Equilibrium Asset Pricing with Time-Varying Pessimism

Sbuelz, A.; Trojani, F.

Publication date:
2002

Citation for published version (APA):
Sbuelz, A., & Trojani, F. (2002). *Equilibrium Asset Pricing with Time-Varying Pessimism*. (CentER Discussion Paper; Vol. 2002-102). Finance.

No. 2002-102

EQUILIBRIUM ASSET PRICING WITH TIME-VARYING PESSIMISM

By Alessandro Sbuelz, Fabio Trojani

November 2002

ISSN 0924-7815

Discussion paper

Equilibrium asset pricing with time-varying pessimism*

Alessandro Sbuelz

Department of Finance, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, NL,

e-mail: A.Sbuelz@kub.nl

Fabio Trojani

Institute of Finance, University of Southern Switzerland, Via Buffi 13, CH-6900 Lugano, CH,

e-mail: Fabio.Trojani@lu.unisi.ch

First Version: April 2001. This Version: October 2002.

Abstract

We study the equilibrium pricing effects of a sentiment for pessimism. Pessimism has the form of Knightian model uncertainty aversion for a neighborhood of indistinguishable model specifications that are constrained in their relative entropy from a given reference model. We

*For many valuable comments and suggestions on an earlier version of this draft we wish to thank Nicholas Barberis, Giovanni Barone Adesi, Suleyman Basak, Steffan Berridge, Simona Kane-Polli, Gabrielle Demange, Darrel Duffie, Larry Epstein, David Feldman, Paolo Ghirardato, Simon Grant, Michael Haliassos, Thorsten Hens, Wilfred Chon Lei, Elisa Luciano, Pascal Maenhout, Massimo Marinacci, Luca Rigotti, Marcel Rindisbacher, Hans Schumacher, Paolo Sodini, Walter Sorana, Pietro Veronesi, Raman Uppal, conference participants at the 2001/2002 CEPR/Studienzentrum Gerzensee European Summer Symposia in Financial Markets and the 2002 EFA Annual Meeting, and seminar participants at the finance seminars of the Copenhagen Business School, Erasmus University, London Business School, Norvegian Business School, Technical University of Vienna, Tilburg University, University of Brescia, University of Cyprus, University of Southern Switzerland, University of Venezia, University of Verona, and University of Zurich. All remaining errors are ours. Alessandro Sbuelz gratefully acknowledges the financial support of the Marie Curie Fellowship HPMF-CT-2000-00703. Fabio Trojani gratefully acknowledges the financial support of the Swiss National Science Foundation (grant 12-65196.01 and CNNR FINRISK).

fully characterise the equilibrium of a pessimistic, representative agent, exchange economy with intertemporal consumption, stochastic opportunity set, and a relative entropy constraint that can depend on the state of the economy. We find that Knightian pessimism generates substantial First Order Risk Aversion (FORA) effects that enhance excess equity returns by pushing riskfree rates down. However, we find that the structure of equity returns is virtually unaffected by a Knightian concern for model uncertainty. We compute and calibrate explicit equilibrium examples of a pessimistic economy whith an amount of pessimism associated to an 11% upper probability bound of confusing the relevant worst-case model and the given reference model. Relative entropy is the key in fixing such a realistic amount of pessimism in our calibrations. Even for log utility, such small amount of pessimism generates some 55 basis points more of unconditional equity premium. Knightian pessimism provides an economically and observationally different description of excess equity returns. Our findings show that realistic amounts of both pessimism and standard risk aversion yield substantial equity premia and low riskfree rates.

JEL Classification: *G11, G12*

Keywords: Asset Pricing, General Equilibrium, Model Misspecification, Knightian Uncertainty, First Order Risk Aversion.

EQUILIBRIUM ASSET PRICING WITH TIME-VARYING PESSIMISM

Abstract

We study the equilibrium pricing effects of a sentiment for pessimism. Pessimism has the form of Knightian model uncertainty aversion for a neighborhood of indistinguishable model specifications that are constrained in their relative entropy from a given reference model. We fully characterise the equilibrium of a pessimistic, representative agent, exchange economy with intertemporal consumption, stochastic opportunity set, and a relative entropy constraint that can depend on the state of the economy. We find that Knightian pessimism generates substantial First Order Risk Aversion (FORA) effects that enhance excess equity returns by pushing riskfree rates down. However, we find that the structure of equity returns is virtually unaffected by a Knightian concern for model uncertainty. We compute and calibrate explicit equilibrium examples of a pessimistic economy whith an amount of pessimism associated to an 11% upper probability bound of confusing the relevant worst-case model and the given reference model. Relative entropy is the key in fixing such a realistic amount of pessimism in our calibrations. Even for log utility, such small amount of pessimism generates some 55 basis points more of unconditional equity premium. Knightian pessimism provides an economically and observationally different description of excess equity returns. Our findings show that realistic amounts of both pessimism and standard risk aversion yield substantial equity premia and low riskfree rates.

