$ for all $x \in Y$. G is weakly upper semicontinuous. In fact, since V is a separable real Banach space and X is bounded, there exists a metric d_w on V such that the weak topology on X is induced by the metric d_w (see, for example, [6, Proposition 8.7]). Let $x \in Y$ and assume that A is a weakly open neighbourhood of $G(x)$. For $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \mid d_w(y, G(x)) < \sigma\}$. Since $G(x)$ is weakly compact, we have that $d_w(Y \setminus A, G(x)) = \inf\{d_w(y, z) \mid y \in Y \setminus A, z \in G(x)\} > 0$, where $Y \setminus A = \{y \in Y \mid y \notin A\}$. So, if $0 < \sigma' < \sigma < d_w(Y \setminus A, G(x))$, we have $G(x) \subset A_{\sigma'} \subset \{y \in Y \mid d_w(y, G(x)) \leqslant \sigma'\} \subset A_\sigma \subset A$. In view of the weakly upper semicontinuity of the multifunction $(1-\delta)F$, there exists an open neighbourhood I of x such that $(1-\delta)F(z) \subset A_{\sigma'}$ for all $z \in I$. Therefore $G(z) = (1-\delta)\overline{F(z)} \subseteq \{y \in Y \mid d_w(y, G(x)) \leqslant \sigma'\} \subset A$ for all $z \in I$. So G is a weakly upper semicontinuous multifunction at x. In the light of [1, Proposition 4, p. 72], G is also a weakly closed multifunction at x. Therefore, in view of Glicksberg's theorem, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$. Hence, there exists $z \in \overline{F(x^*)}$ such that $x^* = (1-\delta)z$, so $\|z - x^*\| = \delta\|z\| \leqslant \delta\alpha \leqslant \varepsilon/2$. Moreover, there is $z' \in F(x^*)$ such that $\|z' - z\| < \varepsilon/2$. Hence $\|z' - x^*\| < \varepsilon$, that is, $x^* \in FIX^{\varepsilon}(F)$. □

Remark 2.1. Even if the assumption of "weakly closed graph on X" looks very strong, it can be obtained for multifunctions whose fixed points are interesting. For example, a solution of a quasi-variational inequality is a fixed point of a suitable set-valued function which is weakly closed under classical assumptions. In fact, following Baiocchi and Capelo [2, p. 240], one can "*reconduce the study of the quasi-variational inequality to the study of a family of variational inequalities and to the finding of a fixed point for an appropriate transformation*." To obtain weak closedness of the graph of this appropriate transformation it is then sufficient to apply two results of Mosco [19, Theorems A and B] or following slight improvement by Lignola and Morgan [12, Corollary 2.2] for $K = \overline{X}$.

In the next theorem the strong topology is involved.

Theorem 2.3. *Let V be a real Banach space and let X be a convex and totally bounded subset of V with non-empty interior. Assume that $F : X \to\to X$ is a closed or upper semicontinuous multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Assume without loss of generality that $0 \in \operatorname{int} X$. Take $\varepsilon > 0$ and $\eta > 0$. Since X is totally bounded there exists $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$ such that $X \subseteq \bigcup_{i=1}^{m} \mathring{B}(x_i, \eta)$ (see, for example, [4]), where $\mathring{B}(x_i, \eta) = \{y \in V \mid \|y - x_i\| < \eta\}$. Moreover, let $h = \max\{\|x_i\| \mid i \in \{1, \ldots, m\}\}$. If $0 < \delta < 1$ the set $Y = (1 - \delta)\overline{X}$ is a non-empty, convex and totally bounded subset of V. Since Y is also closed, Y is complete and therefore compact.

First, we assume that F is a closed multifunction and we take $0 < \delta < 1$ such that $\delta(\eta + h) \leqslant \varepsilon$. Then the multifunction $G : Y \to\to Y$, defined by $G(x) = (1 - \delta)F(x)$ for all $x \in Y$, is closed. This implies by Glicksberg's theorem that G possesses a fixed point x^*. Then there is a point $z \in F(x^*)$ such that $x^* = (1 - \delta)z$. Since $X \subseteq \bigcup_{i=1}^{m} \mathring{B}(x_i, \eta)$, there exists an $r \in \{1, \ldots, m\}$ such that $z \in \mathring{B}(x_r, \eta)$. So $\|x^* - z\| = \delta\|z\| \leqslant \delta(\|z - x_r\| + \|x_r\|) < \delta(\eta + h) \leqslant \varepsilon$. Hence $x^* \in FIX^{\varepsilon}(F)$.

