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COMPROMISE STABLE TU-GAMES

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Compromise stable TU-games

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Abstract

In this paper we characterize the class of games for which the core coincides with the core cover (compromise stable games). Moreover we will develop an easy explicit formula for the nucleolus for this class of games, using an approach based on bankruptcy problems. Also the class of convex compromise stable games is characterized. The relation between core cover and Weber set is studied and it is proved that under a weak condition their intersection is nonempty.

Keywords: Core cover, core, nucleolus.

JEL Classification Number: C71.

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1 Introduction

An important issue in cooperative game theory is the allocation of the value of the grand coalition of a game to the players of this game. To this aim various solution concepts have been developed. They can be categorized in one point solution concepts, e.g. the Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)) and the compromise value (Tijs (1981)), and set-valued solutions concepts, e.g. the core (Gillies (1953)), the core cover (Tijs and Lipperts (1982)) and the Weber set (Weber (1988)). The core is contained in the Weber set and the core cover. And, the nucleolus is an element of the core. It is established that a game is convex (Shapley (1971), Ichiishi (1981)) if and only if the Weber set coincides with the core.

In this paper we characterize the class of games for which the core coincides with the core cover (compromise stable games). This class contains the class of bankruptcy games (Curiel, Maschler, and Tijs (1988)), big boss games (Muto, Nakayama, Potters, and Tijs (1988)) and clan games (Potters, Poos, Muto, and Tijs (1989)). Moreover we will develop an easy explicit formula for the nucleolus for this class of games, using an approach based on bankruptcy problems. As an application we provide an easy proof of the formula for the nucleolus of big boss and clan games as derived by Muto et al. (1988) and Potters et al. (1989). Furthermore the class of convex compromise stable games is characterized. Finally the relation between the core cover and the Weber set is studied. It is proved that under a weak condition their intersection is nonempty.

In section 2 we recall some game theoretic notions. Section 3 deals with the characterization of the class of compromise stable games. Section 4 derives an explicit formula for the nucleolus for compromise stable games and applications to big boss and clan games are provided. In the final section the relation between the core cover and the Weber set is studied.

2 Preliminaries

This section reviews some general notions about transferable utility games. A transferable utility game (TU-game) consists of a pair (N, v) , in which $N = \{1, \dots, n\}$ is a set of players and $v : 2^N \rightarrow \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a payoff $v(S)$, by definition $v(\emptyset) = 0$. The set of all transferable utility games with player set N is denoted by TU^N . A game (N, v) is additive if there exists a vector $a \in \mathbb{R}^N$ such that $v(S) = \sum_{i \in S} a_i$ for all $S \in 2^N$. The game (N, v) is then denoted by (N, a) . A game (N, v)

is strategically equivalent to (N, w) if there exist a positive real number k and an additive game (N, a) such that $w = a + kv$. A game (N, v) is superadditive if for all $S, T \subset N$ with $S \cap T = \emptyset$ it holds that:

$$v(S) + v(T) \leq v(S \cup T).$$

The core of a TU-game (N, v) is given by:

$$C(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

The core of a game consists of those payoff vectors such that no coalition has an incentive to split off. The core of a game might be empty. A game is called balanced if it has a non-empty core.

A special class of TU-games is the class of bankruptcy games (O'Neill (1982)). These games arise from so-called bankruptcy situations. These situations are formalized by a triple (N, E, d) , or a pair (E, d) . $E \geq 0$ is the estate which has to be divided among the claimants. N is the set of claimants and $d \geq 0$ is a vector of claims. By the nature of a bankruptcy problem it holds that:

$$E \leq \sum_{i \in N} d_i.$$

One can associate a bankruptcy game $v_{E,d}$ to a bankruptcy problem (E, d) . The value of a coalition S is determined by the amount of E that is not claimed by $N \setminus S$:

$$v_{E,d}(S) = \max \left\{ 0, E - \sum_{i \in N \setminus S} d_i \right\}.$$

An order of N is a bijective function $\sigma : \{1, \dots, n\} \rightarrow N$. The player at position i in the order σ is denoted by $\sigma(i)$. The set of all orders of N is denoted by $\Pi(N)$. For $\sigma \in \Pi(N)$ the corresponding marginal vector $m^\sigma(v)$ measures the marginal contribution of the players with respect to σ , i.e.

$$m_{\sigma(i)}^\sigma(v) = v(\{\sigma(1), \dots, \sigma(i)\}) - v(\{\sigma(1), \dots, \sigma(i-1)\}), \quad i \in \{1, \dots, n\}.$$

The Shapley value (Shapley (1953)) $\phi(v)$ is computed by taking the average of all marginal vectors:

$$\phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

The Weber set is the convex hull of all marginals vectors:

$$W(v) = \text{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\}.$$

An important relation between core and Weber set is given in the following proposition:

Proposition 2.1 (Weber (1988)) *Let $v \in TU^N$, then $C(v) \subset W(v)$.*

A game (N, v) is convex if for all $i \in N$ and all $S \subset T \subset N \setminus \{i\}$ it holds that:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

For convex games the marginal contribution of a player increases if this player joins a larger coalition. Shapley (1971) and Ichiishi (1981) proved that a game (N, v) is convex if and only if $C(v) = W(v)$. For example bankruptcy games are convex games.

