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Type monotonic allocation schemes for multi–glove games*

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Abstract

Multi–glove markets and corresponding games are considered. For this class of games we introduce the notion of type monotonic allocation scheme. Allocation rules for multi–glove markets based on weight systems are introduced and characterized. These allocation rules generate type monotonic allocation schemes for multi–glove games and are also helpful in proving that each core element of the corresponding game is extendable to a type monotonic allocation scheme. The τ-value turns out to generate a type monotonic allocation scheme with nice extra properties. The same holds true for the nucleolus, for in multi–glove games these two solutions coincide.

JEL Classification Code: C71

Key-words: multi-glove market game, monotonic allocation scheme, τ-value, nucleolus

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1 Introduction

The main purpose of this paper is to introduce a new kind of monotonic allocation schemes for special cooperative games, where there are players of different types. Other sort of monotonic allocation rules are introduced earlier by Sprumont (1990) and by Brânzei et al. (2001). Sprumont considers for the subclass of totally balanced games population monotonic allocation schemes (pmas). Such a pmas is a rule which assigns to the game and to all subgames a core element in such a way that the payoff of a player in a core element assigned to a smaller subgame is not larger than in a larger subgame. Sprumont proves that for each convex game a pmas exists, that each core element is pmas-extendable and that the application of the Shapley value (Shapley, 1953) to all the subgames results also in a pmas. In Brânzei et al. (2001) bi-monotonic allocation schemes (bi-mas) are introduced for the cone of total big boss games (cf. Muto et al., 1988), which are schemes assigning simultaneously to such a game and to those subgames which contain the big boss a core element, in such a way that in the core elements of smaller coalitions the big boss is not better off and the other players are not worse off than in a larger coalition. Here the τ-value (Tijs, 1981) applied to the (non-trivial) subgames results in a bi-mas. Furthermore, each core element is bi-mas extendable. Extensions of bi-mas to clan games are studied in Voorneveld et al. (2000). Another kind of monotonic allocation schemes for sequencing situations, namely drop out monotonic allocation schemes, are introduced in Fernández et al. (2001).

In this paper we consider multi-glove markets and the corresponding cooperative games. For these multi-glove games we introduce type monotonic allocation schemes (tmas), tackle the question of the existence of tmas and of the possibility of extensions of core elements to tmas.

The outline of the paper is as follows. In Section 2 we introduce multi-glove markets (mg-markets), where there are agents of different types each with a type specific good. Then allocation rules for such markets are considered; special attention is paid to the interesting class of weighted allocation rules, which are based on hierarchical weight systems on the type space. In Section 3 multi-glove games (mg-games) corresponding to mg-markets are studied. For these kind of cooperative games we describe the imputation set, the core, the τ-value, and the nucleolus. The τ-value and the nucleolus coincide, and these point-valued solutions turn out to be in the barycenter of the core of such a game. In Section 4 the notion of type monotonic allocation scheme (tmas) for mg-games is introduced. Each weighted allocation rule for mg-markets generates for each mg-game a tmas and all core elements of an mg-game can be extended to a tmas. The τ-value (or the nucleolus) applied to games and subgames turns out to generate also a tmas.

2 Multi-glove markets and allocation rules

In this section we consider well–known special markets in which t different types of goods $1, 2, ..., t$ ($t \geq 2$) are available and where each agent in the market possesses precisely one unit of one of the present goods; moreover, for each type of goods there is at least one owner. Further a profit of one unit can be obtained only if one unit of each of the t types of goods are combined. The agents in the market with one unit of good s are called agents of type s and this set of agents is denoted by N_s. Then $N_1, N_2, ..., N_t$ is a partition of the set N of all agents in the market. Let us denote the set of types $\{1, 2, ..., t\}$ with T and let us denote by n_s the number of agents in N_s. Then the agents in N can make a profit of $\min_{s\in T} n_s$ units. We will denote such a market with $m_T :=< T, N_1, N_2, ..., N_t >$ and, in situations where only the number of agents of the different types matters, we also denote the market by $n = (n_1, n_2, ..., n_t)$. If there are only two types ($t = 2$) such a market is called a glove market, where the agents of type 1 (type 2) possess a left (right) glove and where a left–right pair of gloves has value 1. Inspired by this name we call our markets *multi-glove markets* or *mg-markets* (although a nice name for $t = 4$ could be horse shoe markets). For relevant related literature we refer to Shapley (1959), Shapley and Shubik (1969), Owen (1975), Rosenmüller and Sudhölter (2000), and Apartsin and Holzman (2000).

We are interested in efficient and symmetric allocation rules, which divide for each market $< T, N_1, N_2, ..., N_t >$ the total profit $p(n) := \min_{s\in T} n_s$ among the agents in such a way that agents of the same type get the same share of the profit. Such allocation rules can be described by a function $F : \mathbb{N}^t \to \mathbb{R}^t_+$, where $\mathbb{N} = \{1, 2, ...\}$ is the set of natural numbers, and where for each market $n = (n_1, n_2, ..., n_t) \in \mathbb{N}^t$ and for each $s \in T$ the share of the profit of agents of type s is $F_s(n)$. Then $\sum_{s\in T} n_s F_s(n) = \min_{u\in T} n_u = p(n)$.

In the following we denote the set of scarce types of the market n by

$sc(n)$, so $sc(n) = \{s \in T \mid n_s = p(n)\} = \arg\min_{u \in T} n_u$.