JEL Classification: *G11, G12*

Keywords: Asset Pricing, General Equilibrium, Model Misspecification, Knightian Uncertainty, First Order Risk Aversion.

Contents

1 Introduction

We study the equilibrium asset pricing impact of a time-varying pessimistic concern for model uncertainty in a continuous-time, representative agent, exchange economy with intermediate consumption and stochastic opportunity set. Pessimism is modelled as Knightian max-min expected utility behavior, that is, maximizing expected utility under the worst-case scenario for equity returns, where the worst-case scenario is picked out of a well specified set of probabilistic models. The set is defined by a neighborhood of scenarios around some reference model and arises because of agent's time-varying distrust of the reference model. Conservative consumption/investment policies are implied, especially in states where such distrust is high.

Firstly, we completely characterize the equilibria of Knightian pessimism economies and, in particular, characterize the general functional forms of asset returns (equity returns and riskfree rates) depending on the state variable dynamics. We exploit that characterization to discuss qualitatively the equilibrium pricing impact of pessimism. Secondly, we quantify such impact by calibrating to real data some specific equilibrium examples for which we obtain closed-form solutions.

These are the findings. Knightian pessimism strongly reduces the riskfree rate but only indirectly affects - via the interplay of risk aversion and model uncertainty aversion - equity returns and worst-case equity premia. In fact, log utility equity returns and worst-case equity premia remain completely unaffected by pessimism. These general claims are substantiated by the closed-form asset returns we work out for the specific equilibrium examples. We calibrate such examples to US aggregate consumption data assuming risk aversion close to log utility and tiny amounts of pessimism. Specifically, pessimism is fixed to be compatible with an 11% ex-post upper bound probability of failing to distinguish between the worst-case and the reference model. In these calibrations, conditional equity premia reach peaks of 1.50%, whereas conditional worst-case premia are akin to non-pessimistic premia and never overshoot the 0.25% level. The unconditional equity premium and the worst-case equity premium are about 0.70% and 0.12%, respectively. We conclude that moderate amounts of pessimism yield substantial equity premia, even for the log utility case.

In our setting, pessimism is a continuous-time, dynamically consistent, formulation of the uncertainty aversion proposed by Knight (1921). Its peculiar asset pricing impact stands out because Knightian preferences produce First Order Risk Aversion - FORA - effects. The equity premium remains high even when equity risk is low since FORA yields pessimistic equity premia that are proportional to equity return volatility; see also Dow and Werlang (1992) for a general discussion on FORA effects. FORA is economically and observationally specific to a Knightian pessimism setting. Indeed, preferences without Knightian aversion to model misspecification produce Second Order Risk Aversion - SORA - effects, that is, equity premia are proportional to equity return variance.

Pure parametric model uncertainty is pure estimation risk and does not survive in the long run after a Bayesian learning process. In this work, we focus on genuine, that is non-parametric, model uncertainty. We generate Knightian model uncertainty by setting a possibly time varying costraint on the maximum relative entropy between the reference probabilistic model and any candidate alternative specification. Relative entropy is a log-likelihood ratio, a generalized measure of the discrepancy between two absolutely continuous probability laws, so that two probabilistic models with very different (parametric or non-parametric) structure may have the same relative entropy with respect to the reference model. Constrained relative entropy is the theoretically sound tool in measuring non-parametric model uncertainty, for it defines a non-parametric neighborhood that encompasses a continuum of indistinguishable probabilistic models. A key feature of relative entropy is that, via statistical model detection theory, it provides an objective measure of what is a realistic amount of model uncertainty for the calibration of a pessimistic economy to the available data. Anderson, Hansen, and Sargent (2000) show how relative entropy between two models relates to the probability of failing to distinguish one from the other given the data information available for the model calibration. Marinacci (1999, 2002) and Epstein and Schneider (2002) show by means of asymptotic theory and concrete examples that indistinguishability preserves a concern for model uncertainty in the long run.