The utopia vector $M(v)$ of a game (N, v) consists of the utopia demands of all players. The utopia demand of a player $i \in N$ is given by:

$$M_i(v) = v(N) - v(N \setminus \{i\}).$$

The minimum right of player i corresponds to the minimum value this player can achieve by satisfying all other players in a coalition by giving them their utopia demands:

$$m_i(v) = \max_{S: i \in S} \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \right\}.$$

For a convex game (N, v) it is easily verified that $m_i(v) = v(\{i\})$ for all $i \in N$.

Using these two vectors one can introduce the core cover, $CC(v)$, of a game (N, v) . The core cover consists of all efficient payoff vectors, giving each player at least his minimum right, but no more than his utopia demand:

$$CC(v) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), m(v) \leq x \leq M(v) \right\}.$$

The elements of the core cover can be interpreted as possible allocations of the value of the grand coalition and can be seen as compromise values between $m(v)$ and $M(v)$. Note that the core cover of a game can be empty. A game $v \in TU^N$ is said to be compromise admissible if:

$$m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v).$$

Clearly the core cover of (N, v) is non-empty if and only if (N, v) is compromise admissible. The class of all compromise admissible games with player set N is denoted by CA^N . The following result about the core and the core cover is well known:

Proposition 2.2 (Tijs and Lipperts (1982)) *Let $v \in TU^N$, then $C(v) \subset CC(v)$.*

The extreme points of the core cover can be described by larginal vectors. The definition of a larginal vector is similar to the definition of a marginal vector. For $\sigma \in \Pi(N)$ the larginal $l^\sigma(v)$ is the efficient payoff vector giving the first players in σ their utopia demands as long as it is still possible to satisfy the remaining players with at least their minimum rights.

Definition 2.1 *Let $v \in CA^N$ and $\sigma \in \Pi(N)$. The larginal vector $l^\sigma(v)$ is defined by:*

$$l_{\sigma(i)}^\sigma(v) = \begin{cases} M_{\sigma(i)}(v) & \text{if } \sum_{j=1}^i M_{\sigma(j)}(v) + \sum_{j=i+1}^n m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(i)}(v) & \text{if } \sum_{j=1}^{i-1} M_{\sigma(j)}(v) + \sum_{j=i}^n m_{\sigma(j)}(v) \geq v(N), \\ v(N) - \sum_{j=1}^{i-1} M_{\sigma(j)}(v) - \sum_{j=i+1}^n m_{\sigma(j)}(v) & \text{otherwise,} \end{cases}$$

for every $i \in \{1, \dots, n\}$.

The concept of larginal vectors is also used in González Díaz, Borm, Hendrickx, and Quant (2003). An alternative way to describe the core cover is by means of the larginals:

$$CC(v) = \text{conv}\{l^\sigma(v) \mid \sigma \in \Pi(N)\}.$$

The first player with respect to σ that does not receive his utopia payoff is called the pivot of $l^\sigma(v)$. In case every player gets his utopia payoff, the pivot is the last player. Note that each larginal vector contains exactly one pivot. The following example illustrates the notion of larginal vectors and pivots.

Example 2.1 *Let $v \in CA^N$ be the game defined by:*

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	1	0	0	0	1	2	1	2	2	2	7	4	6	8	10

Then $M(v) = (2, 4, 6, 3)$ and $m(v) = (1, 0, 1, 0)$. For $\sigma = 1234$ the larginal $l^\sigma(v)$ equals $(2, 4, 4, 0)$ and player three is the pivot of this larginal. If $\sigma = 3421$ the corresponding larginal equals $l^\sigma(v) = (1, 0, 6, 3)$ and player two is the pivot. The core cover of (N, v) can be described by:

$$\begin{aligned} CC(v) &= \text{conv}\{l^\sigma(v) \mid \sigma \in \Pi(N)\} \\ &= \text{conv}\{(2, 4, 4, 0), (2, 4, 1, 3), (2, 2, 6, 0), (2, 0, 6, 2), (2, 0, 5, 3), \\ &\quad (1, 4, 5, 0), (1, 4, 2, 3), (1, 0, 6, 3), (1, 3, 6, 0)\}. \end{aligned}$$

Since for all $\sigma \in \Pi(N)$ it holds that $l^\sigma(v) \in C(v)$, we may conclude that $C(v) = CC(v)$.

Tijs (1981) introduced the compromise value, also known as the τ -value, as a one point solution concept based on the utopia vector and the minimum right vector. The compromise value $\tau(v)$ of a compromise admissible game (N, v) is the efficient convex combination of $M(v)$ and $m(v)$:

$$\tau(v) = \alpha M(v) + (1 - \alpha)m(v),$$

with $\alpha \in [0, 1]$ such that:

$$\sum_{i \in N} \tau_i(v) = v(N).$$

The nucleolus $\nu(v)$ of a game (N, v) is introduced by Schmeidler (1969) and is an element of the imputation set. The imputation set of a game (N, v) is defined as:

$$I(v) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}), \forall i \in N\}.$$

For an imputation $x \in I(v)$ the excess of coalition S with respect to x measures the complaint of coalition S :

$$E(S, x) = v(S) - \sum_{i \in S} x_i.$$

The vector $\theta(x)$ contains the complaints of all coalitions with respect to x in decreasing order. The nucleolus $\nu(v)$ is the unique imputation minimizing the maximum complaint, i.e. the nucleolus is the lexicographic minimum of the set $\{\theta(x) \mid x \in I(v)\}$.