Assume now that F is an upper semicontinuous multifunction. We take $0 < \delta < 1$ such that $\delta(\eta + h) \leqslant \varepsilon/2$. Let $G : Y \to\to Y$, defined by $G(x) = (1 - \delta)\overline{F(x)}$ for all $x \in Y$. We claim that G is upper semicontinuous. Let $x \in Y$ and assume that A is an open neighbourhood of $G(x)$. For each $\sigma > 0$, we denote with A_σ the open set $\{y \in Y \mid \inf_{z \in G(x)} \|z - y\| < \sigma\}$. As in the proof of Theorem 2.2, we obtain that G is an upper semicontinuous multifunction at x and is also a closed multifunction at x. In view of Glicksberg's theorem, there exists a point $x^* \in Y$ such that $x^* \in G(x^*)$ and $z \in \overline{F(x^*)}$ such that $x^* = (1 - \delta)z$. Since $X \subseteq \bigcup_{i=1}^{m} \mathring{B}(x_i, \eta)$, there exists $s \in \{1, \ldots, m\}$ such that $z \in \mathring{B}(x_s, \eta)$, so $\|z - x^*\| = \delta\|z\| \leqslant \delta(\|z - x_s\| + \|x_s\|) < \delta(\eta + h) \leqslant \varepsilon/2$. Moreover, there exists a point $z' \in F(x^*)$ such that $\|z' - z\| < \varepsilon/2$, so $\|z' - x^*\| < \varepsilon$, that is, $x^* \in FIX^{\varepsilon}(F)$. □

The next theorems deal with the existence of approximate fixed points for multifunctions on convex regions which are not necessarily bounded. Useful here is the notion of a *tame multifunction*, which we introduce in

Definition 2.1. Let U be a normed space and $X \subseteq U$ with $0 \in X$. A multifunction $F : X \to\to X$ is called a *tame multifunction* if, for each $\varepsilon > 0$, there is an $R > 0$ such that for each $x \in B(0, R) \cap X$ the set $F(x) \cap B(0, R + \varepsilon)$ is non-empty, where $B(0, R) = \{z \in U \mid \|z\| \leqslant R\}$.

Example 2.1. The map $F : [0, \infty[\to\to [0, \infty[$, defined by

$$F(x) = \left[x + (x + 1)^{-1}, \infty\right[\quad \text{for all } x \in [0, \infty[,$$

is a tame multifunction on the unbounded set $[0, \infty[$. Moreover, F has ε-fixed points for each $\varepsilon > 0$ (see Theorems 2.4 and 2.5).

Example 2.2. Let U be a normed space. Let $F : U \to\to U$ be a multifunction with $F(x) \neq \emptyset$ for each $x \in U$. Suppose that the image $F(U) = \{y \in U \mid y \in F(x) \text{ for some } x \in U\}$ of F is a bounded set. Then F is a tame multifunction (for each $\varepsilon > 0$, take $R = 1 + \sup\{\|y\|,\ y \in F(U)\}$).

Remark 2.2. It follows from Example 2.2 that each $F : X \to\to X$, where X is a bounded subset of a normed space U and $F(x)$ is non-empty for all $x \in X$, is a tame multifunction.

Example 2.3. Let U be a normed linear space. The translation $T : U \to U$ given by $T(x) = x + a$, where $a \in U \setminus \{0\}$, is not tame and for small $\varepsilon > 0$, T has no ε-fixed points. The *tame* property for multifunction in the next theorems is a non-superfluous condition for the existence of ε-fixed points.

Theorem 2.4. *Let X be a convex subset with non-empty interior, containing 0, of a reflexive real Banach space. Assume that $F : X \to\to X$ is a tame and weakly closed multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Let $\varepsilon > 0$ and $R > 0$ such that $F(x) \cap B(0, R + \varepsilon/2) \neq \emptyset$ for each $x \in B(0, R) \cap X$, and let $C = B(0, R) \cap X$. C is a non-empty, bounded and convex set. Then $G : C \to\to C$, defined by

$$G(x) = R\left(R + \frac{\varepsilon}{2}\right)^{-1} F(x) \cap B\left(0, R + \frac{\varepsilon}{2}\right) \quad \text{for all } x \in C,$$

