Normal's deconvolution and the independence of sample mean and variance (problem 03.4.1)

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PROBLEMS AND SOLUTIONS

PROBLEMS

03.4.1. Normal's Deconvolution and the Independence of Sample Mean and Variance

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(a) Let x_1 and x_2 be independent variates having m.g.f.s $m_1(t_1)$ and $m_2(t_2)$, respectively, and define $y := x_1 + x_2$. Prove that y is normal if and only if x_1 and x_2 are both normal. Is the existence of m.g.f.s necessary for this result?

(b) Let $\boldsymbol{x} := (x_1, \ldots, x_n)'$ be a vector of independent (but not necessarily identically distributed) components, where $2 \le n < \infty$. Define $\bar{x} := (1/n)\sum_{i=1}^{n} x_i$ and $z := \sum_{i=1}^{n}(x_i - \bar{x})^2$. It is well known that if $\boldsymbol{x} \sim \mathrm{N}(\mu\boldsymbol{\imath}, \sigma^2 \boldsymbol{I}_n)$, then $\bar{x} \sim \mathrm{N}(\mu, \sigma^2/n)$ independently from $z/\sigma^2 \sim \chi^2(n-1)$. For $n \ge 3$, prove that if $\bar{x} \sim \mathrm{N}(\mu, \sigma^2/n)$ and $z/\sigma^2 \sim \chi^2(n-1)$, then $\boldsymbol{x} \sim \mathrm{N}(\mu\boldsymbol{\imath}, \sigma^2 \boldsymbol{I}_n)$.

(c) Why is the last statement in (b) not necessarily true for $n = 2$? What additional conditions are needed to make it hold for $n = 2$?

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