3 Core and core cover

In this section we characterize the class of compromise stable games. Furthermore the class of convex compromise stable games is characterized as well.

We are interested in the class of compromise stable games. For example bankruptcy games, big boss games and clan games (the precise definitions are provided later on) are compromise stable games.

Definition 3.1 *Let $v \in TU^N$, then (N, v) is compromise stable if $C(v) = CC(v)$.*

The following theorem characterizes the class of compromise stable games.

Theorem 3.1 *Let $v \in CA^N$, then $C(v) = CC(v)$ if and only if for all $S \in 2^N \setminus \{\emptyset\}$ the following is true:*

$$v(S) \leq \max \left\{ \sum_{i \in S} m_i(v), v(N) - \sum_{i \in N \setminus S} M_i(v) \right\}. \quad (1)$$

Proof: Let $v \in CA^N$. First suppose that $C(v) = CC(v)$. Then it holds for all $\sigma \in \Pi(N)$ that $l^\sigma(v) \in C(v)$. Let $S \in 2^N \setminus \{\emptyset\}$, we have to show that (1) holds. Let $\sigma \in \Pi(N)$ begin with all players of $N \setminus S$ and end with the players of S . Hence for $i \in \{1, \dots, |N \setminus S|\}$ it holds that $\sigma(i) \in N \setminus S$. Let $l^\sigma(v)$ be the corresponding larginal vector. There are two possibilities:

- The pivot of $l^\sigma(v)$ is an element of $N \setminus S$. Hence each player of S has a payoff equal to his minimum right. We can conclude that:

$$\begin{aligned} v(S) &\leq \sum_{i \in S} l_i^\sigma(v) \\ &= \sum_{i \in S} m_i(v). \end{aligned}$$

- The pivot of $l^\sigma(v)$ is an element of S . This implies that each player in $N \setminus S$ achieves a payoff equal to his utopia demand. It follows that:

$$\begin{aligned} v(S) &\leq \sum_{i \in S} l_i^\sigma(v) \\ &= v(N) - \sum_{i \in N \setminus S} l_i^\sigma(v) \\ &= v(N) - \sum_{i \in N \setminus S} M_i(v). \end{aligned}$$

Combining these two cases yields that:

$$v(S) \leq \max \left\{ \sum_{i \in S} m_i(v), v(N) - \sum_{i \in N \setminus S} M_i(v) \right\}.$$

Secondly, assume that inequality (1) holds for each $S \in 2^N \setminus \{\emptyset\}$. By convexity of the core it suffices to show that for each order $\sigma \in \Pi(N)$, $l^\sigma(v)$ is an element of the core. Let $\sigma \in \Pi(N)$ and $S \in 2^N \setminus \{\emptyset\}$. Then at least one of the following statements is true:

$$\begin{aligned} v(S) &\leq v(N) - \sum_{i \in N \setminus S} M_i(v) \\ &\leq v(N) - \sum_{i \in N \setminus S} l_i^\sigma(v) \\ &= \sum_{i \in S} l_i^\sigma(v), \end{aligned}$$

or,

$$\begin{aligned} v(S) &\leq \sum_{i \in S} m_i(v) \\ &\leq \sum_{i \in S} l_i^\sigma(v). \end{aligned}$$

In both cases the core condition concerning coalition S is satisfied. Hence, $l^\sigma(v)$ is an element of $C(v)$. \square

In the following example Theorem 3.1 is illustrated.

Example 3.1 *Consider the game of Example 2.1. For every coalition S it holds that (1) is valid. For example if $S = \{1, 2\}$, it holds that $v(\{1, 2\}) \leq m_1(v) + m_2(v)$ and if $S = \{2, 3\}$ it holds that $v(\{2, 3\}) \leq v(N) - M_1(v) - M_4(v)$. So according to Theorem 3.1 it holds that $C(v) = CC(v)$.*

The following theorem describes the class of convex compromise stable games. This class contains exactly the games that are strategically equivalent to bankruptcy games. Because each bankruptcy game is convex and compromise stable, this gives a characterization of the class of bankruptcy games.