satisfies the conditions of Theorem 2.1. Hence there is $x^* \in FIX^{\varepsilon/4}(G)$ such that $d(x^*, G(x^*)) \leqslant \varepsilon/4 < \varepsilon/2$ and there exists $x' \in G(x^*)$ such that $\|x' - x^*\| < \varepsilon/2$. Moreover, there exists an element $z \in F(x^*)$ such that $z = R^{-1}(R + \varepsilon/2)x'$. This implies that

$$\|z - x^*\| \leqslant \left\| R^{-1}\left(R + \frac{\varepsilon}{2}\right)x' - x'\right\| + \|x' - x^*\| < \frac{\varepsilon}{2} R^{-1}\|x'\| + \frac{\varepsilon}{2} \leqslant \varepsilon.$$

So $x^* \in FIX^{\varepsilon}(F)$. □

Theorem 2.5. *Let X be a convex subset with non-empty interior, containing 0, of a reflexive and separable real Banach space. Assume that $F : X \to\to X$ is a tame and weakly upper semicontinuous multifunction such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^{\varepsilon}(F) \neq \emptyset$ for each $\varepsilon > 0$.*

Proof. Using the same arguments of the proof of Theorem 2.4, we can show that the multifunction G, defined on $B(0, R) \cap X$ by

$$G(x) = R\left(R + \frac{\varepsilon}{2}\right)^{-1} F(x) \cap B\left(0, R + \frac{\varepsilon}{2}\right),$$

satisfies the conditions of Theorem 2.2 and the conclusion follows as in Theorem 2.4. □

3. Approximate Nash equilibria for strategic games

In Nash [20], Nash equilibria for n-person non-cooperative games have been introduced and using Kakutani's fixed point theorem it has been shown that mixed extensions of finite n-person non-cooperative games possess at least one Nash equilibrium. The aggregate best response multifunction on the Cartesian product of the strategy spaces constructed with the aid of the best response multifunctions for each player possesses fixed points which coincide with the Nash equilibria of the game.

Of course, for many non-cooperative games Nash equilibria do not exist. Interesting are games for which ε-Nash equilibria exist for each $\varepsilon > 0$. Here a strategy profile is called an ε-Nash equilibrium if unilateral deviation of one of the players does not increase his payoff with more than ε. One can try to derive the existence of approximate equilibrium points following the next scheme:

(i) develop ε-fixed point theorems and find conditions on strategy spaces and payoff functions of the game such that the aggregate ε-best response multifunction satisfies conditions in an ε-fixed point theorem;
(ii) add extra conditions on the payoff-functions, guaranteeing that points in the Cartesian product of the strategy spaces nearby each other have payoffs sufficiently nearby.

We will derive in this section a key proposition, which gives the possibility to find various approximate equilibrium theorems.

First we recall some definitions. An *n-person strategic game* is a tuple $\Gamma = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ where for each player $i \in N = \{1, \ldots, n\}$ X_i is the set of strategies and $u_i : \prod_{i \in N} X_i \to \mathbb{R}$ is the payoff function. If players $1, \ldots, n$ choose strategies $x_1, \ldots, x_n$, then $u_1(x_1, \ldots, x_n), \ldots, u_n(x_1, \ldots, x_n)$ are the resulting payoffs for the players $1, \ldots, n$, respectively. Let $\varepsilon > 0$. Then we say that $(x_i^*)_{i \in N} \in \prod_{i \in N} X_i$ is an *ε-Nash equilibrium* if

$$u_i(x_i, x_{-i}^*) \leqslant u_i(x^*) + \varepsilon \quad \text{for all } x_i \in X_i \text{ and for all } i \in N.$$

Here x_{-i}^* is a shorthand for $(x_j^*)_{j \in N \setminus \{i\}}$ and we will denote by $NE^{\varepsilon}(\Gamma)$ the set of ε-Nash equilibria for the game Γ. Note that for an $x^* \in NE^{\varepsilon}(\Gamma)$, a unilateral deviation by a player does not improve the payoff with more than ε. Useful will be for each $i \in N$ the *ε-best response multifunction* $B_i^{\varepsilon} : \prod_{j \in N \setminus \{i\}} X_j \to\to X_i$ defined by

$$B_i^{\varepsilon}(x_{-i}) = \left\{ x_i \in X_i \mid u_i(x_i, x_{-i}) \geqslant \sup_{t_i \in X_i} u_i(t_i, x_{-i}) - \varepsilon \right\}$$

