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ON CORES AND STABLE SETS FOR FUZZY GAMES

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Discussion paper

On cores and stable sets for fuzzy games*

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Abstract

In this paper cores and stable sets for games with fuzzy coalitions are introduced and their relations studied. For convex fuzzy games it turns out that all cores coincide and that the core is the unique stable set. Also relations between cores and stable sets for fuzzy clan games are discussed.

MSC: 90D12; 03E72

Keywords: Cooperative game; Core; Stable set; Decision making; Fuzzy coalition; Fuzzy game; Dominance core; Convex fuzzy game; Fuzzy clan game

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1. Introduction

Let $N = \{1, 2, \dots, n\}$ be a nonempty set of players considering possibilities of cooperation. This kind of situations are modeled in von Neumann and Morgenstern (1944) by means of games in characteristic function form. A cooperative game with player set N is a function $w: 2^N \rightarrow \mathbf{R}$, assigning to each group of players (coalition) $S \subseteq N$, its worth $w(S)$ obtained as a result of achieved cooperation; it is assumed that $w(\emptyset) = 0$. In this paper we will refer to elements of 2^N as crisp coalitions. For each crisp coalition S , its characteristic vector is e^S , with $(e^S)_i = 1$ if $i \in S$ and $(e^S)_i = 0$ otherwise. The set of cooperative crisp games with player set N is denoted by G^N .

Note that in a cooperative crisp game each player may have only two variants of participation in a crisp coalition: full participation or non-involvement at all. However, more freedom may be given to players by considering fuzzy cooperation, that is participation at any level between non-cooperation and full cooperation.

The class of cooperative games with fuzzy coalitions is introduced in Aubin (1974) together with his solution concept of core. Formally, a fuzzy coalition of player set N is a vector s in $[0, 1]^N$, where the i -th coordinate s_i is referred to as the participation level of player i . The empty coalition in a fuzzy setting is $e^\emptyset = (0, \dots, 0)$, and e^S , with $S \in 2^N$, denotes a crisp-like coalition. We call e^S a crisp-like coalition because it corresponds to the situation where the players within S fully cooperate (i.e., they have participation level 1) and the players outside S are not involved at all in cooperation (i.e., they have participation level 0). $e^N = (1, \dots, 1)$ is called the grand coalition. We often write e^i instead of $e^{\{i\}}$. In this paper the set of fuzzy coalitions is denoted by F^N .

A cooperative fuzzy game with player set N is a function $v: F^N \rightarrow \mathbf{R}$, with $v(e^\emptyset) = 0$, assigning to each fuzzy coalition s the value achieved by cooperation. Here we denote the set of fuzzy games with player set N by FG^N .

The purpose of this paper is to study cores and stable sets for fuzzy games. In Section 2, we review definitions of the core, the dominance core, and the stable set (von Neumann-Morgenstern solution) in crisp games and their basic properties. In Section 3 we define solutions like the dominance core and stable sets for fuzzy games and study their interrelations as well as the relation with Aubin's core. Also we analyze relations with the corresponding notions for crisp games, namely the dominance core and the stable set, using the crisp operator $cr: FG^N \rightarrow G^N$. For a fuzzy game $v \in FG^N$, the corresponding crisp game $w = cr(v) \in G^N$ is given by $w(S) = v(e^S)$ for each $S \in 2^N$. Further, for a fuzzy game v we also define a new notion, namely that of proper core, denoted by $C^P(v)$, and study its relations with the other cores of v (including the core of the corresponding crisp games $C^{cr}(v)$) and stable sets for cooperative

fuzzy games.

In Section 4 special attention is paid to the class of convex fuzzy games, introduced by Branzei et al. (2002), and their cores and stable sets. The fact that the core is itself a stable set for convex crisp games, shown in Shapley (1971) is extended to convex fuzzy games. Section 5 is devoted to the special class of fuzzy clan games.

2. Cores and stable sets for crisp games

Let $w \in G^N$ and let $I(w)$ be the imputation set of w , i.e.

$$I(w) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = w(N), x_i \geq w(\{i\}) \text{ for each } i \in N\}.$$

Here we denote by \mathbf{R}^N the n-dimensional Euclidean space.