Theorem 3.2 *(N, v) is a convex compromise stable game if and only if (N, v) is strategically equivalent to a bankruptcy game.*

Proof: Let (N, v) be a convex compromise stable game. Define $a_i = v(\{i\}) = m_i(v)$ (the last equality holds because (N, v) is a convex game) and $w(S) = v(S) - \sum_{i \in S} a_i$ for all $S \in 2^N$. Then (N, w) is a zero-normalized convex game and $C(w) = CC(w)(= W(w))$. Furthermore the following equations hold:

$$\begin{aligned} M(w) &= M(v) - m(v), \\ m(w) &= 0. \end{aligned}$$

We will show that (N, w) is the bankruptcy game $(N, v_{E,d})$ with $E = w(N)$ and $d = M(w)$. For $S \in 2^N \setminus \{\emptyset\}$ it holds that:

$$\begin{aligned} v_{E,d}(S) &= \max \left\{ 0, E - \sum_{i \in N \setminus S} M_i(w) \right\} \\ &= \max \left\{ \sum_{i \in S} m_i(w), E - \sum_{i \in N \setminus S} M_i(w) \right\}. \end{aligned}$$

Theorem 3.1 implies that $w(S) \leq v_{E,d}(S)$ for all $S \subset N$. Now suppose that there exists a coalition $S \in 2^N \setminus \{\emptyset\}$ such that $w(S) < v_{E,d}(S)$. Because (N, w) is a convex game, it holds that $w(S) \geq \sum_{i \in S} w(\{i\}) = \sum_{i \in N} m_i(w)$ and hence:

$$\begin{aligned} w(S) &< E - \sum_{i \in N \setminus S} M_i(w) \\ &= w(N) - \sum_{i \in N \setminus S} M_i(w). \end{aligned}$$

Consider a permutation $\sigma \in \Pi(N)$ that begins with the players of S and ends with the players of $N \setminus S$, i.e. $\sigma(i) \in S$ for $i \in \{1, \dots, |S|\}$. The payoff of coalition $N \setminus S$ according to the marginal vector $m^\sigma(w)$ is given by:

$$\begin{aligned} \sum_{j \in N \setminus S} m_j^\sigma(w) &= w(N) - w(S) \\ &> \sum_{j \in N \setminus S} M_j(w). \end{aligned}$$

This implies that $m^\sigma(w) \notin CC(w)$. This contradicts $CC(w) = C(w) = W(w)$.

The converse is also true because bankruptcy games are convex games and the core of a coincides with the core cover (Curiel et al. (1988)). \square

It is trivial to show that a 3-player TU-game is balanced if and only if it is compromise admissible. Moreover for any 3-player game (N, v) it holds that $C(v) = CC(v)$. From Theorem 3.2 it then follows that each convex three player game is strategically equivalent to a bankruptcy game.

4 Compromise solutions based on bankruptcy

There are several well-known solutions for bankruptcy problems. These solutions are called bankruptcy rules. Let (E, d) be a bankruptcy problem and $i \in N$. The following bankruptcy rules are often used:

- **Constrained equal award rule (CEA):**

$$CEA_i(E, d) = \min\{\alpha, d_i\},$$

with α such that $\sum_{i \in N} \min\{\alpha, d_i\} = E$.

- **Proportional rule (PROP):**

$$PROP_i(E, d) = \frac{d_i}{\sum_{j \in N} d_j} \cdot E.$$

- **Talmud rule (TAL):**

$$TAL_i(E, d) = \begin{cases} CEA_i(E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j \geq 2E \\ d_i - CEA_i(\sum_{j \in N} d_j - E, \frac{1}{2}d) & \text{if } \sum_{j \in N} d_j < 2E. \end{cases}$$

- **Run to the bank rule (RTB):**

$$RTB_i(E, d) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} r_i^\sigma(E, d).$$

The value of $r_{\sigma(j)}^\sigma(E, d)$, $\sigma \in \Pi(N)$, depends on the amount left of E if all players which are before $\sigma(j)$ in σ get their claim (as far as this is possible):

$$r_{\sigma(j)}^\sigma(E, d) = \max \left\{ \min \left\{ d_{\sigma(j)}, E - \sum_{k=1}^{j-1} d_{\sigma(k)} \right\}, 0 \right\}.$$

A bankruptcy rule f is self-dual if:

$$f(E, d) = d - f\left(\sum_{i \in N} d_i - E, d\right).$$

It is easy to see that the proportional rule is self-dual. Curiel (1988) proves that the Talmud rule and the run to the bank rule are self-dual. Note that if f is self-dual and $\sum_{i \in N} d_i = 2E$, then $f(E, d) = \frac{1}{2}d$.

Bankruptcy games have some nice properties. For example Aumann and Maschler (1985) proved that the nucleolus of a bankruptcy game $(N, v_{E,d})$ is given by:

$$\nu(v_{E,d}) = TAL(E, d).$$

Furthermore the Shapley value can be computed by (cf. O'Neill (1982)):

$$\phi(v_{E,d}) = RTB(E, d).$$

This result gives rise to the thought that it is interesting to approach allocation problems in TU-games from the point of view of bankruptcy problems. We will consider an approach based on the core cover, which consists of all efficient compromise solutions between $m(v)$ and $M(v)$. Let $v \in CA^N$ and f be a bankruptcy rule, then one could consider the following type of compromise solution γ :

$$\gamma(v) = m(v) + f\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right). \quad (2)$$

From this point of view, one could rewrite the compromise value as:

$$\tau(v) = m(v) + PROP\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).$$