and the *aggregate ε-best response multifunction* $B^{\varepsilon} : X \to\to X$ defined by

$$B^{\varepsilon}(x) = \prod_{i \in N} B_i^{\varepsilon}(x_{-i}).$$

Obviously, if $x^* \in B^{\varepsilon}(x^*)$, then $x^* \in NE^{\varepsilon}(\Gamma)$, and conversely. So if B^{ε} has a fixed point, then we have an ε-Nash equilibrium. If we do not know whether B^{ε} has a fixed point but we know that B^{ε} has δ-fixed points for each $\delta > 0$, then this leads under extra continuity conditions to the existence of approximate Nash equilibria for the game as we will see.

The next result is called the key proposition because it opens the door to obtain different ε-equilibrium point theorems, using as inspiration source the existing literature on Nash equilibrium point theorems. Many of them contain collections of sufficient conditions on the strategy spaces and payoff functions, guaranteeing that the aggregate best response multifunction has a fixed point. To guarantee the existence of ε-fixed points one has to modify, often in an obvious way, the conditions guaranteeing the existence of δ-fixed points for the aggregate ε-best response multifunction and to replace the condition (iii) in the key proposition by the obtained conditions.

Key proposition. *Let $\Gamma = \langle X_1, \ldots, X_n, u_1, \ldots, u_n \rangle$ be an n-person strategic game with the following three properties:*

(i) *for each $i \in N = \{1, \ldots, n\}$, the strategy space X_i is endowed with a metric d_i;*
(ii) *the payoff functions $u_1, \ldots, u_n$ are uniform continuous functions on $X = \prod_{i=1}^{n} X_i$, where X is endowed with the metric d defined by*

$$d(x, y) = \sum_{i=1}^{n} d_i(x_i, y_i) \quad \text{for all } x, y \in X;$$

(iii) *for each $\varepsilon > 0$ and $\delta > 0$, the aggregate ε-best response multifunction B^{ε} possesses at least one δ-fixed point, i.e., $FIX^{\delta}(B^{\varepsilon}) \neq \emptyset$.*

Then, $NE^{\varepsilon}(\Gamma) \neq \emptyset$ for each $\varepsilon > 0$.

Proof. Take $\varepsilon > 0$. By (ii) we can find $\eta > 0$ such that for all $x, x' \in X$ with $d(x, x') < \eta$ we have $|u_i(x) - u_i(x')| < \varepsilon/2$ for each $i \in N$. We will prove that

$$x^* \in FIX^{\eta/2}(B^{\varepsilon/2}) \quad \Rightarrow \quad x^* \in NE^{\varepsilon}(\Gamma).$$

Take $x^* \in FIX^{\eta/2}(B^{\varepsilon/2})$, which is possible by (iii). Then there exists $\hat{x} \in B^{\varepsilon/2}(x^*)$ such that $d(x^*, \hat{x}) < \eta$, and, consequently, for each $i \in N$, $d((x_i^*, x_{-i}^*), (\hat{x}_i, x_{-i}^*)) < \eta$. This implies that

$$u_i(x_i^*, x_{-i}^*) \geqslant u_i(\hat{x}_i, x_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all } i \in N. \tag{1}$$

Further $\hat{x} \in B^{\varepsilon/2}(x^*)$ implies

$$u_i(\hat{x}_i, x_{-i}^*) \geqslant \sup_{t_i \in X_i} u_i(t_i, x_{-i}^*) - \frac{1}{2}\varepsilon \quad \text{for all } i \in N. \tag{2}$$

Combining (1) and (2) we obtain

$$u_i(x_i^*, x_{-i}^*) \geqslant \sup_{t_i \in X_i} u_i(t_i, x_{-i}^*) - \varepsilon \quad \text{for all } i \in N, \tag{3}$$

that is, $x^* \in NE^{\varepsilon}(\Gamma)$. □

It will be clear that using the key proposition many approximate Nash equilibrium theorems can be obtained. We restrict ourselves here in giving three examples.

Example 3.1 (Games on the open unit square). Let $\langle]0, 1[,]0, 1[, u_1, u_2 \rangle$ be a game with uniform continuous payoff functions u_1 and u_2. Suppose that u_1 is concave in the first coordinate and u_2 is concave in the second coordinate. Then for each $\varepsilon > 0$, the game has an ε-Nash equilibrium point. In fact, apply the key proposition to the above game and note that (i) and (ii) are satisfied by taking the standard metric on $]0, 1[$. Further, (iii) follows from Theorem 2.1 applied to the set-valued function B^{ε}.