The core $C(w)$ of a crisp game w is the subset of imputations which are stable against any possible deviation by a coalition, i.e.

$$C(w) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = w(N), \sum_{i \in S} x_i = w(S) \text{ for each } S \in 2^N\}.$$

Let $x, y \in I(w)$ and let $S \in 2^N$. We say x dominates y via S , denoted by $x \text{ dom}_S y$, if (i) $x_i > y_i$ for all $i \in S$ and (ii) $\sum_{i \in S} x_i \leq w(S)$. The two conditions are interpreted as follows. Let $x \text{ dom}_S y$. Then

- (i) $x_i > y_i$ for each $i \in S$, means that the imputation $x = (x_1, \dots, x_n)$ is better than the imputation $y = (y_1, \dots, y_n)$ for all players $i \in S$;
- (ii) $\sum_{i \in S} x_i \leq w(S)$ means that the payoff $\sum_{i \in S} x_i$ is reachable by the coalition S .

Remark 1. Note that $x \text{ dom}_S y$ implies $S \neq N$ because from $x_i > y_i$ for all $i \in N$ it follows $\sum_{i \in N} x_i > \sum_{i \in N} y_i$, in contradiction with $x, y \in I(w)$. Further $x \text{ dom}_{\{i\}} y$ implies $w(\{i\}) \geq x_i > y_i$ which contradicts the fact that $y \in I(w)$.

We simply say x dominates y , denoted by $x \text{ dom } y$, if there is a coalition S such that $x \text{ dom}_S y$. The negation of $x \text{ dom } y$ is denoted here by $\neg x \text{ dom } y$.

The dominance core (*D-core*) $DC(w)$ of a crisp game w is the set of imputations which are not dominated by any other imputation, i.e.

$$DC(w) = \{x \in I(w) \mid \neg y \text{ dom } x \text{ for all } y \in I(w)\}.$$

A *stable set* of a crisp game w is a nonempty set K of imputations satisfying the properties:

- (i) (*Internal stability*) For all $x, y \in K$, $\neg x \text{ dom } y$.
- (ii) (*External stability*) For all $z \in I(w) \setminus K$, there is an imputation $x \in K$ such that $x \text{ dom } z$.

We briefly recall some well-known facts for a crisp game w , which are interesting for our paper:

(1) The core $C(w)$ is a subset of the D-core $DC(w)$, and both sets are convex sets (Gillies(1953)).

(2) If $DC(w) = \phi$, then, according to (1), $C(w) = DC(w) = \phi$. If $DC(w) \neq \phi$, then a sufficient condition for the coincidence of $C(w)$ and $DC(w)$, is $w(N) \geq w(S) + \sum_{i \in N \setminus S} w(\{i\})$, for each

$S \subseteq N$ (Shapley and Shubik (1969), Rafels and Tijs (1997), Chang (2000)).

(3) Each stable set contains the D-core $DC(w)$.

(4) For each convex crisp game w there is only one stable set, which coincides with the D-core $DC(w)$ (Shapley (1971)).

(5) If $C(w) \neq DC(w)$ then $C(w) = \phi$ (this is a consequence of (2) and the Bondareva-Shapley theorem (Bondareva (1963), Shapley (1967)).

For details we refer the reader to the books of Driessen (1988), Owen (1995), and Tijs (2003).

3. Cores and stable sets for fuzzy games

Let $v \in FG^N$ and let $I(v)$ be the imputation set of v , i.e.

$$I(v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(e^N), x_i \geq v(e^i) \text{ for each } i \in N\}.$$

The core (Aubin (1974)) $C(v)$ of a fuzzy game v is the subset of imputations which are stable against any possible deviation by fuzzy coalitions, i.e.

$$C(v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in F^N\}.$$

Now we introduce two other cores for a fuzzy game v , namely the proper core and the crisp core, by weakening the stability conditions.

Let $s \in F^N$. From now on we use the notation $car(s) = \{i \in N \mid s_i > 0\}$. We call s a *proper fuzzy coalition* if $car(s) \neq N$. The set of proper fuzzy coalitions is denoted by PF^N . To define the *proper core* $C^P(v)$ of a fuzzy game v , we consider only stability regarding proper fuzzy coalitions, i.e.

$$C^P(v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in PF^N\}.$$

Further, if we consider only crisp-like coalitions e^S in the stability conditions, one obtains the

crisp core $C^{cr}(v)$ of the fuzzy game v , i.e.

$$C^{cr}(v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in \text{car}(e^S)} x_i \geq v(e^S) \text{ for each } S \in 2^N\}.$$

Note that the crisp core $C^{cr}(v)$ of a fuzzy game v is also the core of the crisp game $w = cr(v)$.