Theorem 3.2 enables us to establish a relation between the Shapley value and the run to the bank rule for convex compromise stable games. A one point solution f is relative invariant with respect to strategic equivalence if $f(w) = a + kf(v)$ if $w = a + kv$. The Shapley value is relative invariant with respect to strategic equivalence. The relation between the Shapley value and the run to the bank rule for bankruptcy games in combination with Theorem 3.2 yields the following result:

Corollary 4.1 *Let (N, v) be a convex compromise stable game, then:*

$$\phi(v) = m(v) + RTB\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).$$

The following theorem shows that the nucleolus for compromise stable games can be computed by taking the Talmud rule as bankruptcy rule in (2). In the proof the following important result is used:

Theorem 4.1 (Potters and Tijs (1994)) *Let (N, v) and (N, w) be two games such that (N, v) is a convex game and $C(v) = C(w)$. Then the nucleoli of (N, v) and (N, w) coincide:*

$$\nu(v) = \nu(w).$$

Theorem 4.2 *Let $v \in CA^N$ be compromise stable. Then:*

$$\nu(v) = m(v) + TAL\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right). \quad (3)$$

Proof: Let $v \in CA^N$ be compromise stable. Define the additive game (N, a) by taking $a_i = m_i(v)$ for all $i \in N$, and define $w(S) = v(S) - \sum_{i \in S} a_i$, $S \in 2^N$. Because the nucleolus is relative invariant with respect to strategic equivalence, it holds that:

$$\begin{aligned} \nu(v) &= a + \nu(w) \\ &= m(v) + \nu(w). \end{aligned}$$

For (N, w) the following assertions can easily be verified:

$$\begin{aligned} M(w) &= M(v) - m(v), \\ m(w) &= m(v) - m(v) = 0, \\ w(N) &= v(N) - \sum_{i \in N} m_i(v), \\ C(w) &= CC(w) \\ &= \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N), 0 \leq x \leq M(w) \right\}. \end{aligned}$$

Consider the bankruptcy problem defined by $E = w(N)$ and $d = M(w)$. For the corresponding bankruptcy game $(N, v_{E,d})$ it holds that:

$$v_{E,d}(N) = w(N).$$

Using the convexity of $(N, v_{E,d})$, it holds for $i \in N$ that:

$$\begin{aligned}
M_i(v_{E,d}) &= \min\{E, d_i\}, \\
m_i(v_{E,d}) &= v_{E,d}(\{i\}) \\
&= \left(E - \sum_{j \in N \setminus \{i\}} d_j\right)_+ \\
&= \left(w(N) - \sum_{j \in N \setminus \{i\}} M_j(w)\right)_+ \\
&= 0.
\end{aligned}$$

The last equality follows from the fact that $m_i(w) = 0$, and $m_i(w) \geq w(N) - \sum_{j \in N \setminus \{i\}} M_j(w)$. The core of $(N, v_{E,d})$ can now be written as:

$$\begin{aligned}
C(v_{E,d}) &= CC(v_{E,d}) \\
&= \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = E, 0 \leq x_i \leq \min\{E, d_i\}, \forall i \in N\right\} \\
&= \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N), \right. \\
&\quad \left. 0 \leq x_i \leq \min\{w(N), M_i(w)\}, \forall i \in N\right\} \\
&= \left\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N), 0 \leq x \leq M(w)\right\} \\
&= CC(w) \\
&= C(w).
\end{aligned}$$

Since $(N, v_{E,d})$ and (N, w) have the same core, and $(N, v_{E,d})$ is convex, we can apply Theorem 4.1. This yields:

$$\begin{aligned}
\nu(w) &= \nu(v_{E,d}) \\
&= TAL(E, d) \\
&= TAL(w(N), M(w)) \\
&= TAL\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).
\end{aligned}$$

Furthermore, the nucleolus of (N, v) is now given by:

$$\begin{aligned}
\nu(v) &= m(v) + \nu(w) \\
&= m(v) + TAL\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).
\end{aligned}$$

□

For any 3-player game (N, v) it holds that $C(v) = CC(v)$. Hence Theorem 4.2 provides a tool that can be used to compute the nucleolus for 3-player games:

Corollary 4.2 *Let (N, v) be a balanced game with three players. Then $C(v) = CC(v)$ and:*

$$\nu(v) = m(v) + TAL\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right).$$

In the following example Theorem 4.2 is used to compute the nucleolus of Example 2.1.