Example 3.2 (Completely mixed approximate Nash equilibria for finite games). Let A and B be $(m \times n)$-matrices of real numbers. Consider the two-person game $\langle \mathring{\Delta}_m, \mathring{\Delta}_n, u_1, u_2 \rangle$, where

$$\mathring{\Delta}_m = \left\{ p \in \mathbb{R}^m \mid p_i > 0 \text{ for each } i \in \{1, \ldots, m\},\ \sum_{i=1}^{m} p_i = 1 \right\},$$

$$\mathring{\Delta}_n = \left\{ q \in \mathbb{R}^n \mid q_j > 0 \text{ for each } j \in \{1, \ldots, n\},\ \sum_{j=1}^{n} q_j = 1 \right\},$$

$$u_1(p, q) = p^T A q, \quad u_2(p, q) = p^T B q \quad \text{for all } p \in \mathring{\Delta}_m,\ q \in \mathring{\Delta}_n.$$

Then for each $\varepsilon > 0$ this game has an ε-Nash equilibrium. Such an ε-Nash equilibrium is called *completely mixed*, because both players use each of their pure strategies with a positive probability. The proof follows from the key proposition and Theorem 2.1 taking the standard Euclidean metric.

Example 3.3. Let X be a normed linear space such that there exists $a \in X \setminus \{0\}$. Let $\Gamma = \langle X, X, u_1, u_2 \rangle$ be the two-person game with $u_1(x_1, x_2) = -\|x_1 - x_2\|$, $u_2(x_1, x_2) = -\|x_1 - x_2 - a/(1 + \|x_1\|)\|$ for all $(x_1, x_2) \in X \times X$. Then $B_1(x_2) = \{x_2\}$ and $B_2(x_1) = \{x_1 - a/(1 + \|x_1\|)\}$. So $B(x_1, x_2) = \{(x_2, x_1 - a/(1 + \|x_1\|))\}$ for each $(x_1, x_2) \in X \times X$. Hence, $\mathit{FIX}(B) = \emptyset$. However, for each $\delta > 0$, $\mathit{FIX}^{\delta}(B) \neq \emptyset$ since one can take $x \in X$ with $\|x\| \geqslant \delta^{-1}\|a\|$ and, then, $(x, x) \in \mathit{FIX}^{\delta}(B)$ because

$$\left\| (x, x) - \left(x, x - \frac{a}{1 + \|x\|} \right) \right\| = \frac{a}{1 + \|x\|} \leqslant \frac{\|a\|}{\|x\|} \leqslant \delta.$$

Moreover, u_1 and u_2 are uniform continuous functions on $X \times X$. In fact,

$$\begin{aligned} \left| u_2(x_1, x_2) - u_2(y_1, y_2) \right| &\leqslant \left\| (x_1 - y_1) - (x_2 - y_2) + \frac{\|x_1\| - \|y_1\|}{(1 + \|x_1\|)(1 + \|y_1\|)} a \right\| \\ &\leqslant \big(\|x_1 - y_1\| + \|x_2 - y_2\| \big) \big(1 + \|a\| \big). \end{aligned}$$

Therefore, in light of the key proposition we can conclude that $\mathit{NE}^{\varepsilon}(\Gamma) \neq \emptyset$ for each $\varepsilon > 0$. In fact, for $\|x\|$ sufficiently large, $(x, x) \in \mathit{NE}^{\varepsilon}(\Gamma)$, since $u_2(x, x_2) - u_2(x, x) \leqslant \|a\|/(1 + \|x\|)$.