One can easily see that both cores $C^P(v)$ and $C^{cr}(v)$ are convex sets.

Let $x, y \in I(v)$ and let $s \in F^N$. We say x dominates y via s , denoted by $x \text{ dom}_s y$, if (i) $x_i > y_i$ for all $i \in \text{car}(s)$ and (ii) $\sum_{i \in N} s_i x_i \leq v(s)$. The two conditions are interpreted as follows. Let

$x \text{ dom}_s y$. Then

(i) $x_i > y_i$ and thus $s_i x_i > s_i y_i$ for each $i \in \text{car}(s)$, means that the imputation $x = (x_1, \dots, x_n)$ is better than the imputation $y = (y_1, \dots, y_n)$ for all (active) players $i \in \text{car}(s)$;

(ii) $\sum_{i \in N} s_i x_i \leq v(s)$ means that the payoff $\sum_{i \in N} s_i x_i$ is reachable by the fuzzy coalition s .

Remark 2. Note that $x \text{ dom}_s y$ implies $s \in PF^N$ because from $x_i > y_i$ for all $i \in N$ it follows

$\sum_{i \in N} x_i > \sum_{i \in N} y_i$, in contradiction with $x, y \in I(v)$. It is, however, to be noted that $|\text{car}(s)|=1$ is

possible.

We simply say x dominates y , denoted by $x \text{ dom } y$, if there is a (proper) fuzzy coalition s such that $x \text{ dom}_s y$. The negation of $x \text{ dom } y$ is denoted here by $\neg x \text{ dom } y$.

The dominance core (*D-core*) $DC(v)$ of a fuzzy game v is the set of imputations which are not dominated by any other imputation, i.e.

$$DC(v) = \{x \in I(v) \mid \neg y \text{ dom } x \text{ for all } y \in I(v)\}.$$

A stable set of a fuzzy game v is a nonempty set K of imputations satisfying the properties:

(i) (*Internal stability*) For all $x, y \in K$, $\neg x \text{ dom } y$.

(ii) (*External stability*) For all $z \in I(v) \setminus K$, there is an imputation $x \in K$ such that $x \text{ dom } z$.

Theorem 1. Let v be a fuzzy game. Then

(i) $C(v) \subset C^P(v) \subset C^{cr}(v)$;

(ii) $C^P(v) \subset DC(v)$;

(iii) For each stable set $K : DC(v) \subset K$.

Proof. The theorem is trivially true if $I(v) = \emptyset$. So, suppose in the following that $I(v) \neq \emptyset$.

(i) This follows straightforwardly from the definitions.

(ii) Let $x \in I(v) \setminus DC(v)$. Then there are $y \in I(v)$ and $s \in PF^N$ satisfying $y_i > x_i$ for each

$i \in \text{car}(s)$ and $\sum_{i \in N} s_i y_i \leq v(s)$. Then $\sum_{i \in \text{car}(s)} s_i x_i < \sum_{i \in \text{car}(s)} s_i y_i \leq v(s)$. Hence $x \in I(v) \setminus C^P(v)$. We

conclude that $C^P(v) \subset DC(v)$.

(iii) Let K be a stable set. Since $DC(v)$ consists of undominated imputations and each imputation in $I(v) \setminus K$ is dominated by some imputation by the external stability property, it follows that $DC(v) \subset K$. Q.E.D.

In the next theorem we give sufficient conditions for the coincidence of the proper core and the dominance core.

