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Abstract

This paper deals with interactive multiple fund investment situations, in which investors can invest their capital in a number of funds. The investors, however, face some restrictions. In particular, the investment opportunities of an investor depend on the behaviour of the other investors. Moreover, the individual investment returns may differ. We consider this situation from a cooperative game theory point of view. Based on different assumptions modelling the gains of joint investment, we consider corresponding types of games and analyse their properties. We propose an explicit allocation process for the maximal total investment revenues.

1 Introduction

Of the many decisions that a firm has to make, none is likely to have more impact than the decision to invest capital, which often involves large, extended commitments of money and management time. Such investment decisions determine the company's future course and, hence, its market value. It is not surprising, therefore, that firms devote much time and effort to planning capital expenditure.

The importance of investment decisions is also reflected in the enormous amount of attention that is devoted to it in the economic literature. In most of this literature on investment, firms are modelled as individually acting agents, ie, *cooperation* between firms is not taken into account. Another assumption that is predominant in the literature on investment, is that the agents face investment opportunities that are exogenously given. That is, the investment opportunities of an agent are

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not influenced by the investments of other agents; the *strategic* aspects of investment are often overlooked. In this paper, we analyse situations in which investment opportunities of an agent depend on the behaviour of other investors. Moreover, the situations will be analysed by taking into account the consequences of possible cooperative behaviour.

In this paper, we introduce a new class of cooperative situations, called *multiple fund investment (MFI)* situations. In an MFI situation, agents can invest their capital in a certain number of funds. There are restrictions on the funds such that there is a maximum number of capital units that can be invested in each of them. The agents (players) in an MFI situation are characterised by the amount of capital they can invest and by their individual returns on the different funds. That is, we consider the possibility that the return of an investment project depends on the player (eg, firm) that is involved in this project. Furthermore, investment opportunities are limited; we assume that the total capital available exceeds the total investment opportunities.

Associated with each MFI situation, we define three cooperative MFI games in characteristic function form. These games are based on three possible assumptions on the coalitional expectations of the return on their joint investments. These coalitional expectations relate to the behaviour of the players outside the coalition. To actually calculate the coalitional values of the MFI games, one has to solve linear programs. These turn out to be transportation problems, allowing for a fairly quick calculation of these values.

The central question in an MFI situation is how to divide, in an acceptable way, the maximal total investment revenues of the players if they all cooperate and coordinate their investment plans in an optimal way. In this context, we study properties of the associated cooperative games, in particular convexity and (total) balancedness. We also propose a two-stage allocation rule for MFI situations. In the first stage, an *allotment* is made, which gives each player investment rights in the various funds. In the second stage, the players are thought of as facing a linear production situation (cf. Owen (1975)) in which their investment rights and capital stock are resources. Owen vectors of this linear production situation are then seen as solutions of the original MFI situation. Stability of these solutions is shown.

This paper is organised as follows. Section 2 introduces MFI situations and the three corresponding MFI games. In section 3, the properties of convexity and (total) balancedness of these games are studied. In section 4, we introduce the concept of allotment and propose our two-stage solution for MFI situations. In section 5, we elaborate on how our analysis can be extended when some of the assumptions are modified.

2 The MFI model

A *multiple fund investment* or *MFI* situation is a tuple (N, M, e, A, d), where $N = \{1, \ldots, n\}$ is the set of players, $M = \{1, \ldots, m\}$ denotes the set of available funds and $e \in \mathbb{R}^M_{++}$ is the vector of fund restrictions. An element e_j denotes the maximum number of capital units that can be invested in fund j. Furthermore, $A \in \mathbb{R}^{N \times M}_+$ is the return matrix, where an element A_{ij} denotes the revenue player i obtains when he invests one unit of his capital in fund j. Finally, $d \in \mathbb{R}^N_{++}$ is the vector of individual investment capital. We assume that $\sum_{j \in M} e_j < \sum_{i \in N} d_i$ [1].

Let (N, M, e, A, d) be an MFI situation. In order to define corresponding MFI games, we first state the program that determines the maximum revenue a coalition $S \subset N, S \neq \emptyset$ can obtain when the fund restrictions are given by a vector $z \in \mathbb{R}^M_+$. These direct revenues are denoted by $DR(S, z)$ and defined by

$$
\begin{aligned}
DR(S,z) \quad &= \quad \max_{X \in \mathbb{R}^{S \times M}} \sum_{j \in M} \sum_{i \in S} A_{ij} X_{ij} \qquad (2.1)\\
\text{such that} \quad & \quad \sum_{j \in M} X_{ij} \leq d_i \text{ for all } i \in S,\\
& \quad \sum_{i \in S} X_{ij} \leq z_j \text{ for all } j \in M,\\
& \quad X_{ij} \geq 0 \text{ for all } i \in S, j \in M.
\end{aligned}
$$

By introducing a dummy fund or player in order to obtain equality restrictions, this problem is translated into a *balanced transportation problem* (cf. Hitchcock (1941)), which can be solved very efficiently.

Facing fund restrictions $z \in \mathbb{R}^M_+$, the players in S will construct an optimal plan $X^S \in \mathbb{R}^{S \times M}$ according to this program in order to maximise their total revenue. The set of all feasible plans is given by

$$FP(S,z) = \{X^S \in \mathbb{R}^{S \times M}_+ \,|\, \forall_{i \in S} : \sum_{j \in M} X^S_{ij} \leq d_i, \forall_{j \in M} : \sum_{i \in S} X^S_{ij} \leq z_j\}.$$

[1]This assumption will be elaborated upon in section 4.