Example 4.1 *Consider the game (N, v) of Example 2.1. Then $M(v) = (2, 4, 6, 3)$ and $m(v) = (1, 0, 1, 0)$. Using Theorem 4.2, the nucleolus of (N, v) is given by:*

$$\begin{aligned} \nu(v) &= m(v) + TAL\left(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\right) \\ &= (1, 0, 1, 0) + TAL(8, (1, 4, 5, 3)) \\ &= (1, 0, 1, 0) + (1, 4, 5, 3) - CEA(5, (\tfrac{1}{2}, 2, 2\tfrac{1}{2}, 1\tfrac{1}{2})) \\ &= (2, 4, 6, 3) - (\tfrac{1}{2}, 1\tfrac{1}{2}, 1\tfrac{1}{2}, 1\tfrac{1}{2}) \\ &= (1\tfrac{1}{2}, 2\tfrac{1}{2}, 4\tfrac{1}{2}, 1\tfrac{1}{2}). \end{aligned}$$

We now consider an application of Theorem 3.1 and Theorem 4.2 with respect to big boss and clan games. In a clan game a coalition can not make any profit if a certain group (CLAN) is not part of this coalition. A game $v \in TU^N$ is a clan game if $v(S) \geq 0$ for all $S \in 2^N$, $M_i(v) \geq 0$ for all $i \in N$ and if there exists a nonempty coalition $\text{CLAN} \subset N$ such that:

- (i) $v(S) = 0$ if $\text{CLAN} \not\subset S$
- (ii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(v)$, for all S with $\text{CLAN} \subset S$.

The last property is also known as the union property. Clan games for which $\text{CLAN} = \{i^*\}$ are also known as big boss games.¹ In the following corollary

¹This definition differs from the definition of big boss games given in Muto et al. (1988) in the sense that it is now required that $v(S) \geq 0$ for all $S \in 2^N$ and the requirement of monotonicity is weakened to $M(v) \geq 0$. A game (N, v) is monotonic, if $v(S) \leq v(T)$ if $S \subset T$.

several (known) properties of clan games are easily proved with the aid of Theorems 3.1, 3.2 and 4.2. Since for big boss games some additional results can be achieved, the results for this class of games are treated separately.

Corollary 4.3 (cf. Potters et al. (1989)) *Let (N, v) be a clan game with $|CLAN| \geq 2$. Then $v \in CA^N$, $C(v) = CC(v)$ and for the nucleolus of (N, v) it holds that:*

$$\nu(v) = CEA\left(v(N), \frac{1}{2}M(v)\right).$$

Proof: Let (N, v) be a clan game, with $|CLAN| \geq 2$. Then the following is true:

$$M_i(v) = v(N), \quad \text{if } i \in CLAN.$$

Let $i \in N$ and $S \subset N$ such that $i \in S$. Then if $CLAN \subset S$ it can be deduced from the union property that:

$$\begin{aligned} v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) &\leq v(N) - \sum_{j \in N \setminus \{i\}} M_j(v) \\ &\leq 0. \end{aligned}$$

The last inequality follows from $M(v) \geq 0$. Since $v(S) = 0$ if $CLAN \not\subset S$, it follows that (by taking $S = \{i\}$) $m_i(v) = 0$. It holds that $m(v) \leq M(v)$ and because $v(N) \geq 0$ and $M(v) \geq 0$ it is true that $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. Hence $v \in CA^N$.

Let $S \in 2^N \setminus \{\emptyset\}$, if $CLAN \subset S$, then (1) is satisfied by condition (ii). If $CLAN \not\subset S$, then $v(S) = 0$ and formula (1) follows from $m(v) = 0$. Theorem 3.1 yields that $C(v) = CC(v)$. According to Theorem 4.2 the nucleolus of (N, v) can be computed by:

$$\nu(v) = TAL\left(v(N), M(v)\right).$$

Since $|CLAN| \geq 2$, we have that $\sum_{i \in N} M_i(v) \geq 2v(N)$. Hence,

$$\nu(v) = CEA\left(v(N), \frac{1}{2}M(v)\right).$$

□

Corollary 4.4 (cf. Muto et al. (1988)) *Let (N, v) be a big boss game with clan i^* . Then $v \in CA^N$, $C(v) = CC(v)$, $\tau(v) = \nu(v)$ and:*

$$\nu_j(v) = \begin{cases} \frac{1}{2}M_j(v) & \text{if } j \in N \setminus \{i^*\} \\ v(N) - \frac{1}{2} \sum_{k \in N \setminus \{i^*\}} M_k(v) & \text{if } j = i^*. \end{cases}$$

Moreover, if (N, v) is convex then $\tau(v) = \nu(v) = \phi(v)$.

Proof: Let $v \in TU^N$ be a big boss game and let $i^* \in N$ denote the clan of (N, v) . Then it holds that:

$$\begin{aligned} M_{i^*}(v) &= v(N) - v(N \setminus \{i^*\}) \\ &= v(N). \end{aligned}$$

Let $j \in N \setminus \{i^*\}$ and $S \subset N$ such that $j \in S$. If $i^* \in S$ it holds that:

$$\begin{aligned} v(S) - \sum_{k \in S \setminus \{j\}} M_k(v) &\leq v(N) - \sum_{k \in N \setminus \{j\}} M_k(v) \\ &\leq 0, \end{aligned}$$

because of (ii) and the fact that $M(v) \geq 0$ and $M_{i^*} = v(N)$. If $i^* \notin S$, then $v(S) = 0$, in particular this holds for $S = \{j\}$. We conclude that:

$$m_j(v) = 0, \quad \forall j \in N \setminus \{i^*\}.$$

It can be derived from the union property that:

$$v(S) - \sum_{j \in S \setminus \{i^*\}} M_j(v) \leq v(N) - \sum_{j \in N \setminus \{i^*\}} M_j(v),$$

for all $S \subset N$ and $i^* \in S$. Hence:

$$m_{i^*}(v) = v(N) - \sum_{j \in N \setminus \{i^*\}} M_j(v). \quad (4)$$

Since $M(v) \geq 0$ it follows that $m(v) \leq M(v)$ and $\sum_{j \in N} m_j(v) \leq v(N) \leq \sum_{j \in N} M_j(v)$ and hence $v \in CA^N$.