4. Concluding remarks

In Section 2 we developed five new approximate fixed point theorems in infinite dimensional Banach spaces. In Theorem 2.1–2.3 bounded and totally bounded convex regions in Banach spaces are considered, while Theorems 2.4 and 2.5 treat possible unbounded convex regions. Theorems 2.1 and 2.2, and Theorems 2.4 and 2.5, respectively, differ only in that in Theorems 2.1 and 2.4 the multifunction is required to be weakly closed, while in Theorems 2.2 and 2.5 it is required to be weakly upper semicontinuous. Theorem 2.3 considers the situations in Theorems 2.1 and 2.2 in the context of the strong topology instead of the weak topology. It seems important to find more sophisticated approximate fixed point theorems, especially for (tame) multifunctions on unbounded sets. In Section 3 we have indicated, via the key proposition, how approximate fixed point theorems can play a role in non-cooperative game theory to prove the existence of approximate Nash equilibria. For a survey of techniques to prove the existence of (ε-) Nash equilibria see Tijs [21]. For approximate equilibrium theorems using approximations of games with smaller subgames see Lucchetti et al. [14]. Also we refer to Lignola [10] for the existence of Nash equilibria for games with non-compact strategy sets and to Lignola and Morgan [11] for convergence of Nash equilibria. The importance of ε-Nash equilibria is also motivated by well-posedness for Nash equilibria (cf. Lignola and Morgan [13], Margiocco et al. [17]), convergence properties of approximate Nash equilibria (cf. Morgan and Raucci [18]) and approximate solutions for hierarchical games (cf. Mallozzi and Morgan [15,16] for approximate mixed strategies).

Also finding new applications of approximate fixed point theorems in economic theory and in the study of well-posed fixed point problems (Lemaire et al. [9]) could be interesting.

Acknowledgment

The authors thank the anonymous referees for their valuable comments on the previous version of this paper.

References

[1] J.P. Aubin, Mathematical Methods of Game and Economic Theory, North-Holland, Amsterdam, 1979.
[2] C. Baiocchi, A.C. Capelo, Variational and Quasi-Variational Inequalities. Applications to Free Boundary Problems, Wiley, Chichester, 1984.
[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922) 133–181.
[4] C. Berge, Espaces Topologiques. Fonctions Multivoques, Dunod, Paris, 1959.
[5] L.E.J. Brouwer, Uber Abbildungen von Mannigfaltigkeiten, Math. Ann. 71 (1912) 97–115.
[6] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.
[7] I.L. Glicksberg, A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, Proc. Amer. Math. Soc. 3 (1952) 170–174.
[8] S. Kakutani, A generalization of Brouwer's fixed-point theorem, Duke Math. J. 8 (1941) 457–459.
[9] B. Lemaire, C.O.A. Salem, J.P. Revalsky, Well-posedness of variational problems with applications to staircase methods, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001) 943–948.
[10] M.B. Lignola, Ky Fan inequalities and Nash equilibrium points without semicontinuity and compactness, J. Optim. Theory Appl. 94 (1997) 137–145.
[11] M.B. Lignola, J. Morgan, Convergence of solutions of quasi-variational inequalities and applications, Topol. Methods Nonlinear Anal. 10 (1997) 375–385.
[12] M.B. Lignola, J. Morgan, Convergence for variational inequalities and generalized variational inequalities, Atti Sem. Mat. Fis. Univ. Modena 45 (1997) 377–388.
[13] M.B. Lignola, J. Morgan, Approximate solutions and α-well-posedness for variational inequalities and Nash equilibria, in: Decision and Control in Management Science, Kluwer Academic, 2002, pp. 367–378.
[14] R. Lucchetti, F. Patrone, S.H. Tijs, Determinateness of two-person games, Boll. Un. Mat. Ital. 6 (1986) 907–924.
[15] L. Mallozzi, J. Morgan, Hierarchical systems with weighted reaction set, in: G.D. Pillo, F. Giannessi (Eds.), Nonlinear Optimization and Applications, Plenum, 1996, pp. 271–282.
[16] L. Mallozzi, J. Morgan, Mixed strategies for hierarchical zero-sum games, Ann. Int. Soc. Dynam. Games 6 (2001) 65–77.

[17] M. Margiocco, F. Patrone, L. Pusillo Chicco, A new approach to Tikhonov well-posedness for Nash equilibria, Optimization 40 (1997) 385–400.

[18] J. Morgan, R. Raucci, New convergence results for Nash equilibria, J. Convex Anal. 6 (1999) 377–385.

[19] U. Mosco, Convergence of convex sets and solutions of variational inequalities, Lecture Notes in Math. 3 (1969) 510–585.

[20] J.F. Nash, Equilibrium points in n-person games, Proc. Nat. Acad. Sci. USA 36 (1950) 48–49.

[21] S.H. Tijs, Nash equilibria for non-cooperative n-person games in normal form, SIAM Rev. 23 (1981) 225–237.

[22] S.H. Tijs, A. Torre, R. Brânzei, Approximate fixed point theorems, Libertas Math. 23 (2003), to appear.