Example 1. The allocation rule $F : \mathbb{N}^t \to \mathbb{R}^t_+$ defined by $F_s(n) = t^{-1} n_s^{-1} p(n)$ for each $s \in T$, divides the profit in such a way that the total share of each of the types is the same amount $t^{-1}p(n)$. Here F assigns also to non–scarce agents a positive payoff.

Example 2. Let $E : \mathbb{N}^t \to \mathbb{R}^t_+$ be such that for each mg-market n, $E_s(n) = 0$ if $s \notin sc(n)$, and $E_s(n) = |sc(n)|^{-1}$ otherwise. Then E divides the profit equally among the scarce members.

We introduce now a special class of allocation rules induced by hierarchical weight systems. A (*hierarchical*) *weight system* for multi-glove markets with type space $T = \{1, 2, ..., t\}$ is determined by a partition $\{C^1, C^2, ..., C^k\}$ of T in classes of types and corresponding vectors $w^1, w^2, ..., w^k$ in $\mathbb{R}^t_+$, where for each $r \in \{1, 2, ..., k\}$ the carrier $\mathrm{carr}(w^r) = \{s \in T \mid w^r_s > 0\}$ of w^r is equal to C^r and where $\sum_{s \in C^r} w^r_s = 1$. We will denote such a weight system by $(w^1, w^2, ..., w^k)$ or shortly w. The class C^r is called the *class of types of rank* r and the vector w^r is called the *weight vector* for this class.

Given $(w^1, w^2, ..., w^k)$ the induced allocation rule $F^w = F^{(w^1, w^2, ..., w^k)}$ divides the total profit in an mg-market $n \in \mathbb{N}^t$ among the agents which belong to those scarce types which have the lowest rank, and the distribution of the profit for these types is determined by the corresponding weight vector. Formally, if $\ell(n) = \min\{r \in \{1, 2, ..., k\} \mid sc(n) \cap C^r \neq \emptyset\}$, then

$$F_s^{(w^1, w^2, ..., w^k)}(n) = \left(\sum \{w_r^{\ell(n)} \mid r \in sc(n) \cap C^{\ell(n)}\}\right)^{-1} w_s^{\ell(n)}$$

if $s \in sc(n) \cap C^{\ell(n)}$, and $F_s^{(w^1, w^2, ..., w^k)}(n) = 0$ otherwise.

Example 3. Let $T = \{1, 2, 3, 4, 5\}$, $k = 2$, $C^1 = \{1, 2, 3\}$, $C^2 = \{4, 5\}$, $w^1 = \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}, 0, 0\right)$ and $w^2 = \left(0, 0, 0, \frac{1}{4}, \frac{3}{4}\right)$. Then

$$F^{(w^1, w^2)}(6, 4, 4, 4, 4) = \left(0, \frac{3}{4}, \frac{1}{4}, 0, 0\right)$$

because $sc(6, 4, 4, 4, 4) = \{2, 3, 4, 5\}$ and $\ell(6, 4, 4, 4, 4) = 1$. Further

$$F^{(w^1, w^2)}(6, 4, 4, 3, 4) = (0, 0, 0, 1, 0).$$

Note that the rule $E : \mathbb{N}^t \to \mathbb{R}^t_+$ from Example 2 is also a rule based on a hierarchical weight system with only one class $C^1 = T$ and $w^1 = |T|^{-1}(1, 1, ..., 1)$; so $E = F^{w^1}$.

Let us now introduce for allocation rules two properties: the *scarcety property* SCARCE and the *ratio consistency property* CONS. These properties hold for rules based on hierarchical weight systems.

In fact, these properties turn out to be characterizing for such rules as we see in Theorem 1.

We say that $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies SCARCE if $F_s(n) = 0$ for each mg-market $n \in \mathbb{N}^t$ and each $s \notin sc(n)$.

Clearly, each F^w satisfies SCARCE. The rule F in Example 1 does not satisfy SCARCE.

We say that $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies CONS if for all pairs $n, m \in \mathbb{N}^t$, for which there is an $s \in T$ such that $F_s(n) > 0$ and $F_s(m) > 0$, we have

$$\frac{F_u(m)}{F_s(m)} = \frac{F_u(n)}{F_s(n)} \text{ for all } u \in T. \tag{1}$$

This implies, for example, that for two types $s, u \in T$ with positive shares in both mg-markets n and m w.r.t. the rule F, if s gets α times the amount which u gets in mg-market n, then this is also (consistently) the case in mg-market m.

Take $F^w = F^{(w^1,w^2,...,w^k)}$ and mg-markets n, m for which there is an $s \in T$ such that $F^w_s(n) > 0$ and $F^w_s(m) > 0$. Then s is scarce in n as well as in m, and if $s \in C^r$ then $\ell(n) = \ell(m) = r$. So $F^w_u(m) = F^w_u(n) = 0$ if $u \notin C^r$ and for such u equality (1) holds. For $u \in C^r$, we have

$$\frac{F^w_u(n)}{F^w_s(n)} = \frac{F^w_u(m)}{F^w_s(m)} = \frac{w^r_u}{w^r_s}.$$

Theorem 1 (Characterization of rules based on weight systems). *An allocation rule $F : \mathbb{N}^t \to \mathbb{R}^t_+$ satisfies* SCARCE *and* CONS *if and only if there are a $k \in \mathbb{N}$ and a weight system $w = (w^1, w^2, ..., w^k)$ such that $F = F^w$.*

