The central limit theorem for student's distribution (problem 03.6.1)

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PROBLEMS AND SOLUTIONS

SOLUTIONS

03.6.1 The Central Limit Theorem for Student's Distribution—Solution

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Consider the Lindeberg–Feller central limit theorem (CLT), which we state as follows. Let $\{x_n\}$ be a sequence of independent random variables with means $\{\mu_n\}$ and nonzero variances $\{\sigma_n^2\}$ (both existing), and c.d.f.s $\{F_n\}$. Define $\lambda_n > 0$ by $\lambda_n^2 = \sum_{i=1}^n \sigma_i^2$. Then, *Lindeberg's condition*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{|u-\mu_i|\geq\lambda_n\epsilon} \left(\frac{u-\mu_i}{\lambda_n}\right)^2 \mathrm{d}F_i(u) = 0, \qquad \forall\epsilon > 0,$$

is equivalent to

$$z_n := \frac{\sum_{i=1}^{n}(x_i-\mu_i)}{\lambda_n} \overset{a}{\sim} \mathrm{N}(0,1) \quad \text{and} \quad \lim_{n\to\infty} \max_{1\leq i\leq n} \Pr\left(\frac{|x_i-\mu_i|}{\lambda_n} \geq \epsilon\right) = 0,$$

where the latter limit is known as the *uniform asymptotic negligibility* (u.a.n.) condition. One can usually interpret λ_n^2 as the variance of the numerator of z_n. We shall see, however, that there are cases where asymptotic normality holds in spite of $\{x_n\}$ having infinite variances.

Let $\{x_n\}$ be a random sample from Student's $\mathrm{t}(\nu)$. For $\nu < 2$, no λ_n exists that can lead to $z_n := \sum_{i=1}^n x_i/\lambda_n \overset{a}{\sim} \mathrm{N}(0,1)$. This is because the tails of the density of $\mathrm{t}(\nu)$ decay at a rate of $u^{-\nu-1}$ and the stable limit theorem tells us that a nonnormal stable law arises if the tails of the p.d.f. of x_i decay at a rate of u^{-a} where $a < 3$; e.g., see Loève (1977, §25) or Hoffmann-Jørgensen (1994, §5.25). For example, for $\nu = 1$, the average of standard Cauchy variates is standard Cauchy too, so that there exists no λ_n achieving asymptotic normality of z_n.

For $\nu > 2$, both the mean and the variance exist, and the Lindeberg–Feller CLT applies, with $\lambda_n^2 = n\,\mathrm{var}(x_i)$. The interesting part is $\nu = 2$, where we will show that asymptotic normality of z_n holds, in spite of $\mathrm{var}(x_i)$ being infinite, and we will derive the appropriate λ_n. We will require the additional assumption that $\lambda_n^2 \to \infty$ as $n \to \infty$. In the standard CLT, this assumption was unnecessary, as it followed from $\lambda_n^2 = n \operatorname{var}(x_i)$. We will see subsequently that λ_n^2 can be interpreted in terms of truncated variances for $\nu = 2$.

To prove the asymptotic normality of z_n, we need to show that the characteristic function $\varphi(t) := \mathrm{E}(\exp(\mathrm{i}tx_i))$ satisfies

$$\lim_{n\to\infty} n \log \varphi\left(\frac{t}{\lambda_n}\right) = -\frac{t^2}{2} \tag{1}$$

for some choice of λ_n, with $\lambda_n \to \infty$ as $n \to \infty$. Because the sequence $\{x_n\}$ is i.i.d., the uniform asymptotic negligibility condition

$$\lim_{n\to\infty} \max_{1\le i\le n} \Pr\left(\frac{|x_i|}{\lambda_n} \ge \epsilon\right) = 0$$

is satisfied for all $\epsilon > 0$, thus implying

$$\lim_{n\to\infty} \left|\varphi\left(\frac{t}{\lambda_n}\right) - 1\right| = 0.$$

This allows us to take the leading term of the logarithmic expansion of the left-hand side of (1) as

$$\lim_{n\to\infty} n \log \varphi\left(\frac{t}{\lambda_n}\right) = \lim_{n\to\infty} n\left(\varphi\left(\frac{t}{\lambda_n}\right) - 1\right)$$

$$= -\frac{t^2}{2} \lim_{n\to\infty} n \int_{|u|<\lambda_n \epsilon} \frac{u^2}{\lambda_n^2}\, \mathrm{d}F(u), \quad \lambda_n \to \infty,$$

where the linear term in t drops out because the sequence $\{x_n\}$ is centered around zero. Asymptotic standard-normality obtains if we can find the appropriate λ_n^2 that makes the latter limit equal to 1 for all $\epsilon > 0$. Notice that this limit is the complement of Lindeberg's condition, where $\sum_{i=1}^n$ is replaced by n because $\{x_n\}$ is an i.i.d. sequence.

From Student's t(2) density,

$$\int_{-c\sqrt{2}}^{c\sqrt{2}} \frac{u^2}{\sqrt{8}\left(1 + \frac{u^2}{2}\right)^{3/2}}\, \mathrm{d}u = 2\log(\sqrt{1+c^2} + c) - \frac{2c}{\sqrt{1+c^2}}$$

$$= 2\sinh^{-1}(c) - \frac{2c}{\sqrt{1+c^2}}$$

tends to infinity as $c \to \infty$. We need to solve

$$1 = \lim_{n\to\infty} \frac{n}{\lambda_n^2} \int_{|u|<\lambda_n \epsilon} u^2 \mathrm{d}F(u) = \lim_{n\to\infty} \frac{2n \sinh^{-1}(\lambda_n \epsilon / \sqrt{2})}{\lambda_n^2}$$

where we have dropped $2c/\sqrt{1+c^2} \to 2$ that is dominated by $\sinh^{-1}(c) \to \infty$. By using the logarithmic representation of the latter and simplifying,

$$1 = \lim_{n\to\infty} \frac{2n \log(\lambda_n)}{\lambda_n^2}$$

is solved by $\lambda_n = \sqrt{n \log(n)}$ or any other function that is asymptotically equivalent to it (such as $\sqrt{n \log(n)} + \sqrt{n}$). Therefore,

$$z_n = \frac{1}{\sqrt{n \log(n)}} \sum_{i=1}^{n} x_i \overset{a}{\sim} \mathrm{N}(0,1).$$

REFERENCES

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