Theorem 2. Let $v \in FG^N$. Suppose $v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s) \geq 0$ for each $s \in F^N$. Then

$$C^P(v) = DC(v).$$

Proof. Note that $C^P(v) = DC(v) = \emptyset$ if $I(v) = \emptyset$. Suppose $I(v) \neq \emptyset$. From Theorem 1(ii) it follows that $C^P(v) \subset DC(v)$. We show the converse inclusion by proving that $x \notin C^P(v)$ implies $x \notin DC(v)$. Let $x \in I(v) \setminus C^P(v)$. Then there is $s \in PF^N$ such that $\sum_{i \in N} s_i x_i < v(s)$. For each

$i \in \text{car}(s)$ take $\varepsilon_i > 0$ such that $\sum_{i \in \text{car}(s)} s_i (x_i + \varepsilon_i) = v(s)$. Define $y \in R^n$ by

$$y_i = \begin{cases} x_i + \varepsilon_i & \text{for each } i \in \text{car}(s) \\ v(e^i) + |N \setminus \text{car}(s)|^{-1} (v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s)) & \text{for each } i \notin \text{car}(s) \end{cases}$$

Note that $\sum_{i \in N} y_i = v(e^N)$, $y_i > x_i \geq v(e^i)$ for each $i \in \text{car}(s)$ and, since

$v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s) \geq 0$, we have $y_i \geq v(e^i)$ for each $i \in N \setminus \text{car}(s)$. Hence $y \in I(v)$. Now,

since $y_i > x_i$ for all $i \in \text{car}(s)$ and $\sum_{i \in N} s_i y_i = v(s)$ we have $y \text{ dom}_s x$; thus $x \in I(v) \setminus DC(v)$.

Q.E.D.

Remark 3. Let $v \in FG^N$. Take the crisp game $w = cr(v)$. Then $v(e^N) \geq v(s) + \sum_{i \in N \setminus \text{car}(s)} v(e^i)$ for each

$s \in F^N$ implies $w(N) \geq w(S) + \sum_{i \in N \setminus S} w(\{i\})$, for each $S \subseteq N$. So Theorem 2 can be seen as an

extension of property (2) for crisp games.

From Theorem 2 we obtain the following corollary.

Corollary 1. Let $v \in FG^N$ with $v(e^i) \geq 0$ for each $i \in N$ and $C^P(v) \neq DC(v)$. Then $C^P(v) = \emptyset$.

Proof. $C^P(v) \neq DC(v)$ implies that $I(v) \neq \emptyset$ and that there is a $t \in PF^N$ with $v(t) + \sum_{i \in N \setminus \text{car}(t)} v(e^i) > v(e^N)$ by Theorem 2. By $v(e^i) \geq 0$ for each $i \in N$, we have $x \geq 0$ and

$\sum_{i \in N} x_i = v(e^N)$ for each $x \in I(v)$. Hence

$$\sum_{i \in N} t_i x_i = \sum_{i \in \text{car}(t)} t_i x_i \leq \sum_{i \in \text{car}(t)} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus \text{car}(t)} x_i \leq v(e^N) - \sum_{i \in N \setminus \text{car}(t)} v(e^i) < v(t)$$

holds for each $x \in I(v)$. Thus there is no $x \in I(v)$ such that $x \in C^P(v)$. Hence, $C^P(v) = \emptyset$. Q.E.D.

We next prove that for a fuzzy game the dominance core is a convex set.

Lemma 1. Let v be a fuzzy game with $v(e^i) \geq 0$ for each $i \in N$. Let \bar{v} be the fuzzy game given by $\bar{v}(s) = \min\{v(s), v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i)\}$. Then $DC(v) = DC(\bar{v}) = C^P(\bar{v})$.

Proof. Note that $DC(v) = DC(\bar{v}) = C^P(\bar{v}) = \emptyset$ if $I(v) = \emptyset$. Suppose $I(v) \neq \emptyset$. It implies $I(v) = I(\bar{v})$. Thus to prove $DC(v) = DC(\bar{v})$, it is sufficient to show that for $x, y \in I(v)$ and $s \in F^N$, $x \text{ dom}_s y$ in v if and only if $x \text{ dom}_s y$ in \bar{v} . We only have to show that for $x \in I(v)$ and $s \in F^N$, $\sum_{i \in N} s_i x_i \leq v(s)$ if and only if $\sum_{i \in N} s_i x_i \leq \bar{v}(s)$.

The ‘if’ part follows from $\bar{v}(s) \leq v(s)$. For the ‘only if’ part note that for $s \in F^N$ and $x \in I(v)$ we have $\sum_{i \in N} s_i x_i = \sum_{i \in \text{car}(s)} s_i x_i \leq \sum_{i \in \text{car}(s)} x_i = \sum_{i \in N} x_i - \sum_{i \in N \setminus \text{car}(s)} x_i \leq v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i)$, where the first inequality

follows from $x_i \geq v(e^i) \geq 0$ for each $i \in \text{car}(s) \subseteq N$. Hence, $\sum_{i \in N} s_i x_i \leq v(s)$ implies $\sum_{i \in N} s_i x_i \leq \bar{v}(s)$.