Proof. We have already shown that a rule of the form F^w satisfies SCARCE and CONS. Conversely, let F be a rule with the SCARCE and CONS properties. We construct a weight system w as follows. Take the mg-market $n^1 =$

$(1,1,...,1)$, where all types are scarce and take $w^1 = F(n^1)$. If $\text{carr}(w^1) = T$, take $w = (w^1)$. If $\text{carr}(w^1) \neq T$, take an mg-market n^2 with $n_s^2 = 2$ if $s \in \text{carr}(w^1)$ and $n_s^2 = 1$ otherwise. Then $sc(n^2) = T \setminus \text{carr}(w^1)$ and take $w^2 = F(n^2)$. If $\text{carr}(w^1) \cup \text{carr}(w^2) = T$, then take $w = (w^1, w^2)$. Otherwise take $w^3 = F(n^3)$ where $n_s^3 = 2$ for $s \in \text{carr}(w^1) \cup \text{carr}(w^2)$ and $n_s^3 = 1$ otherwise. And so we go on.

This leads then to a weight system, say $w = (w^1, w^2, ..., w^k)$. Note that

$$F(n^h) = w^h = F^w(n^h) \text{ for all } h \in \{1, 2, ..., k\}. \tag{2}$$

It remains to prove that $F(n) = F^w(n)$ for all $n \in \mathbb{N}^t$.

Take $n \in \mathbb{N}^t$. Take among the scarce types in n a type s with $F_s(n) > 0$ and such that there is no s' with $F_{s'}(n) > 0$, which is in a lower class w.r.t. the hierarchy corresponding to w. Let $s \in C^h$. Then the mg-markets n and n^h have both a positive payoff for s. So, by CONS, for all $u \in T$

$$\frac{F_u(n)}{F_s(n)} = \frac{F_u(n^h)}{F_s(n^h)}.$$

Now, from (2), it follows

$$\frac{F_u(n^h)}{F_s(n^h)} = \frac{w_u^h}{w_s^h} = \frac{F_u^w(n)}{F_s^w(n)}.$$

This implies that $F_u(n) = 0$ if $u \notin \text{carr}(w)$ and $F(n) = \alpha F^w(n)$ with $\alpha = \frac{F_s(n)}{F_s^w(n)}$. Since $\sum_{s \in T} F_s(n) = \sum_{s \in T} F_s^w(n) = 1$, we obtain $\alpha = 1$. Hence $F = F^w$.□

The next theorem will be useful in Section 4.

Theorem 2 (Type monotonicity for allocation rules based on weight systems). *Let $w = (w^1, w^2, ..., w^k)$ be a hierarchical weight system. Let $n \in \mathbb{N}^t$ and $u \in T$ with $n_u \geq 2$. Then*

$$F_u^w(n - e^u) \geq F_u^w(n), \; F_s^w(n - e^u) \leq F_s^w(n) \textit{ for } s \in T \setminus \{u\}. \tag{3}$$

(Here $(n - e^u) = (n_1, n_2, ..., n_u - 1, ..., n_t)$, e^u is the u-th basis vector in $\mathbb{R}^t$.)

Interpretation. If gains in mg-markets are distributed using an allocation rule based on weight systems, and in an mg-market one player of a certain type leaves, then in the new mg-market the remaining agents of the same type are not worse off, while agents of other types are not better off.

Proof of Theorem 2. We consider three cases, where the third case consists of three subcases.

(i) Suppose $u \in sc(n)$. Then $sc(n - e^u) = \{u\}$, so

$$F_u^w(n - e^u) = 1 \geq F_u^w(n),\ F_s^w(n - e^u) = 0 \leq F_s^w(n) \text{ for } s \in T \setminus \{u\}.$$

Hence (3) holds.

(ii) Suppose $u \notin sc(n - e^u)$. Then $u \notin sc(n) = sc(n - e^u)$ and

$$F_u^w(n - e^u) = F_u^w(n) = 0,\ F_s^w(n - e^u) = F_s^w(n) \text{ for } s \in T \setminus \{u\}.$$

So (3) holds with equalities.

(iii) Suppose $u \notin sc(n)$ and $u \in sc(n - e^u)$, hence $sc(n - e^u) = sc(n) \cup \{u\}$. Let $u \in C^r$.
If $r < \ell(n)$, then $r = \ell(n - e^u)$ and $C^r \cap sc(n - e^u) = \{u\}$. Thus $F_u^w(n - e^u) = 1 \geq F_u^w(n)$, $F_s^w(n - e^u) = 0 \leq F_s^w(n)$, so (3) holds.
If $r > \ell(n)$, then (3) holds with equalities, because $\ell(n) = \ell(n - e^u)$, $sc(n) \cap C^{\ell(n)} = sc(n - e^u) \cap C^{\ell(n-e^u)}$.
If $r = \ell(n)$, then $\ell(n - e^u) = \ell(n)$, so $F_u^w(n - e^u) < F_u^w(n)$ because $\sum\{w_p^{\ell(n)} \mid p \in sc(n) \cap C^{\ell(n)}\} < \sum\{w_p^{\ell(n-e^u)} \mid p \in sc(n-e^u) \cap C^{\ell(n-e^u)}\}$, and $F_s^w(n - e^u) = F_s^w(n) = 0$ for $s \in T \setminus \{u\}$, so also now (3) holds. □

3 Multi–glove games

We assign in this section to a multi–glove market a cooperative game and discuss how some classical solution concepts look like for such multi–glove games.

