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The central limit theorem for student's distribution (problem 03.6.1)

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PROBLEMS AND SOLUTIONS

SOLUTIONS

03.6.1 The Central Limit Theorem for Student's Distribution—Solution

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Consider the Lindeberg–Feller central limit theorem (CLT), which we state as follows. Let $\{x_n\}$ be a sequence of independent random variables with means $\{\mu_n\}$ and nonzero variances $\{\sigma_n^2\}$ (both existing), and c.d.f.s $\{F_n\}$. Define $\lambda_n > 0$ by $\lambda_n^2 = \sum_{i=1}^n \sigma_i^2$. Then, *Lindeberg's condition*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{|u - \mu_i| \geq \lambda_n \epsilon} \left(\frac{u - \mu_i}{\lambda_n} \right)^2 dF_i(u) = 0, \quad \forall \epsilon > 0,$$

is equivalent to

$$z_n := \frac{\sum_{i=1}^n (x_i - \mu_i)}{\lambda_n} \stackrel{a}{\sim} N(0,1) \quad \text{and} \quad \lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} \Pr \left(\frac{|x_i - \mu_i|}{\lambda_n} \geq \epsilon \right) = 0,$$

where the latter limit is known as the *uniform asymptotic negligibility* (u.a.n.) condition. One can usually interpret λ_n^2 as the variance of the numerator of z_n . We shall see, however, that there are cases where asymptotic normality holds in spite of $\{x_n\}$ having infinite variances.

Let $\{x_n\}$ be a random sample from Student's $t(\nu)$. For $\nu < 2$, no λ_n exists that can lead to $z_n := \sum_{i=1}^n x_i / \lambda_n \stackrel{a}{\sim} N(0,1)$. This is because the tails of the density of $t(\nu)$ decay at a rate of $u^{-\nu-1}$ and the stable limit theorem tells us that a nonnormal stable law arises if the tails of the p.d.f. of x_i decay at a rate of u^{-a} where $a < 3$; e.g., see Loève (1977, §25) or Hoffmann-Jørgensen (1994, §5.25). For example, for $\nu = 1$, the average of standard Cauchy variates is standard Cauchy too, so that there exists no λ_n achieving asymptotic normality of z_n .

For $\nu > 2$, both the mean and the variance exist, and the Lindeberg–Feller CLT applies, with $\lambda_n^2 = n \text{var}(x_i)$. The interesting part is $\nu = 2$, where we will show that asymptotic normality of z_n holds, in spite of $\text{var}(x_i)$ being infinite, and we will derive the appropriate λ_n . We will require the additional assump-

tion that $\lambda_n^2 \rightarrow \infty$ as $n \rightarrow \infty$. In the standard CLT, this assumption was unnecessary, as it followed from $\lambda_n^2 = n \operatorname{var}(x_i)$. We will see subsequently that λ_n^2 can be interpreted in terms of truncated variances for $\nu = 2$.

To prove the asymptotic normality of z_n , we need to show that the characteristic function $\varphi(t) := E(\exp(itx_i))$ satisfies

$$\lim_{n \rightarrow \infty} n \log \varphi\left(\frac{t}{\lambda_n}\right) = -\frac{t^2}{2} \quad (1)$$

for some choice of λ_n , with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Because the sequence $\{x_n\}$ is i.i.d., the uniform asymptotic negligibility condition

$$\lim_{n \rightarrow \infty} \max_{1 \leq i \leq n} \Pr\left(\frac{|x_i|}{\lambda_n} \geq \epsilon\right) = 0$$

is satisfied for all $\epsilon > 0$, thus implying

$$\lim_{n \rightarrow \infty} \left| \varphi\left(\frac{t}{\lambda_n}\right) - 1 \right| = 0.$$

This allows us to take the leading term of the logarithmic expansion of the left-hand side of (1) as

$$\begin{aligned} \lim_{n \rightarrow \infty} n \log \varphi\left(\frac{t}{\lambda_n}\right) &= \lim_{n \rightarrow \infty} n \left(\varphi\left(\frac{t}{\lambda_n}\right) - 1 \right) \\ &= -\frac{t^2}{2} \lim_{n \rightarrow \infty} n \int_{|u| < \lambda_n \epsilon} \frac{u^2}{\lambda_n^2} dF(u), \quad \lambda_n \rightarrow \infty, \end{aligned}$$

where the linear term in t drops out because the sequence $\{x_n\}$ is centered around zero. Asymptotic standard-normality obtains if we can find the appropriate λ_n^2 that makes the latter limit equal to 1 for all $\epsilon > 0$. Notice that this limit is the complement of Lindeberg's condition, where $\sum_{i=1}^n$ is replaced by n because $\{x_n\}$ is an i.i.d. sequence.

From Student's $t(2)$ density,

$$\begin{aligned} \int_{-c\sqrt{2}}^{c\sqrt{2}} \frac{u^2}{\sqrt{8} \left(1 + \frac{u^2}{2}\right)^{3/2}} du &= 2 \log(\sqrt{1+c^2} + c) - \frac{2c}{\sqrt{1+c^2}} \\ &= 2 \sinh^{-1}(c) - \frac{2c}{\sqrt{1+c^2}} \end{aligned}$$

tends to infinity as $c \rightarrow \infty$. We need to solve

$$1 = \lim_{n \rightarrow \infty} \frac{n}{\lambda_n^2} \int_{|u| < \lambda_n \epsilon} u^2 dF(u) = \lim_{n \rightarrow \infty} \frac{2n \sinh^{-1}(\lambda_n \epsilon / \sqrt{2})}{\lambda_n^2}$$

where we have dropped $2c/\sqrt{1+c^2} \rightarrow 2$ that is dominated by $\sinh^{-1}(c) \rightarrow \infty$. By using the logarithmic representation of the latter and simplifying,

$$1 = \lim_{n \rightarrow \infty} \frac{2n \log(\lambda_n)}{\lambda_n^2}$$

is solved by $\lambda_n = \sqrt{n \log(n)}$ or any other function that is asymptotically equivalent to it (such as $\sqrt{n \log(n)} + \sqrt{n}$). Therefore,

$$z_n = \frac{1}{\sqrt{n \log(n)}} \sum_{i=1}^n x_i \stackrel{d}{\sim} N(0,1).$$

REFERENCES

- Hoffmann-Jørgensen, J. (1994) *Probability with a View toward Statistics*, vol. I. Chapman and Hall.
 Loève, M. (1977) *Probability Theory I*, 4th ed. Springer-Verlag.