Since we have $\bar{v}(s) + \sum_{i \in N \setminus \text{car}(s)} \bar{v}(e^i) \leq \bar{v}(s) + \sum_{i \in N \setminus \text{car}(s)} v(e^i) \leq v(e^N) = \bar{v}(e^N)$ by $\bar{v}(e^i) \leq v(e^i)$, we obtain

$DC(\bar{v}) = C^P(\bar{v})$ by Theorem 2. Q.E.D.

Let $v \in FG^N$. Define the fuzzy game v' by $v'(s) = v(s) - \sum_{i \in \text{car}(s)} s_i v(e^i)$ for each $s \in F^N$. Note that

$v'(e^i) = 0$ for each $i \in N$. From Lemma 1 it follows that $DC(v')$ is a convex set, because $C^P(\bar{v}')$ is a convex set. Since for an arbitrary fuzzy game v

$DC(v) = DC(v') + (v(e^1), v(e^2), \dots, v(e^n))$ holds, where $DC(v') + (v(e^1), v(e^2), \dots, v(e^n)) = \{x + y \mid x \in DC(v'), y = (v(e^1), v(e^2), \dots, v(e^n))\}$, we obtain the following theorem.

Theorem 3. For each fuzzy game v the dominance core $DC(v)$ is a convex set.

We end this section with two examples that illustrate the results in the above theorems.

Example 1. Let $N = \{1, 2\}$ and let $v: F^{\{1, 2\}} \rightarrow \mathbf{R}$ be given by $v(s_1, s_2) = s_1 + s_2 - 1$ for each $s \in F^{\{1, 2\}}$. Further, let v_1 and v_2 be given by

$$v_1(s) = v(s) \text{ if } s \neq (0, 1/2) \text{ and } v_1(0, 1/2) = 4,$$

$$v_2(s) = v(s) \text{ if } s \neq (1/2, 1/2) \text{ and } v_2(1/2, 1/2) = 4.$$

Let $\Delta = \{x \in \mathbf{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 1\}$. Then

$$(i) \ C(v) = C^P(v) = DC(v) = I(v) = \Delta$$

$$(ii) \ C(v_1) = C^P(v_1) = \phi, DC(v_1) = I(v_1) = \Delta$$

$$(iii) \ C(v_2) = \phi, C^P(v_2) = DC(v_2) = I(v_2) = \Delta$$

(iv) For v, v_1 , and v_2 , the imputation set Δ is the unique stable set.

Note that $v_2(s) + \sum_{i \in N \setminus \text{car}(s)} v_2(e^i) > v_2(e^N)$ for $s = (1/2, 1/2)$ and $C^P(v_2) = DC(v_2)$. Hence, the sufficient condition in Theorem 2 for the equality $C^P(v) = DC(v)$ is not a necessary condition.

In the next example we give a fuzzy game v with $C(v) \neq DC(v)$ and $C(v) \neq \phi$. For a crisp game w we have according to (5) that $C(w) = \phi$ if $C(w) \neq DC(w)$.

Example 2. Let $N = \{1, 2\}$ and let $v: F^{\{1, 2\}} \rightarrow \mathbf{R}$ be given by $v(s_1, 1) = \sqrt{s_1}$ for all $(s_1, 1) \in F^{\{1, 2\}}$,

and $v(s_1, s_2) = 0$ otherwise. Then $I(v) = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid x_1 + x_2 = 1\}$,

$C(v) = \{x \in I(v) \mid 0 \leq x_1 \leq 1/2\} \neq I(v)$, and $C^P(v) = DC(v) = I(v)$. Further $I(v)$ is the unique stable set.

4. Cores and stable sets for convex fuzzy games

A special class of fuzzy games with a nonempty core is the class of convex fuzzy games introduced in Branzei et al.(2002). Here $v \in FG^N$ is called convex if it satisfies the properties:

(i) (*Supermodularity*) $v(s \vee t) + v(s \wedge t) \geq v(s) + v(t)$ for all $s, t \in F^N$, where $s \vee t$ and $s \wedge t$ are those elements of $[0, 1]^N$ with the i -th coordinate equal to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$,

respectively, for each $i \in N$.