First we recall that a cooperative game is a pair $< N, v >$, where $N = \{1, 2, ..., n\}$ is the set of players and $v : 2^N \rightarrow \mathbb{R}$ is the worth function assigning to each coalition $S \in 2^N$, the worth $v(S)$ with the assumption that $v(\phi) = 0$. The imputation set $I(v)$ is the set of individual rational allocations of the worth of the grand coalition, so

$$I(v) = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N),\ x_i \geq v(\{i\}) \text{ for each } i \in N\}.$$

The core $C(v)$ (cf. Gillies, 1953) is the subset of $I(v)$ of split-off stable allocations:

$$C(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}.$$

The τ-value $\tau(v)$ (cf. Tijs, 1981) for a game $< N, v >$ with non-empty core is the feasible compromise between the marginal vector $M(v) \in \mathbb{R}^n$ and the minimum right vector $m(v) \in \mathbb{R}^n$, i.e., $\tau(v) = \alpha M(v) + (1 - \alpha) m(v)$, where

$$M_i(v) = v(N) - v(N \setminus \{i\}) \text{ for each } i \in N,$$

$$m_i(v) = \max \left\{ v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \mid S \in 2^N, i \in S \right\}, \text{ for each } i \in N,$$

and where $\alpha \in \mathbb{R}$ is such that $\sum_{i \in N} \tau_i(v) = v(N)$.
The nucleolus $\eta(v)$ (cf. Schmeidler, 1969) for a game $< N, v >$ with non-empty imputation set is the (unique) imputation which lexicographically minimizes the vector of non-increasingly ordered excesses of coalitions over the imputation set, i.e.,

$$\eta = \eta(v) \in I(v) \qquad \text{and} \qquad E(\eta) \preceq_{lex} E(y) \text{ for all } y \in I(v),$$

where $E(x) := [\ \ldots, e(S, x) := v(S) - \sum_{i \in S} x_i, \ldots \ \mid S \subset N]$ is the vector of coalitional excesses, ordered such that $e(S, x) \geq e(S', x)$ whenever from left to right the excess of S preceeds the excess of S', and where $\preceq_{lex}$ denotes the usual lexicographic ordering of vectors.

Now let $(n_1, n_2, ..., n_t) \in \mathbb{N}^t$ be an mg-market, where $N_1 = \{1, 2, ..., n_1\}$ is the set of agents of type 1, $N_2 = \{n_1+1, n_1+2, ..., n_1+n_2\}$ the set of agents of type 2,..., and $N_t = \left\{ \sum_{s=1}^{t-1} n_s + 1, \sum_{s=1}^{t-1} n_s + 2, ..., \sum_{s=1}^{t} n_s \right\}$. The corresponding multi–glove game $< N, v >$ is given by $N = \bigcup_{s=1}^{t} N_s$ and for each $S \subset N$,

$$v(S) = \min\{|S \cap N_1|, |S \cap N_2|, ..., |S \cap N_t|\}, \tag{4}$$

(where $|R|$ is the number of elements in set R). Observe that $v(N) = p(n)$.

The imputation set, core, τ-value, and nucleolus of a multi–glove game are discussed in the next two theorems.

Theorem 3. *Let* $(n_1, n_2, ..., n_t) \in \mathbb{N}^t$ *be a multi–glove market with corresponding multi–glove game* $< N, v >$ *. Then*

(i) $I(v) = \mathrm{conv}\{p(n)e^i \mid i \in N\}$. *(Here* e^i *is the i-th basis vector in* $\mathbb{R}^n$*.)*

(ii) $C(v)$ *consists of all vectors of the form* $[\alpha_1, \alpha_2, ..., \alpha_t]$ *where* $\sum_{s=1}^{t} \alpha_s = 1$, $\alpha_s \geq 0$ *for each s and* $\alpha_s = 0$*, if* $s \notin sc(n)$*, and where* $[\alpha_1, \alpha_2, ..., \alpha_t]$ *is a short hand for the vector x in* $\mathbb{R}^n$ *with* $x_i = \alpha_s$ *for each* $s \in T$ *and* $i \in N_s$*.*

Remark 1. The imputation set of a multi–glove game is a simplex in $\mathbb{R}^n$ whose extreme points are those allocations where one player obtains the whole worth of the grand coalition N. Core elements are imputations where players of the same type get the same share and players of a non–scarce type get nothing.

Proof of Theorem 3. (i) Since $t \geq 2$, for each $i \in N$ the worth $v(\{i\}) = 0$. So $I(v) = \left\{ x \in \mathbb{R}^n_+ \mid \sum_{i=1}^{n} x_i = v(N) \right\} = \mathrm{conv}\{v(N)e^i \mid i \in N\}$.