For a plan $X \in \mathbb{R}^{S\times M}$, the corresponding revenues are given by the *direct payoff* vector $O(X) \in \mathbb{R}^S$, where $O_i(X) = \sum_{j\in M} A_{ij}X_{ij}$ for all $i \in N$. The set of all optimal feasible plans X^S is denoted by $OP(S,z)$:

$$OP(S,z) = \{X^S \in FP(S,z) \,|\, \sum_{i\in S} O_i(X) = DR(S,z)\}.$$

Once the members of a coalition S have decided upon a paricular plan X^S, they will invest their capital accordingly, thereby tightening the fund restrictions z. The resulting fund restrictions $z(X^S)$ are given by

$$z_j(X^S) = z_j - \sum_{i\in S} X_{ij}^S$$

for all $j \in M$.

Using this notation, we now introduce three TU games that correspond to an MFI situation. A *TU (transferable utility) game* is a pair (N,v), where $N = \{1,\ldots,n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, assigning to every coalition $S \subset N$ a value $v(S)$, representing the total monetary payoff the members of S can guarantee themselves if they cooperate. By convention, $v(\emptyset) = 0$.

Depending on how the "guarantee" in the last paragraph is interpreted, an MFI situation gives rise to three TU games, which will be denoted by v^1, v^2 and v^3. The common feature is that first the players outside S can invest their capital and afterwards the members of S invest optimally given the resulting (tightened) fund restrictions. The difference between the games lies in the way the players outside S are assumed to behave in the first stage.

Let (N,M,e,A,d) be an MFI situation. The game v^1 is defined by

$$v^1(S) = \min\{DR(S, e(X^{N\setminus S})) \,|\, X^{N\setminus S} \in FP(N\setminus S, e)\}$$

for all $S \subset N, S \neq \emptyset$. That is, the players outside S, facing fund restrictions e, are assumed to choose that feasible plan $X^{N\setminus S}$ for which the resulting revenue for S, facing fund restrictions $e(X^{N\setminus S})$, is minimal.

For our second game, we again take a pessimistic approach, but with the assumption that the choice of the players in $N\setminus S$ is restricted to plans that maximise their own revenue, ie, that they choose an investment plan in $OP(N\setminus S, e)$:

$$v^2(S) = \min\{DR(S, e(X^{N\setminus S})) \,|\, X^{N\setminus S} \in OP(N\setminus S, e)\}$$

for all $S \subset N, S \neq \emptyset$.

For the third game, the players outside S also choose an optimal plan for themselves, giving them a revenue of $DR(N\backslash S, e)$. In the second stage, we assume that the players in S can persuade the members of $N\backslash S$ to change their investment plan as long as they still receive $DR(N\backslash S, e)$. Of course, coalition S will persuade them to choose a plan in such a way that the two coalitions together generate a total revenue of $DR(N, e)$. After giving up the promised $DR(N\backslash S, e)$, the net revenue of coalition S equals

$$v^3(S) = DR(N, e) - DR(N\backslash S, e).$$

Example 2.1 Consider the MFI situation (N, M, e, A, d) with three players (rows) and two funds (columns):

$$\begin{array}{cc|c} 3 & 3 & \\ \hline 10 & 9 & 1 \\ 1 & 4 & 4 \\ 4 & 10 & 3 \end{array}$$

So, $N = \{1, 2, 3\}$, $M = \{1, 2\}$, $e = (3, 3)$, $A = \begin{bmatrix} 10 & 9 \\ 1 & 4 \\ 4 & 10 \end{bmatrix}$ and $d = (1, 4, 3)$.

The unique optimal plan for the grand coalition is

$$X^N = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 0 & 3 \end{bmatrix}$$

with total payoff 42 and direct payoff $O(X^N) = (10, 2, 30)$.

Next, take $S = \{2, 3\}$. In order to compute $v^1(S)$, we have to determine where player 1 should invest his single unit of capital so that the resulting optimal payoff to S is minimal. If player 1 invests his unit in fund 1 ($X^1 = [\ 1 \quad 0\ ] \in FP(N\backslash S, e)$), then $z(X^1) = (2, 3)$ and coalition S can obtain 32 with plan $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \in OP(S, z(X^1))$. If player 1 invests in fund 2, coalition S can get 26 with plan $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$. Hence, $v^1(S) = 26$.

For our second game, player 1 has to invest in fund 1, which is optimal for him. As a result, $v^2(S) = 32$.

For the third game, we first determine $DR(N\backslash S, e)$, which equals 10 with plan $[\ 1 \quad 0\ ]$ for player 1. Hence, $v^3(S) = DR(N, e) - DR(N\backslash S, e) = 42 - 10 = 32$.

In the following table, we list the direct revenues and the three coalitional values of each coalition:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$DR(S,e)$	10	13	30	23	40	33	42
$v^1(S)$	0	2	4	12	14	26	42
$v^2(S)$	0	2	4	12	14	32	42
$v^3(S)$	9	2	19	12	29	32	42

$\triangleleft$

The first game is the most pessimistic, whereas the third game is the most optimistic, as is shown in the following proposition.