(ii) (*Coordinate-wise convexity*) For each $i \in N$ the function $g_{s^{-i}} : [0,1] \rightarrow \mathbf{R}$ with $g_{s^{-i}}(t) = v(s^{-i} \parallel t)$ for each $t \in [0,1]$ is a convex function, where for each $i \in N$, $s \in F^N$ and $t \in [0,1]$, $(s^{-i} \parallel t)$ is the element in F^N such that $(s^{-i} \parallel t)_j = s_j$ for each $j \in N \setminus \{i\}$ and $(s^{-i} \parallel t)_i = t$.

Convex fuzzy games form a convex cone. It is proved in Branzei et al. (2002) that the core of a convex fuzzy game v coincides with the core of the corresponding crisp game $w = cr(v)$.

Lemma 2. Let v be a convex fuzzy game. Take $x, y \in I(v)$ and suppose $x \text{ dom}_s y$ for some $s \in F^N$. Then $|car(s)| \geq 2$.

Proof. Take $x, y \in I(v)$ and suppose $x \text{ dom}_s y$ for some $s \in F^N$ with $car(s) = \{i\}$. Then $x_i > y_i$ and $s_i x_i \leq v(s_i e^i)$. By the convexity of v , we obtain $s_i v(e^i) \geq v(s_i e^i)$. Thus we have $y_i < x_i \leq v(s_i e^i) / s_i \leq v(e^i)$ which is a contradiction with the individual rationality of y . Q.E.D.

Theorem 4. The *dom* relations for a convex fuzzy game v and its corresponding crisp game w coincide, i.e. for all $x, y \in I(v) = I(w)$: $x \text{ dom}_y$ in v if and only if $x \text{ dom}_y$ in $w = cr(v)$.

Proof. First we note that $I(v) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = v(e^N), x_i \geq v(e^i) \text{ for each } i \in N\}$ and

$I(w) = \{x \in \mathbf{R}^N \mid \sum_{i \in N} x_i = w(N), x_i \geq w(\{i\}) \text{ for each } i \in N\}$ coincide because $w(N) = v(e^N)$ and $w(\{i\}) = v(e^i)$ for each $i \in N$.

To prove the ‘if’ part, let $x, y \in I(w) = I(v)$ and $x \text{ dom}_s y$ for some $s \in F^N$. Then it implies $x \text{ dom}_{e^s} y$ in v .

Now we prove the ‘only if’ part. Let $x, y \in I(v) = I(w)$ and $x \text{ dom}_s y$ for some $s \in F^N$. Let $\varphi(s) = |\{i \in N \mid 0 < s_i < 1\}|$. As we noted in Remark 2, $\varphi(s) < n$. It is sufficient to prove, by induction on $\varphi(s) \in \{0, 1, \dots, n-1\}$, that $x \text{ dom}_s y$ implies $x \text{ dom}_y$ in w .

Clearly, if $\varphi(s) = 0$ then $x \text{ dom}_{car(s)} y$, because $\varphi(s) = 0$ implies that s is a crisp-like coalition.

Suppose now that the assertion ‘ $x \text{ dom}_s y$ in v with $\varphi(s) = k$ implies $x \text{ dom}_y$ in w ’ holds for each k with $0 \leq k < r < n$. Take $s \in F^N$ with $\varphi(s) = r$, and $i \in N$ such that $0 < s_i < 1$, and take $x, y \in I(v)$ such that $x \text{ dom}_s y$. Then $x_i > y_i$ for each $i \in car(s)$ and $s \cdot x \leq v(s)$. Further $|car(s)| \geq 2$ by Lemma 2. We note that s can be represented by a convex combination of $a = s - s_i e^i$ and $b = s + (1 - s_i) e^i$, i.e. $s = (1 - s_i) a + s_i b$. Note that $\varphi(a) = r - 1$ and $\varphi(b) = r - 1$. Further $|car(a)| = |car(s)| - 1$, $|car(b)| = |car(s)|$.