(ii) Take first $x = [\alpha_1, \alpha_2, ..., \alpha_t] \in \mathbb{R}^n_+$ with $\sum_{s \in T} \alpha_s = 1$ and $\alpha_u = 0$ if $u \notin sc(n)$. Then $\sum_{i \in N} x_i = \sum_{s \in T} \alpha_s n_s = \sum_{s \in sc(n)} \alpha_s n_s = \sum_{s \in sc(n)} \alpha_s v(N) = v(N)$. For $R \in 2^N$ we have $\sum_{i \in R} x_i = \sum_{s \in T} \alpha_s |R \cap N_s| \geq \sum_{s \in T} \alpha_s \min_{u \in T} |R \cap N_u| = \sum_{s \in T} \alpha_s v(R) = v(R)$. So $x \in C(v)$. Conversely, let $z \in C(v)$. We have to prove that there are $\beta_1, ..., \beta_t$ such that $z = [\beta_1, \beta_2, ..., \beta_t] \in \mathbb{R}^n_+$, where $\sum_{t \in T} \beta_t = 1$ and $\beta_s = 0$ if $s \notin sc(n)$. Let us call a coalition $S \in 2^N$ a *simple coalition* if it consists of t players, all of a different type.

For each simple coalition S we have $\sum_{i \in S} z_i \geq v(S) = 1$. Take one of the many systems consisting of $p(n)$ disjoint simple coalitions, say $S_1, \ldots, S_{p(n)}$. We have $p(n) = v(N) = \sum_{i \in N} z_i \geq \sum_{j=1}^{p(n)} \sum_{i \in S_j} z_i \geq \sum_{j=1}^{p(n)} v(S_j) = p(n)$. This implies that $\sum_{i \in S_j} z_i = 1$ for each $j = 1, \ldots, p(n)$, and that $z_i = 0$ for each $i \in N \setminus (\bigcup_{j=1}^{p(n)} S_j)$. Since each simple coalition is a member of such a family, we have $\sum_{i \in S} z_i = 1$ for each simple coalition S. Similarly, since each non–scarce player is left out from at least one such family, we have $z_i = 0$ for each non–scarce player i. Finally, let $s \in sc(n)$ and let $k, \ell \in N_s$. We claim that $z_k = z_\ell$. Take a simple coalition S with $k \in S$. Then $S' = (S \cup \{\ell\}) \setminus \{k\}$ is also simple. From $\sum_{i \in S} z_i = \sum_{i \in S'} z_i = 1$ it follows that $z_k = z_\ell$.

Now let $\beta_s = z_i$ for each $s \in T$ and some $i \in N_s$. Then z corresponds to $[\beta_1, \beta_2, ..., \beta_t]$. □

Theorem 4. *Let $< N, v >$ be the mg-game corresponding to the mg-market $n \in \mathbb{N}^t$. Then for the τ-value $\tau(v)$ and nucleolus $\eta(v)$ the following holds:*

(i) $\tau_i(v) = |sc(n)|^{-1}$ *for* $r \in sc(n)$ *and* $i \in N_r$*, and* $\tau_i(v) = 0$ *otherwise.*

(ii) $\tau(v) = \eta(v)$

(iii) *The τ-value and the nucleolus, restricted to mg–games, are the only rules which assign to each such a game a core element with coordinates the same for all players of a scarce type and coordinates zero for the players of a non–scarce type.*

Proof. (i) Note that $M_i(v) = 1$ for each $i \in N_r$ with $r \in sc(n)$, and $M_i(v) = 0$ for the other players. We consider two subcases corresponding to $|sc(n)| = 1$ and $|sc(n)| > 1$. Suppose that $|sc(n)| = 1$ and $sc(n) = \{r\}$, hence $v(S) = |S \cap N_r|$. Then for $i \notin N_r$ and $i \in S$ we have $v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq v(S) - \sum_{j \in S \cap N_r} M_j(v) \leq 0$, which implies that $m_i(v) = v(\{i\}) = 0$ and also $M_i(v) = 0$. For $i \in N_r$ and $S \ni i : v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq v(S) - (v(S) - 1) = 1$ and for a coalition S containing precisely $v(S)$ elements of the set N_r including i, we get $v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) = 1$. Hence, $m_i(v) = M_i(v) = 1$ for $i \in N_r$ and $r \in sc(n)$. This implies that $\tau(v) = \alpha M(v) + (1 - \alpha) m(v) = M(v)$; so $\tau_i(v) = M_i(v) =$

$1 = |sc(n)|^{-1}$ for i scarce, and $\tau_i(v) = M_i(v) = 0$ for i non-scarce. Suppose secondly that $|sc(n)| > 1$. If $v(S) = 0$, then $v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq 0$. A coalition S with $v(S) > 0$ contains at least $v(S)$ players of each scarce type. So for each $i \in S$ we get $v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq v(S) - |sc(n)|v(S) + 1 \leq 0$. This implies that $m_i(v) = v(\{i\}) = 0$ for all $i \in N$. Hence, $\tau(v) = \alpha M(v)$ with $\alpha = |sc(n)|^{-1}$.

(ii) Let $\mathcal{B}$ denote the collection of single-player and of simple coalitions. It immediately follows from (4) that for each $S \notin 2^N \setminus \mathcal{B}$ there is a non-trivial partition $\mathcal{R}$ of S such that $\mathcal{R} \subseteq \mathcal{B}$ and $v(S) = \sum_{R \in \mathcal{R}} v(R)$. That is, the coalitions not in $\mathcal{B}$ are all inessential in the mg-game v. It was proved by Huberman (1980) that in general, if a collection $\mathcal{B}$ contains all essential coalitions in a game with a nonempty core then the nucleolus is the (unique) core-allocation which lexicographically minimizes over the core the restricted vector $E_{\mathcal{B}}(x) := [\ \ldots, e(S,x), \ldots \ \mid S \in \mathcal{B}]$ of non-increasingly ordered excesses of coalitions taken only from $\mathcal{B}$, i.e.,

$$\eta = \eta(v) \in C(v) \qquad \text{and} \qquad E_{\mathcal{B}}(\eta) \preceq_{lex} E_{\mathcal{B}}(y) \text{ for all } y \in C(v).$$