Proposition 2.1 *Let (N, M, e, A, d) be an MFI situation. Then for the three corresponding games we have that $v^1(N) = v^2(N) = v^3(N) = DR(N, e)$ and $v^1 \leq v^2 \leq v^3$.*

Proof:
The first part of the proposition follows immediately from the definitions. The relation between v^1 and v^2 is obvious. It remains to show that $v^2 \leq v^3$. Let $S \subset N, S \neq \emptyset$ and let $X^{N\backslash S} \in OP(N\backslash S, e)$, resulting in the revenue of $DR(N\backslash S, e)$ for coalition $N\backslash S$. Let $X^S \in OP(S, e(X^{N\backslash S}))$, resulting in the revenue of $DR(S, e(X^{N\backslash S}))$ for coalition S. If we combine the plans $X^{N\backslash S}$ and X^S, we obtain a feasible plan for the grand coalition. Therefore,

$$v^2(S) + DR(N\backslash S, e) \leq DR(S, e(X^{N\backslash S})) + DR(N\backslash S, e) \leq DR(N, e).$$

Hence, $v^2(S) \leq v^3(S)$. $\square$

3 Properties of MFI games

In this section, we analyse some properties of our three MFI games. In particular, we consider convexity and (total) balancedness.

A TU game (N, v) is called convex if

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T) \tag{3.1}$$

for all $i \in N, S \subset T \subset N\backslash\{i\}$.

In order to prove convexity of v^3 we first show that in terms of direct revenues, larger coalitions benefit more from a rise in the fund restrictions than do smaller coalitions. For this, we consider the following class of linear programs for $S \subset N, S \neq \emptyset$:

$$\begin{aligned}\overline{DR}(S,k,\ell) &= \min_{y\in\mathbb{R}^S, z\in\mathbb{R}^M} \sum_{i\in S} d_i y_i + \sum_{j\in M} e_j z_j \\ \text{such that} \quad & y_i + z_j \geq A_{ij} \text{ for all } i\in S, j\in M, \\ & y_i \geq k_i \text{ for all } i \in S, \\ & z_j \geq \ell_j \text{ for all } j \in M,\end{aligned} \tag{3.2}$$

where $k \in \mathbb{R}^S_+$ and $\ell \in \mathbb{R}^M_+$. If $k = 0$ and $\ell = 0$, then (3.2) is the dual of (2.1).

First, we show how the solution of (3.2), $(\bar{y}(k,\ell), \bar{z}(k,\ell))$ depends on ℓ.

Lemma 3.1 *Let $\ell, \ell' \in \mathbb{R}^M_+$ be such that $\ell' \geq \ell$. Then*

$$\bar{z}(k,\ell') \geq \bar{z}(k,\ell) \tag{3.3}$$

for all $k \in \mathbb{R}^N_+$.

Proof: If $m = 1$, then (3.3) is trivial. If $n = 1$, then we cannot have $\bar{y}(k,\ell') > \bar{y}(k,\ell)$, since this would contradict optimality of $(\bar{y}(k,\ell), \bar{z}(k,\ell))$. From this, (3.3) immediately follows. So, assume that $m > 1, n > 1$.

Clearly, it suffices to show that (3.3) holds for ℓ and ℓ' which differ in only one coordinate, so without loss of generality assume that $\ell_j = \ell'_j$ for all $j \in M\backslash\{1\}$. Then we can immediately conclude that $\bar{z}_1(k,\ell') \geq \bar{z}_1(k,\ell)$. Substituting $\bar{z}_1(k,\ell)$ back into the program for $\overline{DR}(S,k,\ell)$ we obtain

$$\begin{aligned}\overline{DR}(S,k,\ell) &= e_1\bar{z}_1(k,\ell) + \min_{y\in\mathbb{R}^S, z\in\mathbb{R}^{M\backslash\{1\}}} \sum_{i\in S} d_i y_i + \sum_{j\in M\backslash\{1\}} e_j z_j \\ \text{such that} \quad & y_i + z_j \geq A_{ij} \text{ for all } i\in S, j\in M\backslash\{1\}, \\ & y_i \geq \hat{k}_i \text{ for all } i \in S, \\ & z_j \geq \ell_j \text{ for all } j \in M\backslash\{1\},\end{aligned}$$

where $\hat{k}_i = \max\{k_i - \bar{z}_1(k,\ell), 0\}$ for all $i \in S$. So,

$$\overline{DR}(S,k,\ell) = e_j\bar{z}_1(k,\ell) + \overline{DR}(S,\hat{k},\ell_{M\backslash\{1\}}),$$

where $\ell_{M\backslash\{1\}} = (\ell_j)_{j\in M\backslash\{1\}}$. Similarly,

$$\overline{DR}(S,k,\ell') = e_j\bar{z}_1(k,\ell') + \overline{DR}(S,\hat{k}',\ell_{M\backslash\{1\}})$$