We have that (1) holds for all S with $i^* \in S$, since (N, v) satisfies the union property. If $i^* \notin S$, then $v(S) = 0$ and (1) holds, because $m_j(v) = 0$ for all $j \in N \setminus \{i^*\}$. It follows from Theorem 3.1 that $C(v) = CC(v)$ and from Theorem 4.2 that:

$$\nu(v) = m(v) + TAL(E, d),$$

with $E = v(N) - \sum_{i \in N} m_i(v)$ and $d = M(v) - m(v)$. Substituting the value of $M(v)$ and $m(v)$ yields:

$$\begin{aligned} E &= v(N) - \sum_{j \in N} m_j(v) \\ &= v(N) - m_{i^*}(v) \\ &= \sum_{j \in N \setminus \{i^*\}} M_j(v), \end{aligned}$$

where the last equality holds because of (4). Furthermore it holds that:

$$d_j = \begin{cases} M_j(v) & \text{if } j \in N \setminus \{i^*\} \\ \sum_{k \in N \setminus \{i^*\}} M_k(v) & \text{if } j = i^*. \end{cases}$$

Observe that it holds that $\sum_{j \in N} d_j = 2E$. By self-duality of the Talmud rule and the proportional rule it follows that:

$$\tau(v) = \nu(v) = m(v) + \frac{1}{2}d.$$

Substituting the value of d yields for $j \in N$:

$$\nu_j(v) = \begin{cases} \frac{1}{2}M_j(v) & \text{if } j \in N \setminus \{i^*\} \\ v(N) - \frac{1}{2} \sum_{k \in N \setminus \{i^*\}} M_k(v) & \text{if } j = i^*. \end{cases}$$

If (N, v) is convex, then it follows from Corollary 4.1 and the self-duality of the run to the bank rule that:

$$\tau(v) = \nu(v) = \phi(v).$$

□

5 Core cover and Weber set

In this section the relation between the core cover and the Weber set is examined.

For a balanced TU-game the intersection of the core cover and the Weber set always contains the core. Hence, the core cover and the Weber set have points in common. This inspires us to investigate whether the intersection of the core cover and the Weber set is non-empty. We will show that under a weak condition this holds true. For the proof of this theorem the following lemma is needed:

Lemma 5.1 *For all $n \in \mathbb{N}$ and all $d, y \in \mathbb{R}^n$ such that:*

$$y_1 \leq \dots \leq y_n, \tag{5}$$

$$\sum_{i=1}^k d_i \leq 0 \quad \text{for all } k \in \{1, \dots, n-1\}, \tag{6}$$

$$\text{and } \sum_{i=1}^n d_i = 0, \tag{7}$$

it holds that:

$$d \cdot y = \sum_{i=1}^n d_i y_i \leq 0.$$

Proof: The proof is given by an induction argument to n . For $n = 1$ the assertion is true, since $d_1 = 0$. Assume that the lemma holds for $k = n - 1$. Let $y, d \in \mathbb{R}^n$ such that the formulas (5)–(7) are true. One can conclude that:

$$\begin{aligned} \sum_{i=1}^n d_i y_i &= \sum_{i=1}^{n-2} d_i y_i + d_{n-1} y_{n-1} + d_n y_{n-1} + d_n (y_n - y_{n-1}) \\ &= \left(\sum_{i=1}^{n-2} d_i y_i + (d_{n-1} + d_n) y_{n-1} \right) + d_n (y_n - y_{n-1}) \\ &\leq 0 + d_n (y_n - y_{n-1}) \\ &\leq 0. \end{aligned}$$

The first inequality follows from the induction hypothesis and the second inequality follows from the fact that $d_n \geq 0$ and $y_n - y_{n-1} \geq 0$. \square

Theorem 5.1 Let $v \in CA^N$ such that for all $S \in 2^N$ it holds that:

$$v(S) + \sum_{j \in N \setminus S} m_j(v) \leq v(N), \quad (8)$$

then $CC(v) \cap W(v) \neq \emptyset$.