By Theorem 3 (ii), for each simple coalition S and for each non-scarce player j we have $e(S,z) = e(\{j\},z) = 0$ for all $z \in C(v)$. Thus, in $E_{\mathcal{B}}(.)$ the corresponding coordinates are constant over the core, so the nucleolus is actually determined by the coordinates associated with the single-player coalitions of the scarce players. That is, the nucleolus is the core-allocation which lexicographically minimizes over the core the further-restricted vector $[\ \ldots, e(\{i\},x) = -x_i, \ldots \ \mid i \in N_r, r \in sc(n)]$ of non-increasingly ordered excesses of single-player coalitions of scarce players. It follows from $\sum_{r \in sc(n)} \sum_{i \in N_r} x_i = p(n)$ that only the uniform distribution achieves the lexicographical minimum, so $\eta_i(v) = |sc(n)|^{-1}$ for $r \in sc(n)$ and $i \in N_r$, and $\eta_i(v) = 0$ otherwise. Therefore, by (i), $\eta(v) = \tau(v)$.

(iii) follows directly from (i), (ii) and Theorem 3. □

Remark 2. Let us make some geometric observations following from the two preceding theorems, where we deal with a game $< N, v >$ corresponding to the mg-market $n \in \mathbb{N}^t$. For each $r \in sc(n)$ we can consider the simplex

$I^r(v) = \text{conv}\{v(N)e^i \mid i \in N^r\}$, which is a face of the simplex $I(v)$ with barycenter $b^r = \sum_{i \in N^r} e^i$.

The core $C(v)$ of the game $< N, v >$ equals the convex hull of these barycenters i.e. $C(v) = \text{conv}\{b^r \mid r \in sc(n)\}$, so the core is also a simplex. The barycenter of the core equals the τ-value and the nucleolus: $\tau(v) = \eta(v) = |sc(n)|^{-1} \sum_{r \in sc(n)} b^r$. The following example may be illustrative.

Example 4. Let $< N, v >$ be the game corresponding to the multi-glove market $n = (2, 2, 3)$.

Then $N = \{1, 2, ..., 7\}$ and $I(v) = \text{conv}\{2e^i \mid i \in N\}$. Further $sc(n) = \{1, 2\}$, $I^1(v) = \text{conv}\{2e^1, 2e^2\} \subset I(v)$, $I^2(v) = \text{conv}\{2e^3, 2e^4\} \subset I(v)$, $b^1(v) = e^1 + e^2$, $b^2(v) = e^3 + e^4$, and $C(v) = \{(\alpha_1, \alpha_1, \alpha_2, \alpha_2, 0, 0, 0) \in \mathbb{R}^7_+ \mid \alpha_1 + \alpha_2 = 1\} = \text{conv}\{(1, 1, 0, 0, 0, 0, 0), (0, 0, 1, 1, 0, 0, 0)\} = \text{conv}\{b^1, b^2\}$, $\tau(v) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right\} = \frac{1}{2}(b^1(v) + b^2(v))$.

4 Type monotonic allocation schemes for multi-glove games

In the following for a game $< N, v >$ also its subgames $< S, v >$ play a role. Here $S \in 2^N \setminus \{\emptyset\}$ and $v : 2^S \to \mathbb{R}$ is the restriction of $v : 2^N \to \mathbb{R}$ to the set 2^S of subcoalitions of S. Special attention is paid to so–called allocation schemes, which assign to a game and its subgames a collection of payoff vectors, one for each subgame. Such an allocation scheme can be denoted by $[x_i^S]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$, where $x^S = (x_i^S)_{i \in S} \in \mathbb{R}^S$. Such a scheme can be seen as a (defect) matrix with n columns corresponding to the players and $2^n - 1$ rows corresponding to the coalitions. An allocation scheme for a game $< N, v >$ is called stable if for each $S \in 2^N \setminus \{\emptyset\}$ the vector x^S is an element of the core of the subgame $< S, v >$, which core we denote here by $C(S, v)$. So $C(v) = C(N, v)$.

In the literature one finds stable allocation schemes with special monotonicity properties. Sprumont (1990) studied for totally balanced games $< N, v >$ population monotonic allocation schemes $[x_i^S]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ where $x^S \in C(S, v)$ for each $S \in 2^N \setminus \{\emptyset\}$ and where for each $S \in 2^N \setminus \{\emptyset\}$, $i \in S$, $j \in N \setminus S : x_i^{S \cup \{j\}} \geq x_i^S$. So a player i gets a better payoff in the larger coalition $S \cup \{j\}$ than in S. In Brânzei et al. (2001) for total big boss game $< N, v >$ with player 1 as big boss bi–monotonic allocation schemes (bi-mas) $[x_i^S]_{S \in B^N, i \in S}$ were introduced, where $B^N = \{S \in 2^N \mid 1 \in S\}$, $x^S \in C(S, v)$ for each $S \in B^N$ and for all $S \in B^N$ with $i \in S \setminus \{1\}$, $j \in N \setminus S : x_i^{S \cup \{j\}} \leq x_i^S$, $x_1^{S \cup \{j\}} \geq x_1^S$. So in such a scheme the big boss is not worse off in a larger coalition and the other players are not better off. Sprumont proved that for convex games each core element $x \in C(N, v)$ is extendable to a population monotonic allocation scheme $[x_i^S]_{S \in 2^N \setminus \{\emptyset\}, i \in S}$, where $x^N = x$. Further, if x^S is the Shapley value of $< S, v >$ then $[x^S]_{S \in 2^N \setminus \{\emptyset\}}$ is a population monotonic allocation scheme for the convex game $< N, v >$. For total big boss games also each core element is bi-mas extendable and the τ-value applied to the subgames containing the big boss generates a bi-mas (cf. Brânzei et al., 2001).

