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Cooperation and competition in inventory games*

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Abstract. Inventory cost games are introduced in Meca et al. (1999). These games arise when considering the possibility of joint ordering in n-person EOQ inventory situations. Moreover, the SOC-rule is introduced and analysed as a cost allocation rule for this type of situations. In the current paper it is seen that n-person EPQ situations with shortages lead to exactly the same class of cost games. Furthermore, an alternative characterization of the SOC-rule is offered, primarily based on a transfer property which constitutes a special form of additivity. Necessary input variables for the SOC-rule are the (optimal) individual average number of orders per time unit in case there is no cooperation. Assuming that these average numbers are observable but not verifiable, we allow the players to select them strategically, while knowing that the SOC-rule will be (consecutively) applied as the cost allocation principle. Necessary and sufficient conditions are provided for the existence (and uniqueness) of a so-called constructive equilibrium in which all players make joint orders.

Key words: inventory models, inventory cost games, SOC-rule, constructive equilibria

1 Introduction

A promising research field is that of operations research games, which studies aspects of joint cost allocation in operations research models. These models

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are designed to optimise the operation of a complex system, in which, commonly, several agents are involved. Clearly, the effects of cooperation and/or competition of the agents who interact in an operations research problem play a prominent role here. In the last years, some surveys on this topic have been written, most of them stressing the connection between cooperative games and operations research: see, for instance, Curiel (1997) and Borm et al. (2001).

Very recently, also inventory models have been approached from this point of view. In particular, cooperation in a news-vendor problem has been treated in Hartman et al. (2000), Müller et al. (2001), and Slikker et al. (2001). In Meca et al. (1999) and Meca (2000) inventory cost games are introduced and studied. In an inventory cost game, a group of players dealing with the ordering and holding of a certain commodity (every individual agent's problem being an EOQ problem), decide to cooperate and make their orders jointly. This kind of cooperation is not unusual in the economic world: for instance, the runners of pharmacy offices usually make groups which order together. Meca et al. (1999) introduce and characterize the SOC-rule (Share the Ordering Costs) as an allocation rule for this class of games.

In this paper we revisit inventory cost games and the SOC-rule. It is seen that the wider class of n-person EPQ inventory situations with shortages lead to exactly the same class of cost games. Moreover, an alternative characterization of the SOC-rule is provided. The most important ingredient in this characterization is a so-called transfer property, which constitutes a kind of additivity.

Implicitly, the cooperative approach and the SOC-rule in particular, preassume that the optimal individual number of orders per time unit (if the players all order individually, without further coordination) is not only observable but also verifiable. Leaving out the verifiability condition, we take a strategic approach in which the SOC-rule still is the leading cost allocation principle, but where the players can strategically select a, possibly untruthful, individual number of orders as the decisive input variable for calculating the SOC-rule cost shares. A necessary and sufficient condition is provided such that a (unique) constructive equilibrium exists, in which all agents make joint orderings. This condition somehow requires the agents to have a rather similar individual ordering behavior.

2 Inventory cost games

An inventory cost game, as introduced in Meca et al. (1999), is a TU-game arising from an inventory cost situation. In an inventory cost situation, a group of agents $N = \{1, \dots, n\}$ agree to make jointly the orders of a certain good which all them need, so that they spend a instead of na ($a > 0$ being the fixed cost of an order) every time an order is placed. Each agent i, if ordering alone, would make $m_i \geq 0$ orders per time unit. A triplet (N, a, m), where $m = (m_1, \dots, m_n)$, characterizes an inventory cost situation and, for every coalition S, $c(S) = 2a\sqrt{\sum_{i \in S} m_i^2}$ is the *average inventory cost per time unit* for the agents in S if they place their orders jointly. The inventory cost game associated with the inventory cost situation (N, a, m) is (N, c). We denote by I^N the class of inventory cost games with player set N.