with $\hat{k}'_i = \max\{k_i - \bar{z}_1(k,\ell'),0\}$ for all $i \in S$. Note that $\hat{k}'_i \leq \hat{k}_i$ for all $i \in S$. In order to prove (3.3), we must show that $\bar{z}(\hat{k}',\ell_{M\backslash\{1\}}) \geq \bar{z}(\hat{k},\ell_{M\backslash\{1\}})$. Again, it suffices to show this for all $\hat{k}$ and $\hat{k}'$ which differ only in one coordinate, so, without loss of generality assume that $\hat{k}_i = \hat{k}'_i$ for all $S\backslash\{1\}$. Then we can immediately conclude that $\bar{y}_1(\hat{k}',\ell_{M\backslash\{1\}}) \leq \bar{y}_1(\hat{k},\ell_{M\backslash\{1\}})$. Substituting $\bar{y}_1(\hat{k},\ell_{M\backslash\{1\}})$ back into the program for $\overline{DR}(S,\hat{k},\ell_{M\backslash\{1\}})$ we get

$$\begin{aligned}\overline{DR}(S,\hat{k},\ell_{M\backslash\{1\}}) \quad &= \quad d_1\bar{y}_1(\hat{k},\ell_{M\backslash\{1\}}) + \min_{y\in\mathbb{R}^{S\backslash\{1\}},z\in\mathbb{R}^{M\backslash\{1\}}} \sum_{i\in S\backslash\{1\}} d_i y_i + \sum_{j\in M\backslash\{1\}} e_j z_j \\ \text{such that} \quad & \quad y_i + z_j \geq A_{ij} \text{ for all } i \in S\backslash\{1\}, j \in M\backslash\{1\}, \\ & \quad y_i \geq \hat{k}_i \text{ for all } i \in S\backslash\{1\}, \\ & \quad z_j \geq \hat{\ell}_j \text{ for all } j \in M\backslash\{1\},\end{aligned}$$

where $\hat{\ell}_j = \max\{\ell_j - \bar{y}_1(\hat{k},\ell_{M\backslash\{1\}}),0\}$ for all $j \in M\backslash\{1\}$. So,

$$\overline{DR}(S,\hat{k},\ell_{M\backslash\{1\}}) = d_1\bar{y}_1(\hat{k},\ell_{M\backslash\{1\}}) + \overline{DR}(S\backslash\{1\},\hat{k}_{S\backslash\{1\}},\hat{\ell}_{M\backslash\{1\}}).$$

Similarly,

$$\overline{DR}(S,\hat{k}',\ell_{M\backslash\{1\}}) = d_1\bar{y}_1(\hat{k}',\ell_{M\backslash\{1\}}) + \overline{DR}(S\backslash\{1\},\hat{k}_{S\backslash\{1\}},\hat{\ell}'_{M\backslash\{1\}})$$

with $\hat{\ell}'_j = \max\{\ell_j - \bar{y}_1(\hat{k}',\ell_{M\backslash\{1\}}),0\}$ for all $j \in M\backslash\{1\}$. Note that $\hat{\ell}'_j \geq \hat{\ell}_j$ for all $j \in M\backslash\{1\}$. So, in order to prove (3.3), we have to show that $\bar{z}(\hat{k}_{S\backslash\{1\}},\hat{\ell}'_{M\backslash\{1\}}) \geq \bar{z}(\hat{k}_{S\backslash\{1\}},\hat{\ell}_{M\backslash\{1\}})$. This is the same problem that we started with, but with one player and one fund less. Repeating the same procedure until either $m = 1$ or $n = 1$, we arrive at the result. □

With the previous lemma, we can now show that larger coalitions benefit more from an increase in the fund restrictions (in terms of direct revenues) than do smaller coalitions.

Theorem 3.2 *Let (N,M,e,A,d) be an MFI situation. Then for all $e' \in \mathbb{R}^M$ such that $e' \geq e$, we have*

$$DR(T,e') - DR(T,e) \geq DR(S,e') - DR(S,e)$$

for all $S \subset T \subset N, S \neq \emptyset$.

Proof: Clearly, it suffices to show the statement for all $S \subset T \subset N, S \neq \emptyset$ such that $|T \backslash S| = 1$. So, let $t \in N, S \subset N \backslash \{t\}, S \neq \emptyset$ and define $T = S \cup \{t\}$.
The dual program for coalition S is given by (3.2) with $k = 0$ and $\ell = 0$. The dual for coalition T is as follows:

$$\begin{aligned}
\overline{DR}(T, 0, 0) \quad &= \quad \min_{y \in \mathbb{R}^T, z \in \mathbb{R}^M} \sum_{i \in S} d_i y_i + d_t y_t + \sum_{j \in M} e_j z_j \\
\text{such that} \quad & \quad y_i + z_j \geq A_{ij} \text{ for all } i \in S, j \in M, \\
& \quad y_t + z_j \geq A_{tj} \text{ for all } j \in M, \\
& \quad y_i \geq 0 \text{ for all } i \in T, \\
& \quad z_j \geq 0 \text{ for all } j \in M.
\end{aligned}$$