The objective of this section is to introduce a new sort of stable monotonic allocation schemes for the class of multi–glove games with type set T.

Note first that for a multi–glove game $< N, v >$ corresponding to the mg-market $< T, N_1, N_2, ..., N_t >$, the subgame $< S, v >$ with $S \in 2^N \setminus \{\emptyset\}$ corresponds to the submarket $< T, N_1 \cap S, N_2 \cap S, ..., N_t \cap S >$ if $v(S) \neq 0$. So non–trivial subgames of an mg-game are also mg-games.

We are interested only in allocation schemes for mg-games with special properties. First we want that such an allocation scheme assigns to the non–trivial subcoalitions core elements of the subgame, which implies according to Theorem 3 that in the subgame players of the same type obtain the same payoff, and non-scarce players in the subgame obtain zero. Secondly, we want that the allocation scheme is type distribution consistent i.e. if for two subgames $< S_1, v >$ and $< S_2, v >$ we have $|S_1 \cap N_s| = |S_2 \cap N_s|$ for each $s \in T$, then, for each $s \in T$, the allocation scheme assigns to players of type s in both subgames the same. This implies that such an allocation scheme for $< N, v >$ can be represented by a map $\alpha^n : \{m \in \mathbb{N}^t | m \leq n\} \rightarrow \Delta$, where $\Delta = \left\{ \beta \in \mathbb{R}_+^t \mid \sum_{s \in T} \beta_s = 1 \right\}$ and $n = (|N_1|, |N_2|, ..., |N_t|)$. For a non-trivial coalition $S \in 2^N$ with positive type distribution vector $t(S) = (|N_1 \cap S|, ..., |N_t \cap S|)$, $\alpha_u^n(t(S))$ is the payoff assigned to players of type u in S. Note that $n = t(N)$.

Now we come to our main notion.

Definition 1. A *type monotonic allocation scheme* for an mg-game $< N, v >$ with type set T and type distribution vector $n = t(N)$ is a map $\alpha^n : \{m \in \mathbb{N}^t \mid m \leq n\} \to \Delta$ with the following two properties:

(*Stability*) $\alpha^n_s(m) = 0$ for each $m \leq n$ and $s \in T \setminus sc(m)$.

(*Type monotonicity*) For all $m \in \mathbb{N}^t$ and $s \in T$ such that $m - e^s \in \mathbb{N}^t$ we have $\alpha^n_s(m - e^s) \geq \alpha^n_s(m)$ and $\alpha^n_u(m - e^s) \leq \alpha^n_u(m)$ for each $u \in T \setminus \{s\}$.

So a type monotonic allocation scheme has the property that in case a player leaves a coalition, in the new core element of the remaining coalition the players of the same type as the player who left are not worse off and the others are not better off than in the old core element.

We illustrate this notion with two examples.

Example 5. Let $< N, v >$ be the 5-person game with $N_1 = \{1\}$, $N_2 = \{2, 3\}$, and $N_3 = \{4, 5\}$. Then assigning to each non–trivial subgame the τ-value leads to a type monotonic allocation scheme which can be represented by the matrix

$$\begin{array}{c} \\ (1,2,2) \\ (1,1,2) \\ (1,2,1) \\ (1,1,1) \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \end{array}$$

where the columns correspond to the types $1, 2, 3$ and the rows to the type distribution vectors of the non–trivial subcoalitions. In $< N \setminus \{2\}, v >$ the remaining player of type 2 is, in comparison to $< N, v >$, not worse off $\left(\frac{1}{2} > 0\right)$ and the players of types 1 and 3 are not better off $\left(\frac{1}{2} < 1, 0 = 0\right)$.

It turns out (Theorem 5) that the τ-value and the nucleolus generate for each mg-game a type monotonic allocation scheme. Also other weighted allocation schemes for mg-markets generate type monotonic allocation schemes for mg-games as we see in Example 6 and Theorem 5.

Example 6. Consider the game in Example 5 and apply the weighted allocation scheme $F^{\left(\left(\frac{1}{3},\frac{2}{3},0\right),(0,0,1)\right)}$ to the type vectors of the game and the subgames. This leads to the type monotonic allocation scheme represented by the matrix

$$\begin{array}{c} \\ (1,2,2) \\ (1,1,2) \\ (1,2,1) \\ (1,1,1) \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{bmatrix} \end{array}$$

where in the third row we find $(1,0,0) = F^{((\frac{1}{3},\frac{2}{3}),(1))}(1,2,1)$.