Ellsberg's (1961) paradox documents that misspecification risk is an actual concern, for investors seem to dislike gambling with unknown probability law. Knightian aversion to misspecification risk generates portfolio choices that are state-dependent in a non-standard way. FORA makes the agent index her equity portfolio rebalances to her existing exposure to equity risk. Small existing exposure signals tiny equity premia and, via model uncertainty about equity premia, triggers even smaller desired exposure than the standard one. Small exposure enhances agent's effective risk aversion. In addition to this key FORA element we flank two further elements of a pessimistic state-dependent portfolio choice. The first is the assumption of a stochastic opportunity set, which generates pessimistic hedging motives. The second is the assumption that agent's distrust in her reference model depends on the economic fundamentals that make the opportunity set stochastic. This link between the degree of distrust / uncertainty and the state of the economy is supported by empirical evidence. Veronesi (1999) shows that financial economists tend to be more uncertain about the future growth of the economy during recessions. The predictability literature based on business-cycle indicators documents that equity premia and equity return volatility are substantially higher during business cycle troughs (see Cochrane (2000) for a survey). Our flexible setting of time varying pessimism can generate countercyclical effective risk aversion behavior by positing that the agent fears equity precisely when fundamentals are not good.

Countercyclical risk aversion arises also from settings without pessimism. Recently, Campbell and Cochrane (1999) and Chan and Kogan (2001) have proposed non-pessimistic settings with countercyclical state-dependent risk aversion to explain many of the observed features of asset prices. However, there equity premia exhibit SORA effects, which need very high levels of relative risk aversion to match observed asset returns. Those high levels of relative risk aversion require high levels of the local curvature of the instantaneous utility function. This makes our framework economically and observationally different. Such difference also persists with respect to existing settings of non-Knightian pessimism. Anderson, Hansen, and Sargent (2000), Maenhout (1999), and Liu, Pan, and Wang (2002) introduce a form of pessimism that yields optimal policies observationally equivalent to those implied by Epstein and Zin (1989) preferences. This freezes the willingness of substituting consumption through time but generates SORA pricing effects. Similarly, the discrete-time pessimism and doubt of Abel (2002) and the multi-asset setting with model uncertainty in Uppal and Wang (2000) generate SORA effects.

Our setting is consistent with the general setting of Chen and Epstein (2002) who consider an intertemporal extension of Gilboa and Schmeidler's (1989) multiple-priors atemporal ambiguity model. We depart from Chen and Epstein (2002) by embracing relative entropy to define the bounded set of indistinguishable priors. Via statistical model detection theory, this crucially helps in fixing a reasonable level for the pessimism parameter when we calibrate concrete examples of pessimistic economy. Indistinguishability of multiple priors and FORA aversion to them makes our setting different also from intertemporal Bayesian asset pricing like Veronesi (1999), Veronesi (2000), and Veldkamp (2001). There, agents learn about the parametric structure of the economy using a standard Bayesian updating rule and only SORA effects arise in equilibrium.

Our calibrations show that realistic amounts of pessimism give a substantial lift to the equity premium via FORA already at low levels of standard risk aversion. This happens without upward pressure on the riskfree rate, since the elasticity of intertemporal substitution is left immaculate. Thus, our combination of Knightian preference axioms and statistical model detection tools tackles the equity premium and riskfree rate puzzles without an ad-hoc use of the preference parameters. Closed forms for equilibrium asset prices make the FORA effects visible and ready for real data calibration without much numerical intermediation. Closed forms can be obtained because the homogeneity in preferences and beliefs of our representative agent economy simplifies the analytical study of the equilibrium quantities. Pessimistic equilibrium quantities can be characterized in terms of the solution, g, say, of an equilibrium differential equation. The function g captures the effect that the opportunity set state variables have on the equilibrium value function and is essential in determining equilibrium asset returns. In the pessimistic log utility equilibria of our examples we fully characterise g in closed form. For general power utility, we work out a set of asymptotics for the equilibrium function g that give more insight into the equilibrium economics for the whole preference parameter space. This goes beyond the equilibrium description provided by exact numerical solutions, which are confined to specific preference parameter levels. More particularly, we employ first order expansions of g in the risk aversion parameter around the solution for the pessimistic log utility equilibrium. This first order expansion buys higher order asymptotics for the power utility equilibrium asset returns. We compare our asymptotic solutions with the exact numerical solutions for several levels of CRRA. This ascertains that the stochastics left out in the expansion is trivial for a broad range of CRRAs. For instance, levels around 2 (log utility exhibits unit CRRA) still give very good analytical approximations. Moreover, even though g is the solution to a highly non-linear problem under Knightian uncertainty asymptotics, our asymptotics show that it admits a remarkably simple polynomial approximation.