The inequality $s \cdot x \leq v(s)$ implies $(1 - s_i) a \cdot x + s_i b \cdot x \leq v(s)$. The (coordinate-wise) convexity of v induces $v(s) \leq (1 - s_i) v(a) + s_i v(b)$. Hence $(1 - s_i) a \cdot x + s_i b \cdot x \leq (1 - s_i) v(a) + s_i v(b)$ which implies $(1 - s_i)(a \cdot x - v(a)) + s_i(b \cdot x - v(b)) \leq 0$; thus $a \cdot x \leq v(a)$ or

$b \cdot x \leq v(b)$. We want to show that

$$x \text{ dom}_a y \text{ or } x \text{ dom}_b y. \quad (*)$$

The following three cases should be considered:

- (1) $b \cdot x \leq v(b)$. Then $x \text{ dom}_b y$, since $|car(b)| \geq 2$.
- (2) $b \cdot x > v(b)$ and $|car(s)| \geq 3$. Then $a \cdot x \leq v(a)$; thus $x \text{ dom}_a y$, since $|car(a)| \geq 2$.
- (3) $b \cdot x > v(b)$ and $|car(s)| = 2$. Then we have $a \cdot x \leq v(a)$ and $|car(a)| = 1$. By the convexity of v and the individual rationality, we obtain $a \cdot x \geq v(a)$. In fact, let $a = s_j e^j$. Then the convexity of v induces $s_j v(e^j) \geq v(s_j e^j)$. By the individual rationality, we obtain $x_j \geq v(e^j)$. Hence $a \cdot x = s_j x_j \geq s_j v(e^j) \geq v(s_j e^j) = v(a)$. So $a \cdot x = v(a)$, which is contradictory to $(1 - s_i)(a \cdot x - v(a)) + s_i(b \cdot x - v(b)) \leq 0$, implying that case (3) does not occur.

Hence (*) holds. Since $\varphi(a) = \varphi(b) = r - 1$ the induction hypothesis implies that $x \text{ dom } y$ in w .

Q.E.D.

Theorem 5. Let v be a convex fuzzy game and $w = cr(v)$. Then

- (i) $C(v) = C^P(v) = C^{cr}(v)$;
- (ii) $DC(v) = DC(w)$;
- (iii) $C(v) = DC(v)$.

Proof. (i) For convex fuzzy games $C(v) = C(w)$ (see Theorem 7 in Branzei et al. (2002)). Now, we use Theorem 1(i).

(ii) From Theorem 4 we conclude that $DC(v) = DC(w)$.

(iii) Since v is a convex fuzzy game we have $v(e^N) \geq v(s) + \sum_{i \in N \setminus car(s)} v(e^i)$ for each $s \in F^N$. We

obtain by Theorem 2: $C^P(v) = DC(v)$. Now we use (i). Q.E.D.

The next theorem extends the result of Shapley (1971) that each crisp convex game has a unique stable set coinciding with the dominance core.

Theorem 6. Let v be a convex fuzzy game. Then there is a unique stable set, namely $DC(v)$.

Proof. Let $w = cr(v)$. Then by Shapley's result, $DC(w)$ is the unique stable set of w . In view of Theorem 4, the set of stable sets of v and w coincide, and by Theorem 5(ii) $DC(v) = DC(w)$. So, the unique stable set of v is $DC(v)$. Q.E.D.

Note that the game v in Example 1 is convex, but v_1 and v_2 are not.

5. Cores and stable sets for fuzzy clan games

In this section games of the form $v: [0,1]^{N_1} \times \{0,1\}^{N_2} \rightarrow \mathbf{R}$ with $N_1 \cap N_2 = \emptyset$, $v(e^\emptyset) = 0$ are considered, where players in N_1 have participation levels which may vary between 0 and 1, while the players in N_2 are crisp players in the sense that they can fully cooperate or not at all. We denote $N_1 \cup N_2$ by N .

Let $v: [0,1]^{N_1} \times \{0,1\}^{N_2} \rightarrow \mathbf{R}$. We can define in an obvious way the proper core $C^P(v)$, the dominance core $DC(v)$, the crisp core $C^{cr}(v)$ and stable sets K . Then modifying the proofs in Theorems 1,2 and 3 we obtain

Theorem 7. Let $v: [0,1]^{N_1} \times \{0,1\}^{N_2} \rightarrow \mathbf{R}$. Then

(i) $C(v) \subset C^P(v) \subset C^{cr}(v)$;

(ii) $C^P(v) \subset DC(v)$; further, if $v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s) \geq 0$ for each $s \in [0,1]^{N_1} \times \{0,1\}^{N_2}$, then

$$C^P(v) = DC(v);$$

(iii) $DC(v)$ is a convex set;

(iv) For each stable set K , we have $DC(v) \subset K$.