Take an arbitrary weighted allocation rule F^w for mg-markets, where $w = (w_1, w_2, ...)$. Then for each mg-game $< N, v >$, the vector $F^w(t(N))$ is a core element for $< N, v >$. This together with Theorem 2 implies that $\alpha^w : \{m \in \mathbb{N}^n \mid m \leq t(N)\} \to \Delta$ with $\alpha^w(m) = F^w(m)$ for each m is a type monotonic allocation scheme for the mg-game $< N, v >$. The weight system (w^1) with $w^1 = t^{-1}(e^1 + e^2 + \cdots + e^t)$ corresponds to $\alpha^{(w^1)}$ and $\alpha^{(w^1)}$ assigns to each subgame its τ-value (or nucleolus). So we have proved the following theorem.

Theorem 5. *Let $< N, v >$ be an mg-game with type set T.*

- (i) *For each hierarchical weight system w, $\alpha^w : \{m \in \mathbb{N}^t \mid m \leq t(N)\} \to \mathbb{R}^t_+$ is a type monotonic allocation scheme for $< N, v >$.*
- (ii) *An allocation scheme for $< N, v >$, which assigns to each subgame the τ-value (or the nucleolus) is a type monotonic allocation scheme.*

The following theorem implies that each core element z of an mg-game $< N, v >$ is extendable to a type monotonic allocation scheme, where the element assigned to $< N, v >$ corresponds to z.

Theorem 6. *Let $z \in C(N, v)$ and $t(N) \in \mathbb{N}^t$. Then there is a hierarchical weight system w such that $\alpha^w(t(N)) = z$.*

Proof. Construct the vector $w^1 \in \mathbb{R}^t_+$ as follows. For each type $s \in T$ take an $i \in N_s$ and put $w^1_s = z_i$. If $\text{carr}(w^1) = T$, then take $w = (w^1)$. If $\text{carr}(w^1) \neq T$, take as weight system $w = (w^1, w^2)$, where w^2 is an arbitrarily chosen vector $w^2 \in \mathbb{R}^T_+$, with the properties that $\text{carr}(w^2) = T \setminus \text{carr}(w^1)$ and that the sum of the coordinates is 1. Then $\alpha_s^w(t(N)) = z_i$ for each $i \in N_s$. So $\alpha^w(t(N))$ corresponds to the core element z of $< N, v >$ and α^w is a type monotonic allocation scheme which 'extends' z. □

Example 7. Let $< N, v >$ be the 4-person mg-game with $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$. Then $z = (1, 1, 0, 0) \in C(N, v)$, which corresponds to $\alpha^w(2, 2)$, where $w = (w^1, w^2)$ with $w^1 = (1, 0)$ and $w^2 = (0, 1)$. Note that $\alpha^w(2, 2) = \alpha^w(1, 2) = \alpha^w(1, 1) = (1, 0)$, $\alpha^w(2, 1) = (0, 1)$.

References

1. Y. Apartsin and R. Holzman (2000), *The core and the bargaining set in glove-market games.* Discussion Paper, Department of Mathematics, Technion–Israel Institute of Technology, Haifa, Israel.

2. R. Brânzei, S. Tijs, and J. Timmer (2001), *Information collecting situations and bi–monotonic allocation schemes.* Mathematical Methods of Operations Research 54, 303–313.

3. C. Fernández, P. Borm, R. Hendrickx, and S. Tijs (2001), *Drop out monotonic rules for sequencing situations.* DP 2001-51 Tilburg University, The Netherlands.

4. D.B. Gillies (1953), *Some theorems on n-person games.* Ph.D. Thesis, Princeton University Press, Princeton, New Jersey.

5. G. Huberman (1980), *The nucleolus and the essential coalitions.* In: *Analysis and Optimization of Systems*, Lecture Notes in Control and Information Sciences 28, Springer, Berlin, 416–422.

6. S. Muto, M. Nakayama, J. Potters, and S. Tijs (1988), *On big boss games.* The Economic Studies Quarterly 39, 303–321.

7. G. Owen (1975), *On the core of linear production games.* Mathematical Programming 9, 358–370.

8. J. Potters, R. Poos, S. Muto, and S. Tijs (1989), *Clan games.* Games and Economic Behavior 1, 275–293.

9. J. Rosenmüller, and P. Sudhölter (2000), *Formation of cartels in glove markets and the modiclus.* DP 319, Institute of Mathematical Economics, University of Bielefeld.

10. D. Schmeidler (1969), *The nucleolus of a characteristic function game.* SIAM Journal of Applied Mathematics 17, 1163–1170.

11. L.S. Shapley (1953), *A value for n-person games.* Annals of Mathematical Studies 28, 307–317.

12. L.S. Shapley (1959), *The solution of a symmetric market game.* In: *Contributions to the theory of games* (Ed. A.W. Tucker and R.D. Luce), vol. IV, Annals of Mathematics Studies, Princeton University Press, 40, 145–162.

13. L.S. Shapley (1971), *Cores of convex games.* International Journal of Game Theory 1, 11–26.

14. L.S. Shapley, and M. Shubik (1969), *On market games.* Journal of Economic Theory, 1, 9–25.

15. Y. Sprumont (1990), *Population monotonic allocation schemes for cooperative games with transferable utilities.* Games and Economic Behavior 2, 378–394.

16. S.H. Tijs (1981), *Bounds for the core and the τ-value.* In: *Game Theory and Mathematical Economics* (O. Moeschlin and D. Pallaschke eds.), North Holland Publ. Cie, Amsterdam, 123–132.

17. M. Voorneveld, S. Tijs, and S. Grahn (2000), *Monotonic allocation schemes in clan games.* CentER DP 2000–80, Tilburg University.