How is above expression derived for an n-person EOQ problem? The Economic Order Quantity (EOQ) problem is a well-known and simple operational research model. It considers an agent i who makes orders of a certain good that he sells. The demand that he must fulfil is deterministic and equal to d_i units per time unit ($d_i \geq 0$). The cost of keeping in stock one unit of this good per time unit is h_i ($h_i > 0$). If the fixed ordering cost is a and the lead time (the time between the placement of an order and the delivery of that order) is deterministic and constant (in which case it can be supposed to be zero without loss of generality), it can be easily checked that the average inventory cost per time unit is a function of Q_i (the order size) given by

$$c(Q_i) = a\frac{d_i}{Q_i} + h_i\frac{Q_i}{2}$$

and that the optimal order size $\hat{Q}_i$ is $\sqrt{2ad_i/h_i}$, so the optimal average inventory cost per time unit is

$$c(\hat{Q}_i) = \sqrt{2ad_ih_i} = 2a\hat{m}_i$$

where $\hat{m}_i = d_i/\hat{Q}_i$ is the optimal number of orders per time unit. Now, if the agents in $S \subset N$ decide to cooperate in order to save inventory costs, Meca et al (1999) proved that:

- In order to minimize the sum of the average inventory costs per time unit, the agents must coordinate their orders, so $Q_i^*/d_i = Q_j^*/d_j$ for all $i, j \in N$, Q_i^* and Q_j^* denoting the optimal order sizes for i and j if agents in S cooperate.
- $Q_i^* = \sqrt{\frac{2ad_i^2}{\sum_{j\in S} d_jh_j}}$, for all $i \in S$.
- The minimal sum of the average inventory costs per time unit is

$$\frac{ad_i}{Q_i^*} + \sum_{j\in S}\frac{h_jQ_j^*}{2} = 2a\sqrt{\sum_{j\in S}\hat{m}_j^2}.$$

Hence, the classes of inventory cost games and situations as defined above, arise in a very natural way as a tool to analyse cooperation in inventory EOQ models. The next step is to use game theory to allocate $c(N)$ among the agents. This issue will be treated in Section 3.

A remarkable feature that should be pointed out here is that, in order that the agents in N cooperate, they do not need to reveal their demands or holding costs to the other agents. It is enough that they know $\hat{m} = (\hat{m}_1, \ldots, \hat{m}_n)$ to be able to cooperate. Of course, to make their orders jointly, they must indicate their Q_i^*, and knowing $\hat{m}_i$ and Q_i^*, d_i and h_i can be computed by the other players. However, if it is important for the agents not to reveal their demands or holding costs, they can use an intermediary to make their orders.

In the remainder of this section we see that, surprisingly enough, the same class of inventory cost games arises from an inventory model which is more general than the EOQ model. This feature makes more interesting the study of this class of games.

Consider a set of agents N making orders of a certain good that they need; the fixed cost of an order is a. Again, every agent i needs d_i units of the good per time unit, and has a holding cost h_i for keeping in stock one good during one time unit. The changes with respect to the EOQ model treated in Meca et al (1999) are that now, for every $i \in N$:

- Agent i considers the possibility of not fulfilling all the demand in time, but allowing a shortage of the good. The cost of a shortage of one unit of the good for one time unit is $s_i > 0$.
- When an order is placed, after a deterministic and constant lead time (which can be assumed to be zero, without loss of generality), agent i receives the order gradually; more precisely, r_i units of the good are received per time unit. It is assumed that $r_i > d_i$ (otherwise the model makes little sense). We call r_i the *replacement rate* of agent i.

The inventory model we are dealing with for every agent is the Economic Production Quantity (EPQ) with shortages. It is a well-known model in inventory management (which generalizes the EOQ model; an EOQ model can be seen as an EPQ model for which the replacement rate and the shortage cost are infinity). In Tersine (1994), for instance, the analysis of this model (that we will summarize below) can be found. Note that every player i must choose $\hat{Q}_i$ (order size) and $\hat{M}_i$ (maximum shortage) minimizing his average inventory cost per time unit given by:

$$c(Q_i, M_i) = a\frac{d_i}{Q_i} + h_i\frac{\left(Q_i\left(1-\frac{d_i}{r_i}\right) - M_i\right)^2}{2Q_i\left(1-\frac{d_i}{r_i}\right)} + s_i\frac{M_i^2}{2Q_i\left(1-\frac{d_i}{r_i}\right)}.$$

Then, it results that

$$\hat{Q}_i = \sqrt{\frac{2ad_i}{h_i\left(1-\frac{d_i}{r_i}\right)}\frac{h_i+s_i}{s_i}}$$

$$\hat{M}_i = \sqrt{\frac{2ad_ih_i}{s_i(h_i+s_i)}\left(1-\frac{d_i}{r_i}\right)}.$$