Now we are going to increase the fund restrictions from e to e' and show that the increase in direct revenues for coalition T is larger than the increase for coalition S. It suffices to show this for an increase in only one fund restriction, so assume without loss of generality that $e'_1 > e_1$ and $e'_j = e_j$ for all $j \in M \backslash \{1\}$.
First note that when the first fund restriction increases from e_1 to e'_1, the feasible regions of the two linear programs remain the same. The only thing that can happen by altering the objective functions is that the optimal face in one (or both) of the programs changes. When gradually increasing the first fund restriction from e_1 to e'_1, the optimal faces of the two minimisation programs may change several times. However, the number of such changes is finite. So, we divide the increase from e_1 to e'_1 into a finite number of smaller increases for which the optimal face is constant (except at the boundary, where the optimal face may be larger), and show for each of these smaller increases that coalition T benefits more than coalition S. Hence, without loss of generality, we can assume that there exist solutions $(\bar{y}^S, \bar{z}^S)$ and $(\bar{y}^T, \bar{z}^T)$ for the programs for S and T, respectively, which remain solutions if we go from e_1 to e'_1.
Since by assumption, the solutions of the two programs do not change and the first fund restriction only appears in the objective function, it suffices to show that its coefficient z_1 is larger in the solution for T that in the solution for S, ie,

$$\bar{z}^T_1 \geq \bar{z}^S_1.$$

Given $\bar{y}^T_t$, the program for T can be rewritten as follows:

$$\overline{DR}(T, 0, 0) \quad = \quad d_t \bar{y}^T_t + \min_{y \in \mathbb{R}^T, z \in \mathbb{R}^M} \sum_{i \in S} d_i y_i + \sum_{j \in M} e_j z_j$$

$$\text{such that} \quad \begin{aligned} & y_i + z_j \geq A_{ij} \text{ for all } i \in S, j \in M, \\ & y_i \geq 0 \text{ for all } i \in S, \\ & z_j \geq c_j \text{ for all } j \in M, \end{aligned}$$

where $c_j = \max\{A_{tj} - \bar{y}_t^T, 0\}$ for all $j \in M$. So, $\overline{DR}(T, 0, 0) = d_t \bar{y}_t^T + \overline{DR}(S, 0, c)$. Hence, the difference with the program for S only lies in the right hand sides of the inequalities corresponding to the $z_j, j \in M$. If $c = 0$, then we immediately have $\bar{z}_1^T = \bar{z}_1^S$. For other $c \in \mathbb{R}_+^M$, the result follows from Lemma 3.1. □

Theorem 3.3 *Let (N, M, e, A, d) be an MFI situation. Then the corresponding game v^3 is convex.*

Proof: Let $i \in N, S \subset T \subset N \backslash \{i\}$. If $S = \emptyset$, then (3.1) is trivial. Otherwise, it suffices to show that

$$DR(S \cup \{i\}, e) - DR(S, e) \geq DR(T \cup \{i\}, e) - DR(T, e). \tag{3.4}$$

Let $X^{T \cup \{i\}} \in OP(T \cup \{i\}, e)$ and denote $x = (X_{ij}^{T \cup \{i\}})_{j \in M}$. Then we have

$$DR(T \cup \{i\}, e) - DR(T, e) = DR(T, e - x) + O_i(X^{T \cup \{i\}}) - DR(T, e).$$

Suppose that player i invests according to x and that the players in S invest according to some plan $X^S \in OP(S, e - x)$. Combining these, we obtain a plan $X^{S \cup \{i\}} \in FP(S \cup \{i\}, e)$. Therefore,

$$DR(S \cup \{i\}, e) - DR(S, e) \geq DR(S, e - x) + O_i(X^{T \cup \{i\}}) - DR(S, e).$$

So, in order to prove (3.4), it suffices to show that

$$DR(S, e - x) + O_i(X^{T \cup \{i\}}) - DR(S, e) \geq DR(T, e - x) + O_i(X^{T \cup \{i\}}) - DR(T, e),$$

or equivalently,

$$DR(T, e) - DR(T, e - x) \geq DR(S, e) - DR(S, e - x).$$

This is a direct consequence of Theorem 3.2, so v^3 is a convex game. □

From convexity of v^3 and from Proposition 2.1 it follows that all three games are balanced, ie, that their respective cores are nonempty, where the core of a game (N, v) is defined by

$$C(v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N), \forall_{S\subset N} : \sum_{i\in S} x_i \geq v(S)\}.$$

The games v^1 and v^2 need not be convex. However, the game v^1 is totally balanced, ie, for each $S \subset N, S \neq \emptyset$, the subgame (S, v^1_S) defined by $v^1_S(T) = v^1(T)$ for all $T \subset S$ is balanced.

Proposition 3.4 *Let (N, M, e, A, d) be an MFI situation. Then the corresponding game v^1 is totally balanced.*

Proof: Let $S \subset N, S \neq \emptyset$. In calculating $v^1(S)$, first the players in $N\backslash S$ use their investment capital to lower the fund restrictions in such a way that the revenues that thereafter can be obtained by S are as low as possible. Let $X^{N\backslash S} \in FP(N\backslash S, e)$ denote an investment plan that is chosen by $N\backslash S$ for that reason. Consider the MFI situation $(S, M, e(X^{N\backslash S}), A_S, d_S)$ with $A_S = (A_{ij})_{i\in S, j\in M}$ and $d_S = (d_i)_{i\in S}$. Denote the corresponding most pessimistic game by $v^{1,S}$. Trivially, we have that

$$v^{1,S}(S) = v^1_S(S).$$

Moreover, we have that

$$v^{1,S}(T) \geq v^1_S(T)$$

for all $T \subset S$. To see this, notice that the fund restrictions faced by a coalition T in calculating $v^{1,S}(T)$ are equal to the restrictions faced by T in calculating $v^1_S(T)$. Since in the case of $v^1_S(T)$ this sum is distributed over the funds in the most pessimistic way for coalition T, we have the stated inequality. Since $v^{1,S}$ is balanced, we have that the subgame v^1_S is balanced and hence, v^1 is totally balanced. □

4 MFI solutions: a linear production approach

In this section, we present a procedure for solving MFI situations, ie, we propose a method of dividing $DR(N, e)$ among the players. This procedure consists of two stages. In the first stage, a division of the investment rights in the available funds (an allotment) is made. In the second stage, this allotment is used as an input vector of a related linear production process and the eventual allocation for the grand coalition is an Owen vector of this process.