Special attention is given to a subclass of such games, which we have called fuzzy clan games (Tijds et al. (2002)), where the clan members are the crisp players. Fuzzy clan games are defined using veto power of clan members, monotonicity, and a condition reflecting the fact that an increase in participation level of a non-clan member in growing coalitions containing at least all clan members with full participation level results in a decrease of the average marginal return of that player (DAMR-property).

In the following F_C^N stands for $[0,1]^{N \setminus C} \times \{0,1\}^C$.

Formally, a game $v: F_C^N \rightarrow \mathbf{R}$ is a *fuzzy clan game* with clan C if v satisfies the following three properties:

(i) (*Veto-power of clan members*) $v(s) = 0$ if $s_C \neq 1_C$;

(ii) (*Monotonicity*) $v(s) \leq v(t)$ for all $s, t \in F_C^N$ with $s \leq t$;

(iii) (*DAMR-property for non-clan members*) for each $i \in N \setminus C$, all $s^1, s^2 \in F_{1_C}^N$ and all $\varepsilon_1, \varepsilon_2 > 0$ such

that $s^1 \leq s^2$ and $0 \leq s^1 - \varepsilon_1 e^i \leq s^2 - \varepsilon_2 e^i$ we have

$$\varepsilon_1^{-1} (v(s^1) - v(s^1 - \varepsilon_1 e^i)) \geq \varepsilon_2^{-1} (v(s^2) - v(s^2 - \varepsilon_2 e^i)).$$

Property (i) expresses the fact that the full participation level of all clan members is a necessary condition for generating a positive reward for coalitions.

Fuzzy clan games for which the clan consists of a single player are called fuzzy big boss games, with the single clan member as the big boss. For fuzzy clan games we have additionally

Theorem 8. Let v be a fuzzy clan game with player set N and clan C . Then $DC(v) = C^P(v)$.

Proof. From the veto-power property we have that $v(e^i) = 0$ for each $i \in N$ if $|C| > 1$. Then the monotonicity of v implies $v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s) = v(e^N) - v(s) \geq 0$ for each $s \in F^N$. One can

easily check that in the case $|C| = 1$, with $C = \{n\}$ $v(e^N) - \sum_{i \in N \setminus \text{car}(s)} v(e^i) - v(s) \geq 0$ for each $s \in F^N$,

too. The equality $DC(v) = C^P(v)$ follows then from Theorem 7(ii). Q.E.D.

Now we give two examples of fuzzy clan games v to illustrate situations like $DC(v) \neq C(v)$ and $DC(v) \neq K$, respectively.

Example 3. Let $N = \{1,2\}$ and let $v: [0,1] \times \{0,1\} \rightarrow \mathbf{R}$ be given for all $s_1 \in [0,1]$ by $v(s_1,1) = \sqrt{s_1}$ and $v(s_1,0) = 0$. This is a big boss game with player 2 as the big boss, so $C(v) \neq \emptyset$. Moreover, as in Example 2, we obtain

$C(v) = \{x \in I(v) \mid 0 \leq x_1 \leq 1/2\}$, $DC(v) = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid x_1 + x_2 = 1\}$, so $DC(v) \neq C(v)$. Note that $I(v) = \{(x_1, x_2) \in \mathbf{R}_+^2 \mid x_1 + x_2 = 1\}$ is the unique stable set.

The following example shows that $DC(v)$ can be a proper subset of a stable set.

Example 4. Let $N = \{1,2,3\}$ and let v be given by $v(s_1, s_2, 0) = 0$ and $v(s_1, s_2, 1) = \min\{s_1 + s_2, 1\}$ for all $(s_1, s_2) \in [0,1]^2$. Then $DC(v) = \{(0,0,1)\}$ and no element in $I(v)$ is dominated by $(0,0,1)$. So $DC(v)$ is not a stable set. The set $K^{a,b} = \{(\varepsilon a, \varepsilon b, 1 - \varepsilon) \mid 0 \leq \varepsilon \leq 1\}$ when $a, b \in \mathbf{R}_+$ with $a + b = 1$ is a stable set of v .

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