It is easy to check that, if we denote $\hat{m}_i = d_i/\hat{Q}_i$ (the optimal number of orders that i must make per time unit), then

$$c(\hat{Q}_i, \hat{M}_i) = \sqrt{2ad_ih_i\frac{s_i}{h_i+s_i}\left(1-\frac{d_i}{r_i}\right)} = 2a\hat{m}_i.$$

Now, assume that the agents in $S \subset N$ decide to make their orders jointly to save part of the order costs. Following the ideas in Meca et al. (1999), it can be easily checked that, in order to minimize the sum of the average inventory costs per time unit, the agents must coordinate their orders, so $Q_i^*/d_i = Q_j^*/d_j$ for all $i, j \in N$, Q_i^* and Q_j^* denoting the optimal order sizes for i and j if agents in S cooperate. Then, the total average cost per time unit is given by:

$$c(Q_i,(M_j)_{j\in S}) = \frac{ad_i}{Q_i} + \sum_{j\in S} h_j \frac{\left(Q_j\left(1-\frac{d_j}{r_j}\right) - M_j\right)^2}{2Q_j\left(1-\frac{d_j}{r_j}\right)} + \sum_{j\in S} s_j \frac{M_j^2}{2Q_j\left(1-\frac{d_j}{r_j}\right)}$$

$$= \frac{ad_i}{Q_i} + \frac{1}{2}\sum_{j\in S} h_j \left(\frac{d_j}{d_i} Q_i\left(1-\frac{d_j}{r_j}\right) - 2M_j + \frac{d_i M_j^2}{d_j Q_i\left(1-\frac{d_j}{r_j}\right)}\right)$$

$$+ \frac{1}{2}\sum_{j\in S} s_j \frac{d_i M_j^2}{d_j Q_i\left(1-\frac{d_j}{r_j}\right)}.$$

Using standard techniques of differential analysis, it can be checked that the values $(Q_i^*)_{i\in S}$ and $(M_i^*)_{i\in S}$ which minimize c are given by:

$$Q_i^* = \sqrt{\frac{2ad_i^2}{\sum_{j\in S} d_j h_j \frac{s_j}{h_j+s_j}\left(1-\frac{d_j}{r_j}\right)}}$$

$$M_i^* = Q_i^* \frac{h_i\left(1-\frac{d_i}{r_i}\right)}{h_i+s_i}$$

for all $i \in S$. From this, it follows that

$$c(Q_i^*,(M_j^*)_{j\in S}) = 2a\sqrt{\sum_{j\in S} \hat{m}_j^2}.$$

Hence, if a group of agents N, all facing individual EPQ problems with shortages, agree to cooperate and to make their orders jointly, the minimal average inventory cost of every possible coalition gives rise to an inventory cost game, which can be characterized by the corresponding inventory cost situation $(N, a, \hat{m})$, where $\hat{m} = (\hat{m}_i)_{i\in N}$ and $\hat{m}_i$ is the optimal number of orders per time unit for agent i if he does not cooperate. Note that the optimal number of orders that the coalition of agents N must make when ordering jointly is d_i/Q_i^* (for all $i \in N$). Let us denote this quantity by $\hat{m}_N$. It is easy to check that $\hat{m}_N = \sqrt{\sum_{j\in N} \hat{m}_j^2}$.

To finish this section, we want to make two additional comments. First, like in the problem considered in Meca et al. (1999), agents do not need to reveal their demand, replacement rate, holding cost or shortage cost if they want to cooperate; it is only needed that $\hat{m}$ is known by all them. Second, r_i is interpreted as a production rate, in the sense that every time an order is made, the supplier has to produce the ordered units, and serve them gradually. Under this interpretation and taking into account that all the agents probably negotiate with the same supplier (otherwise it would be, perhaps, senseless that they order jointly), normally r_i will be the same for all i.

3 A proportional allocation rule for inventory cost games

Now we have revisited the class of inventory cost games and reconsidered its interest, we reexamine an allocation rule for this class of games proposed in Meca et al. (1999). This rule, which we now call *the SOC-rule* (Share the Ordering Costs), turns out to be a proportional-like rule, although this is not completely obvious from its definition. Let $(N, a, \hat{m})$ be an inventory cost situation, and $c \in I^N$ its corresponding inventory cost game. The SOC-rule σ proposes that agent i pays:

- A part of the fixed ordering cost proportional to his input parameter $\hat{m}_i^2$.
- His own holding and stock-out costs.