Let (N, M, e, A, d) be an MFI situation. An *allotment* is an investment plan $Y \in FP(N, e)$ satisfying

$$\sum_{i \in N} Y_{ij} = e_j$$

for all $j \in M$.

An element Y_{ij} represents the amount that player i is allowed to invest in fund j. When the players individually invest in the funds according to the investment rights they receive from an allotment Y, a payoff vector $O(Y) \in \mathbb{R}^N_+$ results.

Example 4.1 One way to construct an allotment is simply to divide the investment rights of each fund proportional to the investment capital of the players, ie,

$$Y_{ij} = \frac{d_i}{\sum_{k \in N} d_k} e_j$$

for all $i \in N, j \in M$. For the MFI situation in Example 2.1, this yields

$$Y = \frac{3}{8} \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \end{bmatrix}.$$

Note that the corresponding direct payoff $O(Y) = \frac{1}{8}(57, 60, 126)$ is not efficient with respect to $DR(N, e) = 42$. ◁

The payoff $O_i(Y)$ to player $i \in N$ according to Y can be viewed as the direct revenue of coalition $\{i\}$ with fund restrictions $(Y_{ij})_{j \in M}$, ie,

$$O_i(Y) = DR(\{i\}, (Y_{ij})_{j \in M})$$

for all $i \in N$.

The players may decide to merge their investment rights and thereafter maximise their joint revenues. Suppose a coalition $S \subset N, S \neq \emptyset$ of players decides to work together. Define

$$Y^S = (\sum_{i \in S} Y_{ij})_{j \in M}.$$

The joint revenues that coalition S can obtain when working together is then given by $DR(S, Y^S)$. So, after an allotment Y is made, a new situation arises, which can be modelled as a TU game. This game, denoted by v_Y, is defined by

$$v_Y(S) = DR(S, Y^S)$$

for all $S \subset N, S \neq \emptyset$.

This process of joining the investment rights according to an allotment turns out to be a linear production process. A *linear production situation* (cf. Owen (1975) and Van Gellekom et al. (2000)) is a tuple (N, R, P, Q, B, c), where N is a finite set of players, R is a finite set of resources, P is a finite set of products, $Q \in \mathbb{R}^{R \times P}$ is a technology matrix where Q_{rp} represents the number of units of resource $r \in R$ necessary to produce one unit of product $p \in P$, $B \in \mathbb{R}^{R \times N}$ is a resource matrix where B_{ri} represents the amount player $i \in N$ possesses of resource $r \in R$ and $c \in \mathbb{R}^P$ is a market price vector of the products.

The maximal profit that can be made from a resource bundle $b \in \mathbb{R}^R$ equals the maximum of the linear program

$$\max_{x \in \mathbb{R}^P_+} \{c^\top x \mid Qx \leq b\},$$

where the coordinate x_p denotes the amount of product p that is produced.

A linear production situation $L = (N, R, P, Q, B, c)$ gives rise to a corresponding *linear production game* v_L, defined by

$$v_L(S) = \max_{x \in F(S)} c^\top x$$

for every $S \subset N$, where $F(S) = \{x \in \mathbb{R}^P_+ \mid Qx \leq (\sum_{i \in S} B_{ri})_{r \in R}\}$.

From duality theory we know that

$$v_L(S) = \min_{y \in F^*} \sum_{r \in R} \sum_{i \in S} y_r B_{ri}$$

with $F^* = \{y \in \mathbb{R}^R_+ \mid y^\top Q \geq c^\top\}$, since it is readily checked that the feasible regions $F(S)$ and F^* are both nonempty.

Let (N, M, e, A, d) be an MFI situation and let Y be an allotment. Each player i is allowed to invest Y_{ij} units of his capital d_i in fund j, resulting in revenues of A_{ij} per invested unit. This is equivalent with saying that each player i can produce one unit of a product p_{ij} by using one unit of his “capital resource” (of which he possesses d_i) and one unit of his “investment right in fund j resource” (of which he possesses Y_{ij}), with a market price for one unit of p_{ij} equal to A_{ij}. So the situation that arises after making the allotment Y can be characterised as a linear production process (N, R, P, Q, B, c) in the following way:

- N: the set of agents coincides with the player set of the underlying MFI situation.
- R: we define $|M| + |N|$ different resources consisting of $|M|$ "fund" resources $\{r_1^f, \ldots, r_m^f\}$ and $|N|$ "capital" resources $\{r_1^c, \ldots, r_n^c\}$.
- P: each player makes $|M|$ different products corresponding to the resources, so we define $|N||M|$ products $(p_{ij})_{i\in N, j\in M}$.
- Q: the technology matrix is constructed in the following way. Every unit of product is made by using one unit of a "fund" resource as well as one unit of a "capital" resource. The first $|M|$ columns of A represent the products made by the first player, the following $|M|$ columns represent the products of the second player, and so on. The first $|M|$ rows of Q represent the "fund" resources, the other rows represent the "capital" resources. For $n = 3, m = 2$, the technology matrix Q looks as follows:

$$\begin{array}{c} \\ r_1^f \\ r_2^f \\ r_1^c \\ r_2^c \\ r_3^c \end{array} \begin{array}{c} \begin{array}{cccccc} p_{11} & p_{12} & p_{21} & p_{22} & p_{31} & p_{32} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