The paper is organized as follows. Section 2 defines the elements of the exchange economy with Knightian pessimism. Section 3 characterises the general features of equilibrium asset returns. Section 4 provides equilibrium closed-form solutions for the log utility case and equilibrium asymptotics for the general power utility case. Section 5 calibrates two concrete equilibrium examples to US consumption data and compares the quality of our asymptotics for power utility to the corresponding exact numerical solution. Section 6 concludes.

2 The economy

In this section we introduce the reference probabilistic model for the economy dynamics, we define via constrained relative entropy the set of alternative models indistinguishable from the reference one, and we lay the problem of max-min expected utility optimization that generates pessimistic optimal policies.

A. Assets, cash flow, and state variables

There are two long-lived assets, a discount bond - the riskfree asset with instantaneous rate r - and equity - the risky asset with ex-dividend price P -, which is a claim on a dividend process e. Riskfree rate, expected dividend growth and dividend growth volatility constitute agent's opportunity set. The opportunity set is governed by the opportunity set process X. Pessimism is described by a state variable correlated to the process X. This variable could be either a meta-process (the agent has no ambiguity about it) or the opportunity set process itself. We choose the second option by identifying it with X. The first option leaves derivation intact.

B. Beliefs

Agent's beliefs include the reference probabilistic model for the economy dynamics and the alternative models indistinguishable from the reference one.

1. The reference model

The reference model on the dynamics of the opportunity set process, dividend growth, and cumulative returns to equity, respectively, is

$$\begin{aligned} dX &= \zeta(X)\,dt + \xi(X)\,dZ^X, \\ \frac{de}{e} &= \alpha_e(X)\,dt + \sigma_e(X)\left[\rho_e(X)\,dZ^X + \sqrt{1-\rho_e^2(X)}dZ^e\right], \\ \frac{dP+edt}{P} &= \alpha_P dt + \sigma_P\left[\rho_P dZ^X + \sqrt{1-\rho_P^2}dZ^e\right], \end{aligned}$$

where $Z = \left(Z^X, Z^e\right)'$ is a bivariate standard Brownian Motion with mutually independent components. Cumulative returns on equity have conditional expectation α_P, conditional volatility

σ_P, and conditional correlation coefficient ρ_P to be determined in equilibrium. The instantaneous riskfree rate r is determined in equilibrium as well. The couple (α_P, σ_P) represents the risk / return profile offered by equity to the agent. The opportunity set process X represents the exogenous primitives of the economy. This is why we write innovations as driven by dZ^X, the standardized news on the opportunity set changes dX, and by idiosyncratic news. The agent places a fraction w of her wealth W to equity and a fraction c to current consumption flow. Thus, the dynamics of the vector $Y = (X, W)'$ is

$$dY = \mu dt + \Lambda dZ,$$

where

$$\mu = \left(\begin{array}{c} \zeta \\ wW(\alpha_P - r) + W(r - c) \end{array} \right), \quad \Lambda = \left[\begin{array}{cc} \xi & 0 \\ \rho_P w W \sigma_P & \sqrt{1-\rho_P^2} w W \sigma_P \end{array} \right].$$

2. The alternative models

The alternative models are scenarios generated by local contaminations of the reference model. Let ν be the Radon-Nikodym derivative of a contaminated probability law with respect to the reference probability law. η is ν's best forecast at time t so that

$$\frac{d\eta}{\eta} = h'dZ, \quad h = \left(h^X, h^P\right)', \quad E(\eta) = 1.$$

E (E_t) is the unconditional (conditional) expectation operator under the reference probability law. η is the scaling factor that generates scenarios around the reference model. Once Y is scaled by η, its annualized reference drift is added with a vector of possibly time-varying contaminations with unspecified structure:

$$\frac{1}{dt} E_t \left(\frac{1}{\eta} d(\eta Y) \right) = \frac{1}{dt} E_t^h (dY) = \mu + \Lambda h.$$

E^h (E_t^h) is the unconditional (conditional) expectation operator under the ν-scenario probability law. h is premultiplied by the volatilities matrix Λ because it comes about as a Girsanov-Cameron-Martin change of drift. This is a technical detail of our continuous-time setting which by no means determines FORA. FORA derives from the max-min expected utility optimization with constrained relative entropy.