Hence:

$$\sigma_i(c) = a\frac{d_i}{Q_i^*}\frac{\hat{m}_i^2}{\sum_{j\in N}\hat{m}_j^2} + h_i\frac{\left(Q_i^*\left(1-\frac{d_i}{r_i}\right)-M_i^*\right)^2}{2Q_i^*\left(1-\frac{d_i}{r_i}\right)} + s_i\frac{M_i^{*2}}{2Q_i^*\left(1-\frac{d_i}{r_i}\right)}.$$

After some easy algebra, one derives

$$\sigma_i(c) = \frac{2a\hat{m}_i^2}{\sqrt{\sum_{j\in N}\hat{m}_j^2}}$$

or, equivalently,

$$\sigma_i(c) = \frac{\hat{m}_i^2}{\sum_{j\in N}\hat{m}_j^2}c(N).$$

In Meca et al. (1999), some other interesting properties of this rule are presented. For instance, it provides core allocations, i.e.

- $\sum_{i\in N}\sigma_i(c) = c(N)$,
- $\sum_{i\in S}\sigma_i(c) \le c(S)$, for all $S \subset N$,

for every $c \in I^N$. Moreover, σ can be reached through a PMAS (see Sprumont (1990)) and can be characterized in I^N using a monotonicity property, jointly with efficiency and symmetry.

The properties of the SOC-rule mentioned above show that it is an important and interesting rule for this class of games. In the remainder of this section we present a new characterization of the SOC-rule. It is inspired by the classical characterization of the Shapley value for TU-games, provided in Shapley (1953), which is based on its additivity. Obviously, the SOC-rule is not additive. However, we may wonder if it satisfies some kind of additivity which allows an axiomatic characterization *à la Shapley*. The answer is positive; next we present this special additivity, which is based on a quadratic sum.

Let $c, c' \in I^N$ two inventory cost games. The quadratic sum of c and c' (denoted by $c \oplus c'$) is the TU-game with player set N given by:

$$(c \oplus c')(S) = \sqrt{c(S)^2 + c'(S)^2}$$

for all $S \subset N$. Note that $c \oplus c' \in I^N$. If c, c' are the inventory cost games associated with (N, a, m) and (N, a', m'), respectively, it is easy to check that $c \oplus c'$ is associated with, for instance, the situation $(N, 1, \bar{m})$, where $\bar{m}_i = \sqrt{a^2 m_i^2 + a'^2 m_i'^2}$, for all $i \in N$. It is also easy to check that, for all $i \in N$, the unanimity game $u_{\{i\}}$ (defined by $u_{\{i\}}(S) = 1$ if $\{i\} \subset S$, $u_{\{i\}}(S) = 0$ if $\{i\} \not\subset S$) belongs to I^N, and that all unanimity games u_T with $|T| > 1$ do not belong to I^N. Let us prove now a proposition which will be useful later.

Proposition 1. *For every $c \in I^N$, there exist uniquely determined non-negative real numbers $\alpha_1, \ldots, \alpha_n$ such that $c = \bigoplus_{i \in N} \alpha_i u_{\{i\}}$.*

Proof. In order to prove the existence, take $\alpha_i = c(i) \geq 0$ for all $i \in N$. Then, for every $S \subset N$,

$$\left(\bigoplus_{i \in N} \alpha_i u_{\{i\}}\right)(S) = \sqrt{\sum_{i \in N} (c(i) u_{\{i\}}(S))^2} = \sqrt{\sum_{i \in S} c(i)^2} = 2a\sqrt{\sum_{i \in S} m_i^2} = c(S).$$

To prove the unicity, suppose that there exist $\beta_1, \ldots, \beta_n$ with $c = \bigoplus_{i \in N} \beta_i u_{\{i\}}$. Then, for every $S \subset N$,

$$c(S) = \sqrt{\sum_{i \in S} \beta_i^2} = \sqrt{\sum_{i \in S} c(i)^2}. \tag{1}$$

Taking $S = \{i\}$, for every $i \in N$, the uniqueness follows from (1). □

Let us introduce now the properties we need to characterize the SOC-rule. Let $\pi : I^N \rightarrow \mathbb{R}^n$ be an allocation rule for inventory cost games.

Transfer Property (TP). We say that π satisfies the transfer property if, for all $c, c' \in I^N$ and for all $i \in N$, $(c \oplus c')(N)\pi_i(c \oplus c') = c(N)\pi_i(c) + c'(N)\pi_i(c')$.