Proof: Let $v \in CA^N$ such that for all $S \in 2^N$ (8) is satisfied. Suppose that $CC(v) \cap W(v) = \emptyset$. Since $CC(v)$ and $W(v)$ are both closed and convex sets we can separate these sets with a hyperplane. This means that there exists a vector $y \in \mathbb{R}^N$ such that:

$$m \cdot y > l \cdot y \quad \text{for all } m \in W(v), \quad l \in CC(v). \quad (9)$$

Let $\sigma \in \Pi(N)$ an order such that:

$$y_{\sigma(1)} \leq y_{\sigma(2)} \leq \dots \leq y_{\sigma(n)}$$

Consider the larginal $l^\sigma(v)$ and the marginal $m^\sigma(v)$, then:

$$\begin{aligned} m^\sigma(v) \cdot y - l^\sigma(v) \cdot y &= (m^\sigma(v) - l^\sigma(v)) \cdot y \\ &= \sum_{i=1}^n (m_{\sigma(i)}^\sigma(v) - l_{\sigma(i)}^\sigma(v)) y_{\sigma(i)}. \end{aligned}$$

Because (N, v) is compromise admissible and hence $m(v) \leq M(v)$, it holds that for all $i \in N$ and for all $S \subset N$ with $i \in S$:

$$\begin{aligned} v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) &\leq \max_{T: i \in T} \left\{ v(T) - \sum_{j \in T \setminus \{i\}} M_j(v) \right\} \\ &= m_i(v) \\ &\leq M_i(v). \end{aligned}$$

This yields that for all $S \in 2^N$:

$$v(S) \leq \sum_{i \in S} M_i(v). \quad (10)$$

From (8) it follows that:

$$v(S) \leq v(N) - \sum_{j \in N \setminus S} m_j(v). \quad (11)$$

Define $d_{\sigma(i)} = m_{\sigma(i)}^\sigma(v) - l_{\sigma(i)}^\sigma(v)$, then it holds that:

$$\begin{aligned} \sum_{i=1}^k d_{\sigma(i)} &= \sum_{i=1}^k (m_{\sigma(i)}^\sigma(v) - l_{\sigma(i)}^\sigma(v)) \\ &= v(\{\sigma(1), \dots, \sigma(k)\}) - \sum_{i=1}^k l_{\sigma(i)}^\sigma(v) \\ &\leq 0. \end{aligned}$$

The inequality follows from inequalities (10) and (11). Furthermore it holds that:

$$\sum_{i=1}^n d_{\sigma(i)} = v(N) - v(N) = 0.$$

Applying Lemma 5.1 gives:

$$\begin{aligned} \sum_{i=1}^n d_{\sigma(i)} y_{\sigma(i)} &= \sum_{i=1}^n (m_{\sigma(i)}^\sigma(v) - l_{\sigma(i)}^\sigma(v)) y_{\sigma(i)} \\ &\leq 0. \end{aligned}$$

Hence $m^\sigma(v) \cdot y \leq l^\sigma(v) \cdot y$. This contradicts (9). \square

Theorem 5.1 can be used to show that for semi-convex games the intersection of the core cover and the Weber set is nonempty. A game $v \in TU^N$ is semi-convex if v is superadditive and $m_i(v) = v(\{i\})$ for all $i \in N$.

Corollary 5.1 *Let $v \in CA^N$ be semi-convex. Then $CC(v) \cap W(v) \neq \emptyset$.*

Proof: Let $v \in CA^N$ be semi-convex. Superadditivity of (N, v) implies that for all $S \in 2^N$:

$$v(S) + \sum_{j \in N \setminus S} v(\{j\}) \leq v(N).$$

This is equivalent with (8), because $m_i(v) = v(\{i\})$ for all $i \in N$. Hence $CC(v) \cap W(v) \neq \emptyset$.

The following example shows that it is possible that the core cover and the Weber set do not have any points in common.

Example 5.1 *Let (N, v) be a game such that $N = \{1, \dots, 5\}$ and such that the players 1, 2 and 3 are symmetric and the players 4 and 5. To simplify notations we state that the players 1, 2 and 3 are of type a and 4 and 5 of type b. For example the coalition $\{abb\}$ can be the coalitions $\{145\}$, $\{245\}$ or $\{345\}$. Define the game (N, v) by:*

S	a	b	aa	ab	bb	aaa	aab	abb	$aaab$	$aabb$	N
$v(S)$	0	0	-1	0	2	-1	-1	2	-1	1	1

Then it is easily verified that $M(v) = (0, 0, 0, 2, 2)$ and $m(v) = (0, 0, 0, 0, 0)$. Hence the core cover of (N, v) is given by:

$$CC(v) = \{x \in \mathbb{R}^5 \mid x \geq 0, x_4 + x_5 = 1, x_a = 0\}.$$

Note that if $x \in CC(v)$, then $x_1 = x_2 = x_3 = 0$.

Because of symmetry, one does not need to calculate all marginal vectors to compute the Weber set. There are only six marginal vectors which each correspond to twenty different orders. The Weber set is given by:

$$W(v) = \text{conv}\{(-1, 0, 0, 2, 0), (-1, 0, 0, 0, 2), (0, -1, 0, 2, 0), (0, -1, 0, 0, 2), (0, 0, -1, 2, 0), (0, 0, -1, 0, 2)\}.$$

We can conclude that for all $\sigma \in \Pi(N)$ it holds that $m_1^\sigma + m_2^\sigma + m_3^\sigma = -1$. Hence for all $m \in W(v)$ it holds that $m_1 + m_2 + m_3 = -1$, this yields that $CC(v) \cap W(v) = \emptyset$.

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