- B: the "capital" resource bundles are provided directly by the MFI situation, whereas the fund resources are given by Y. For $n = 3, m = 2$, the resource matrix looks as follows:

$$\begin{array}{c} \\ r_1^f \\ r_2^f \\ r_1^c \\ r_2^c \\ r_3^c \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} Y_{11} & Y_{21} & Y_{31} \\ Y_{12} & Y_{22} & Y_{32} \\ d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \end{array}$$

- c: a product that is made of fund resource r_j^f and capital resource r_i^c has price A_{ij}. So, given the structure of Q, c looks as follows:

$$c = [A_{11}, \ldots, A_{1m}, A_{21}, \ldots, A_{2m}, \ldots, A_{n1}, \ldots, A_{nm}].$$

The *Owen set* of a linear production situation $L = (N, R, P, Q, B, c)$ is defined by

$$Owen(L) = \{y^\top B \in \mathbb{R}^N \mid y \in F^*(N), v_L(N) = \sum_{r\in R}\sum_{i\in N} y_r B_{ri}\},$$

where the vector y, being an optimal solution of the dual program, reflects the shadow prices of the resources. An element of the Owen set is called an *Owen vector*. Every Owen vector is an element of the core of the corresponding linear production game:

$$Owen(L) \subset C(v_L).$$

In particular, this implies that every linear production game is balanced. Also, since every subgame corresponds in a natural way to a linear production situation which is a "subsituation" of the original one, every linear production game is totally balanced.

So, when an allotment Y is made, the situation that arises can be viewed as a linear production process. We will refer to this process as $L(Y)$. It is easily verified that the corresponding linear production game $v_{L(Y)}$ coincides with v_Y.

Theorem 4.1 *Let (N, M, e, A, d) be an MFI situation and let $Y \in FP(N, e)$ be an allotment. Then $v_{L(Y)} = v_Y$.*

As a consequence, v_Y is totally balanced for every allotment Y.

Given an allotment Y, we propose $Owen(L(Y))$ as solution for the MFI situation, where every Owen vector is an efficient division of $DR(N, e)$ $(= v_{L(Y)}(N))$. Irrespective of the allotment that is chosen, the resulting allocation lies in the core of the most pessimistic MFI game v^1, as is shown in the following theorem.

Theorem 4.2 *Let (N, M, e, A, d) be an MFI situation and let $Y \in FP(N, e)$ be an allotment. Then $Owen(L(Y)) \subset C(v^1)$.*

Proof: Let $S \subset N, S \neq \emptyset$. Then

$$v_{L(Y)}(S) = v_Y(S) = DR(S, Y^S)$$

and

$$v^1(S) = DR(S, e(X^{N\setminus S}))$$

for some $X^{N\setminus S}$ such that the resulting direct revenue for coalition S is minimal. So, by construction, we have

$$\sum_{j\in M} e_j(X^{N\setminus S}) = \max\{0, \sum_{j\in M} e_j - \sum_{i\in N\setminus S} d_i\}.$$

Also, we have that

$$\sum_{j\in M} Y_j^S \geq \max\{0, \sum_{j\in M} e_j - \sum_{i\in N\setminus S} d_i\},$$

and so,

$$\sum_{j\in M} Y_j^S \geq \sum_{j\in M} e_j(X^{N\setminus S}).$$

Moreover, we have that given the total of fund restrictions $\sum_{j\in M} e_j(X^{N\setminus S})$, the division of this sum over the funds is such that $DR(S, e(X^{N\setminus S}))$ is as low as possible. Hence,

$$v_{L(Y)}(S) \geq v^1(S).$$

Now, since any Owen vector of $L(Y)$ is in the core of the corresponding linear production game $L(Y)$ and, trivially, $v_{L(Y)}(N) = v^1(N)$, the statement follows. $\square$

Example 4.2 Consider the MFI situation (N, M, e, A, d) of Example 2.1. Solving the (dual) linear production program for the grand coalition yields a solution set with two extreme points: $(1, 7, 9, 0, 3)$ and $(1, 4, 9, 0, 6)$, where the first two coordinates correspond to the "fund" resources and the other three to the "capital" resource. Using the resource matrix corresponding to the proportional allotment of Example 4.1,

$$B = \begin{bmatrix} \frac{3}{8} & \frac{12}{8} & \frac{9}{8} \\ \frac{3}{8} & \frac{12}{8} & \frac{9}{8} \\ 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

this yields

$$Owen(L(Y)) = Conv\{(12,12,18), (\frac{87}{8}, \frac{60}{8}, \frac{189}{8})\}.$$

Note that $Owen(L(Y)) \subset C(v^1)$. However, the Owen solution $(12, 12, 18)$ is not an element of $C(v^2)$, since $12 + 18 = 30 < 32 = v^2(\{2, 3\})$. ◁

Suppose that for an allotment Y the corresponding direct division of revenues is already efficient with respect to $DR(N, e)$, ie,

$$\sum_{i \in N} O_i(Y) = DR(N, e),$$

or equivalently,

$$Y \in OP(N, e).$$

Then the corresponding direct division of the revenues $O(Y)$ coincides with the allocation that is proposed by the Owen set of the corresponding linear production game. So according to this process there is no need to redistribute the allocation of revenues as given by $O(Y)$. This is the result of the following theorem.