Efficiency (EFF). We say that π satisfies efficiency if, for all $c \in I^N$, $\sum_{i \in N} \pi_i(c) = c(N)$.

Null Player Property (NPP). We say that π satisfies the null player property if, for all $c \in I^N$ and all $i \in N$ with $c(i) = 0$, it holds that $\pi_i(c) = 0$.

The transfer property is the special additivity for inventory cost games that we announced earlier. The efficiency and null player property are standard properties of allocation rules for TU-games. Next we show that these three properties characterize the SOC-rule. It is readily verified that any pair of these properties is not sufficient to characterize this rule, so next theorem provides a tight characterization.

Theorem 1. *The SOC-rule σ is the unique allocation rule for inventory cost games which satisfies (TP), (EFF) and (NPP).*

Proof. Obviously σ satisfies (EFF) and (NPP). Let us check that it also satifies (TP). Let $c, c' \in I^N$ given by

$$c(S) = 2a\sqrt{\sum_{i \in S} m_i^2} \quad \text{and} \quad c'(S) = 2a'\sqrt{\sum_{i \in S} m_i'^2}$$

for all $S \subset N$. Then, for all $i \in N$,

$$(c \oplus c')(N)\sigma_i(c \oplus c') = (c \oplus c')(N)\frac{((c \oplus c')(i))^2}{(c \oplus c')(N)}$$

$$= c(i)^2 + c'(i)^2 = c(N)\sigma_i(c) + c'(N)\sigma_i(c').$$

Suppose now that π is an allocation rule on I^N satisfying (TP), (EFF) and (NPP), and take $c \in I^N$. In view of Proposition 1, $c = \bigoplus_{i \in N} c(i)u_{\{i\}}$. Take $j \in N$. Hence, since π satisfies (TP),

$$c(N)\pi_j(c) = \sum_{i \in N} c(i)\pi_j(c(i)u_{\{i\}}).$$

Now, taking into account that π satisfies (EFF) and (NPP),

$$\pi_j(c(i)u_{\{i\}}) = \begin{cases} c(j) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and, then, $c(N)\pi_j(c) = c(j)^2$, so $\pi_j(c) = \sigma_j(c)$. □

4 A non-cooperative approach to inventory games

In the previous two sections we analysed the cost allocation problem arising from inventory cost situations from a cooperative point of view. We assumed that the $\hat{m}_i$ (which play a crucial role in our model and in the SOC-rule) are observable and verifiable for all agents. In this section we adopt a non-cooperative point of view. We still assume that the $\hat{m}_i$ are observable for all agents (from e.g. longstanding practice), but now we consider that they are not verifiable. Therefore we propose a strategic round taking stock of the individual number of orders per time unit, before the cooperative implementation of the SOC-rule takes place.

More precisely, the non-cooperative situation we are going to study is the following. We consider a group of agents $N = \{1, \ldots, n\}$, $n \geq 2$. Each $i \in N$ faces an EPQ model with shortages, which is characterized by a, d_i, h_i, r_i and s_i. The individual optimal number of orders per time unit for every i, $\hat{m}_i$, is computed as in Section 2. We assume that $\hat{m} = (\hat{m}_1, \ldots, \hat{m}_n)$ is observable for all agents and that the agents in N decide to make their orders jointly and to allocate the costs using the SOC-rule, accordingly to the following two-step procedure:

1. Each agent $i \in N$ goes to an intermediary and selects $m_i \in [0, \infty)$ without knowing the choices of the other agents.
2. If only one agent has chosen a positive m_i in the first step, then all agents order alone. Otherwise, all $i \in N$ who have chosen a positive m_i in the first step go to the intermediary, who tells them what m has finally been

chosen and how many orders are going to be made per time unit ($m_N = \sqrt{\sum_{j\in N} m_j^2}$). Then i specifies Q_i to the intermediary (these Q_i are kept in secret to the other agents). Finally, all the agents i who have chosen $m_i = 0$ in step one order alone.

In this section we perform a non-cooperative analysis of this situation. To start with, let us obtain the normal form of this game G. The set of strategies of every agent $i \in N$ is $X_i = [0, +\infty)$, i.e., every agent can choose any non-negative m_i. Now we compute agent i's cost function, taking into account that the SOC-rule is used as an allocation mechanism.

$$c_i(m) = \begin{cases} c(i) = 2a\hat{m}_i & \text{if } m_i = 0 \text{ or } m_{-i} = 0, \\ \frac{am_i^2}{m_N} + \frac{h_i\left(Q_i(m)\left(1-\frac{d_i}{r_i}\right)-M_i(m)\right)^2}{2Q_i(m)\left(1-\frac{d_i}{r_i}\right)} + \frac{s_i M_i(m)^2}{2Q_i(m)\left(1-\frac{d_i}{r_i}\right)} & \text{otherwise,} \end{cases}$$

where $m_{-i} = 0$ means that all components of m different from the i-th are zero; $Q_i(m)$ and $M_i(m)$ denote the optimal order and optimal maximum shortage for player i if m has been chosen (i.e., if m_N orders will be made per time unit). Obviously, $Q_i(m) = d_i/m_N$. It can be checked that

$$M_i(m) = Q_i(m)\frac{h_i\left(1-\frac{d_i}{r_i}\right)}{h_i + s_i}.$$

Hence, if $m_i \neq 0$ and $m_{-i} \neq 0$,

$$\begin{aligned} c_i(m) &= \frac{am_i^2}{m_N} + \frac{h_i\left(\frac{d_i}{m_N}\left(1-\frac{d_i}{r_i}\right) - \frac{d_i}{m_N}\frac{h_i\left(1-\frac{d_i}{r_i}\right)}{h_i+s_i}\right)^2}{2\frac{d_i}{m_N}\left(1-\frac{d_i}{r_i}\right)} + \frac{s_i\frac{d_i}{m_N}\frac{h_i\left(1-\frac{d_i}{r_i}\right)^2}{h_i+s_i}}{2\frac{d_i}{m_N}\left(1-\frac{d_i}{r_i}\right)} \\ &= \frac{am_i^2}{m_N} + \frac{1}{2}\frac{d_i}{m_N}\left(1-\frac{d_i}{r_i}\right)\left[h_i\left(1-\frac{h_i}{h_i+s_i}\right)^2 + s_i\left(\frac{h_i}{h_i+s_i}\right)^2\right] \\ &= \frac{am_i^2}{m_N} + \frac{1}{2}\frac{d_i}{m_N}\left(1-\frac{d_i}{r_i}\right)\left(\frac{s_i h_i}{h_i+s_i}\right) \\ &= \frac{a(m_i^2 + \hat{m}_i^2)}{m_N}, \end{aligned}$$

where the last equality follows from the fact that $2a\hat{m}_i = \sqrt{2ad_ih_i\frac{s_i}{h_i+s_i}\left(1-\frac{d_i}{r_i}\right)}$. Summarizing, the game we are considering is $G = (X_1, \ldots, X_n, c_1, \ldots, c_n)$ where, for all $i \in N$,

$$c_i(m) = \begin{cases} 2a\hat{m}_i & \text{if } m_i = 0 \text{ or } m_{-i} = 0, \\ \frac{a(m_i^2+\hat{m}_i^2)}{m_N} & \text{otherwise.} \end{cases}$$

In the remainder of this section we study the Nash equilibria of this strategic cost game. Clearly, $m = (0, \dots, 0)$ is a Nash equilibrium of G. However, we are interested in what we call *constructive equilibria*. We say that $m \in X$ is a constructive equilibrium of G if it is a Nash equilibrium and, moreover, $m_i > 0$ for all $i \in N$.

Lemma 1. *If m is a constructive equilibrium of G then $m_i^2 = \hat{m}_i^2 - 2\sum_{j \in N \setminus i} m_j^2$, for all $i \in N$.*

Proof. Take into account that, for all $i \in N$,

$$\frac{\partial c_i}{\partial m_i}(m) = \frac{am_i\left(2\sum_{j \in N \setminus i} m_j^2 + m_i^2 - \hat{m}_i^2\right)}{m_N^3}.$$

This means that, for $m_i > 0$, c_i is decreasing when $m_i^2 < \hat{m}_i^2 - 2\sum_{j \in N \setminus i} m_j^2$ and increasing when $m_i^2 > \hat{m}_i^2 - 2\sum_{j \in N \setminus i} m_j^2$. □

In the next theorem we give a necessary and sufficient condition for G having a constructive equilibrium. This condition can be roughly interpreted as players being not too different. Thus, a group of players facing this non-cooperative situation will be able to adopt constructive behaviour when they have comparable optimal numbers of orders per time unit. Moreover, it is seen that if G has a constructive equilibrium then it is unique. Besides, an expression of this equilibrium is provided.