Theorem 4.3 *Let (N, M, e, A, d) be an MFI situation. Let $Y \in OP(N, e)$. Then*

$$Owen(L(Y)) = \{O(Y)\}.$$

Proof: Consider the linear production process $L(Y)$ and let $S \subset N, S \neq \emptyset$. For the corresponding linear production game we have

$$v_{L(Y)}(S) = \sum_{i \in S} O_i(Y),$$

because $Y \in OP(N, e)$. Let $y \in Owen(L(Y))$. Since the Owen vector is in the core of the linear production game, we have

$$\sum_{i \in S} y_i \geq v_{L(Y)}(S) = \sum_{i \in S} O_i(Y).$$

Efficiency of both y and $O(Y)$ with respect to $DR(N, e)$ implies

$$\sum_{i \in S} y_i = \sum_{i \in S} O_i(Y)$$

for all $S \subset N$ and hence,

$$Owen(L(Y)) = \{O(Y)\}.$$

□

For an optimal allotment $Y \in OP(N, e)$, the resulting allocation $O(Y)$ belongs to the core of the most optimistic game v^3, as is shown in the next theorem.

Theorem 4.4 *Let (N, M, e, A, d) be an MFI situation and let $Y \in OP(N, e)$. Then $O(Y) \in C(v^3)$.*

Proof: Let $S \subset N, S \neq \emptyset$. Then

$$\begin{aligned} v^3(S) &= DR(N, e) - DR(N \backslash S, e) \\ &\leq DR(N, e) - \sum_{i \in N \backslash S} O_i(Y) \\ &= \sum_{i \in S} O_i(Y). \end{aligned}$$

Together with efficiency, we obtain $O(Y) \in C(v^3)$. □

Because of Proposition 2.1, $O(Y)$ also belongs to $C(v^1)$ and $C(v^2)$.

To compare the various solutions, consider again the MFI situation of Example 2.1. As we saw in Example 4.1, the direct payoff corresponding to the proportional allotment is not efficient and hence, not an element of any of the three cores. After constructing the corresponding linear production game and applying the Owen procedure we obtained a solution set which is part of the core of v^1, although not not of v^2 (and hence, v^3). According to Theorem 4.4, the direct division corresponding to the optimal plan X^N in Example 2.1 should be in all three cores, which is indeed the case.

5 Extensions

One of the assumptions in our MFI model is that the total capital available is larger than the sum of the fund restrictions, ie, $\sum_{j \in M} e_j < \sum_{i \in N} d_i$. Note that this assumption is common in the bankruptcy literature (cf. O'Neill (1982)), where the total amount of the "claims" (capital) exceeds the available "estate" (investment opportunities). If we do not impose this assumption, we can still compute the three corresponding games in the same way and the results of section 2 still hold.

The problem with dropping this assumption, however, lies in the concept of allotment. An allotment is a feasible plan which is efficient with respect to the fund restrictions. If the sum of the fund restrictions is larger than the total capital, such a feasible plan does not exist. If we drop the requirement of feasibility of an allotment and allow a player to have more investment rights than his total capital, Theorem 4.2 no longer holds. However, an allotment Y for which the direct division of revenues $O(Y)$ is efficient with $DR(N, e)$ is always in the core of all three games, regardless whether we require an allotment to be feasible or not.

Another (implicit) assumption in our MFI model is that when a coalition S of players decides to cooperate, they can coordinate their investment actions, but they cannot pool their capital. If we allow capital to be transferable, the direct revenues of S would be given by

$$\begin{aligned}
\widetilde{DR}(S, z) \quad &= \quad \max_{X \in \mathbb{R}^{S \times M}} \sum_{j \in M} \sum_{i \in S} A_{ij} X_{ij} \\
\text{such that} \quad & \quad \sum_{j \in M} \sum_{i \in S} X_{ij} \leq \sum_{i \in S} d_i, \\
& \quad \sum_{i \in S} X_{ij} \leq z_j \text{ for all } j \in M, \\
& \quad X_{ij} \geq 0 \text{ for all } i \in S, j \in M.
\end{aligned}$$

This maximisation problem, however, is quite trivial to solve. For each fund $j \in M$, the players in S determine $\tilde{A}_{Sj} = \max_{i \in S} A_{ij}$ and invest their capital in those funds with the highest $\tilde{A}_{Sj}$, taking the fund restrictions into account.

For the transferable capital case, we can define the same three corresponding games as for the nontransferable capital case. Again, we have that $v^1 \leq v^2 \leq v^3$ and that v^1 is totally balanced. However, the game v^3 need not be convex (or even balanced).

Allotments can be defined in the same way as for nontransferable MFI situations, but the constructions of the corresponding linear production game is different and involves the introduction of an additional resource representing "total capital". With this adjusted linear production situation, the analysis of Section 3 can be fully translated to the transferable capital setting.

A more detailed discussion of these and other extensions can be found in Wintein (2002).

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