Theorem 2. *G has a constructive equilibrium if and only if, for all $i \in N$,*

$$\hat{m}_i^2 < \frac{2}{2n-3} \sum_{j \in N \setminus i} \hat{m}_j^2. \tag{2}$$

Moreover, if G has a constructive equilibrium $\bar{m}$, it is unique and, for all $i \in N$, it is defined by:

$$\bar{m}_i^2 = \frac{2}{2n-1} \sum_{j \in N \setminus i} \hat{m}_j^2 - \frac{2n-3}{2n-1} \hat{m}_i^2. \tag{3}$$

Proof. Let us first prove that, if G has a constructive equilibrium, then it is unique and given by (3). By Lemma 1, if $\bar{m}$ is a constructive equilibrium, then it is a solution of the system

$$A \begin{pmatrix} m_1^2 \\ \cdots \\ m_n^2 \end{pmatrix} = \begin{pmatrix} \hat{m}_1^2 \\ \cdots \\ \hat{m}_n^2 \end{pmatrix} \tag{4}$$

where

$$A = \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 & 2 \\ 2 & 1 & 2 & \cdots & 2 & 2 \\ & & & \cdots & & \\ 2 & 2 & 2 & \cdots & 2 & 1 \end{pmatrix}.$$

It is easy to check that

$$A^{-1} = \frac{2}{2n-1} J_n - I_n,$$

where J_n is the $n \times n$ matrix of ones and I_n is the $n \times n$ identity matrix. Hence, if $\bar{m}$ is a constructive equilibrium of G, $\bar{m}$ is uniquely given by (3). Let us now prove the necessary and sufficient condition. Clearly, if $\bar{m}$ is a constructive equilibrium of G, then for all $i \in N$,

$$0 < \bar{m}_i = \frac{2}{2n-1} \sum_{j \in N \setminus i} \hat{m}_j^2 - \frac{2n-3}{2n-1} \hat{m}_i^2$$

which implies that

$$\hat{m}_i^2 < \frac{2}{2n-3} \sum_{j \in N \setminus i} \hat{m}_j^2.$$

Hence, to finish the proof it is enough to demonstrate that, if (2) holds, then $\bar{m}$, defined as in (3), is a constructive equilibrium. Clearly, (2) implies that $\bar{m} > 0$. Moreover, since $(\bar{m}_1^2, \ldots, \bar{m}_n^2)$ is a solution to (4), we only have to prove that, for all $i \in N$,

$$c_i(\bar{m}) \leq c_i(\bar{m}_{-i}, 0) = 2a\hat{m}_i.$$

Note that

$$c_i(\bar{m}) = \frac{a\left(\frac{2}{2n-1}\sum_{j \in N \setminus i} \hat{m}_j^2 + \frac{3-2n}{2n-1}\hat{m}_i^2 + \hat{m}_i^2\right)}{\sqrt{\sum_{j \in N}\left(\frac{2}{2n-1}\sum_{k \in N \setminus j} \hat{m}_k^2 + \frac{3-2n}{2n-1}\hat{m}_j^2\right)}} = \frac{2a}{\sqrt{2n-1}} \sqrt{\sum_{j \in N} \hat{m}_j^2}. \tag{5}$$

Then, $c_i(\bar{m}) \leq 2a\hat{m}_i$ if and only if

$$\hat{m}_i^2 \geq \frac{\sum_{j \in N \setminus i} \hat{m}_j^2}{2n-2}. \tag{6}$$

But taking into account that $\hat{m}_i^2$ satisfies (2) for all $i \in N$

$$\sum_{j \in N \setminus i} \hat{m}_j^2 < \sum_{j \in N \setminus i} \frac{2}{2n-3} \sum_{l \in N \setminus j} \hat{m}_l^2 = \frac{2(n-1)}{2n-3} \hat{m}_i^2 + \sum_{j \in N \setminus i} \frac{2(n-2)}{2n-3} \hat{m}_j^2$$

and then

$$\frac{2n-2}{2n-3} \hat{m}_i^2 > \sum_{j \in N \setminus i} \frac{1}{2n-3} \hat{m}_j^2$$

which implies that condition (6) holds. □

To complete this section, we present some remarks concerning the Nash equilibria of game G.

Remark 1. *Denote* $N_0 = \{i \in N \mid \hat{m}_i = 0\}$ *and assume that* $N_0 = \varnothing$. *Then* G *has, at most, two Nash equilibria:* $(0, \dots, 0)$ *and a constructive equilibrium* $\bar{m}$ *(the latter may not exist). For, suppose that* m^* *is a Nash equilibrium and that* $m_k^* = 0$ *and* $m_l^* > 0$ *for some* $k, l \in N$. *Then*

$$c_k(m^*) = 2a\hat{m}_k > \frac{2a\hat{m}_k^2}{\sqrt{\hat{m}_k^2 + \sum_{j \in N \setminus k} m_j^{*2}}} = c_k(m_{-k}^*, \hat{m}_k),$$

which contradicts the fact that m^* *is a Nash equilibrium. Next assume that* $N_0 \neq \varnothing$ *and* $|N \setminus N_0| = \{i\}$. *In this case, it is clear that the set of Nash equilibria of the game is* $\{m \in [0, \infty)^n \mid m_{-i} = 0\}$. *Finally assume that* $N_0 \neq \varnothing$ *and* $|N \setminus N_0| \geq 2$. *Then there are not constructive equilibria because, otherwise, if* m *is a constructive equilibrium, then every player* $i \in N_0$ *would have an incentive to deviate from* m_i *to zero. In this case it is clear that* $(m_{-N_0}, 0)$ *is a Nash equilibrium of the game if and only if* $m_{-N_0} = 0$ *or* m_{-N_0} *is a constructive equilibrium of the game restricted to the set of players* $N \setminus N_0$.

Remark 2. *Note that, if* $\bar{m}$ *is a constructive equilibrium of* G, *then* $\bar{m}$ *is a strict equilibrium, i.e.,* $c_i(\bar{m}) < c_i(\bar{m}_{-i}, m_i')$, *for all* $m_i' \in [0, +\infty) \setminus \{\bar{m}_i\}$ *and all* $i \in N$.

To finish this section we compare the payoffs for the players in the different scenarios considered in this paper.

Proposition 1. *Let* $\bar{m}$ *be a constructive equilibrium of game* G *and let* (N, c) *be the corresponding inventory cost game. If* $n \geq 3$ *then*

$$\sigma_i(c) < c_i(\bar{m}) < c(i)$$

for all $i \in N$.

Proof. The inequality $c_i(\bar{m}) < c(i)$ follows from Remark 2 and the fact that $c_i(\bar{m}_{-i}, 0) = c(i)$, for all $i \in N$. The inequality $\sigma_i(c) < c_i(\bar{m})$ is equivalent to

$$(\sqrt{2n-1})\hat{m}_i^2 < \sum_{j \in N} \hat{m}_j^2 \tag{7}$$

using (5). Note that (7) is equivalent to

$$(\sqrt{2n-1} - 1)\hat{m}_i^2 < \sum_{j \in N \setminus i} \hat{m}_j^2, \tag{8}$$

which can be readily checked using (2) and the fact that $2\sqrt{2n-1} < 2n - 1$ if $n \geq 3$. □

It is easy to check that, in case $n = 2$, maybe $\sigma_i(c) > c_i(\bar{m})$ for some i, although it holds that $\sigma_1(c) + \sigma_2(c) < c_1(\bar{m}) + c_2(\bar{m})$.

References

Borm P, Hamers H, Hendrickx R (2001) Operations research games: A survey. Top 9:139–216

Curiel I (1997) Cooperative game theory and applications. Kluwer Academic Publishers

Hartman B, Dror M, Shaked M (2000) Cores of inventory centralization games. Games and Economic Behavior 31:26–49

Meca A (2000) Juegos cooperativos asociados a problemas de inventario. PhD dissertation, Miguel Hernández University

Meca A, Timmer J, García-Jurado I, Borm P (1999) Inventory games. Discussion Paper 9953, Tilburg University

Müller A, Scarsini M, Shaked M (2001) The newsvendor game has a non-empty core. To appear in Games and Economic Behavior

Shapley LS (1953) A value for n-person games. In: Kuhn H, Tucker AW (eds.) Contributions to the Theory of Games II. Princeton University Press, pp. 307–317

Slikker M, Fransoo J, Wouters M (2001) Joint ordering in multiple news-vendor problems: a game theoretical approach. Preprint

Sprumont Y (1990) Population monotonic allocation schemes for cooperative games with transferable utility. Games and Economic Behavior 2:378–394

Tersine RJ (1994) Principles of inventory and materials management. Elsevier