C. Knightian model uncertainty

Knightian model uncertainty is described by a maximal bound $\varphi f^2(X)$ on the size of the contaminating vector h,

$$\frac{1}{2}h'h \leq \varphi f^2(X) \quad , \tag{1}$$

where φ is a non-negative constant and f a function of the current state X of the economy. Such bound defines a neighborhood of transition probability densities that the agent is unable to distinguish from the reference one. Equivalently, $\varphi f^2(X)$ is a state-dependent maximal bound on the rate at which relative entropy of a relevant model misspecification is allowed to increase over time,

$$\frac{1}{dt}E_t\left(\frac{1}{\eta}d(\eta\ln\eta)\right) = \frac{1}{dt}E_t^h(d\ln\eta) = \frac{1}{2}h'h \leq \varphi f^2(X).$$

Anderson, Hansen, and Sargent (1998), Lei (2001), and Trojani and Vanini (2001a, 2002) allow for a time-invariant maximal bound on the growth rate of relative entropy over time.

For φ decreasing to 0, the agent tends to have full confidence in her reference model of asset returns. For φ greater than 0, the agent considers a continuum of local scenarios that cannot be statistically distinguished from the reference model. These scenarios form a neighborhood centered on the reference model. The maximal bound $\varphi f^2(X)$ defines a state-dependent maximal radius for the drift contamination h implied by any scenario in the neighborhood. The radius depends on the fundamentals X via the function f. The free structure of f can capture possible asymmetries of the impact of fundamentals on agent's faith in her reference model. Candidates for $f(X)$ can be, for instance, indicators of expected dividend growth, dividend growth volatility, or other forward-looking indicators of the state of the economy.

D. Knightian pessimism and the worst case

The agent has time preference rate δ and gets the following CRRA utility $u(\cdot)$

$$u(cW) = \frac{(cW)^\gamma - 1}{\gamma} \quad ; \quad \gamma < 1 \quad , \tag{2}$$

out of current consumption flow cW. As usual for $\gamma \to 0$ the log utility case arises. The coefficient of relative risk aversion is constant and equal to $1-\gamma$. Agent's value function is given by

$$J(W,X)=\begin{cases}\max_{c,w}\min_h E_0^h\left[\int_0^\infty e^{-\delta t}u(cW)\,dt\right]\\ \text{s.t.}\\ \frac{1}{2}h'h\le\varphi f^2(X)\quad,\quad dY=\mu dt+\Lambda dZ\end{cases}\quad. \tag{3}$$

Given a reference model on Y's transition density, the agent maximizes the worst-case expected utility. The worst-case scenario is associated with the worst-case drift contamination h^*. h^* belongs to the neighborhood of allowed contaminations and minimizes expected utility. The neighborhood is defined by the relative entropy constraint and is parametrised by φ. This is Knightian max-min expected utility behavior, namely, Knightian pessimism. φ is the pessimism parameter. When φ tends to zero, pessimism disappears as max-min expected utility behavior converges to standard max expected utility behavior. The Knightian feature of our max-min expected utility optimization comes from the constraint on the relative entropy of local drift contaminations. Such a constraint formulation only corresponds to a Lagrangean penalization of the difference between relative entropy growth and its upper bound, where the constraint remains visible via the first order conditions for optimization. In continuous-time intertemporal settings, constrained relative entropy has been recently used in partial equilibrium by Lei (2001) and in general equilibrium by Trojani and Vanini (2001a, 2002). In an atemporal setting, Kogan and Wang (2002) employ it for a parametric neighborhood of Gaussian models. Non-Knightian max-min behavior comes about when relative entropy is penalized rather than constrained, like in Anderson, Hansen, and Sargent (2000), Maenhout (1999), and Uppal and Wang (2002). No FORA pricing effects are obtained via a penalty formulation of relative entropy; see also Trojani and Vanini (2001a). With the constraint on relative entropy we also give specific form to the rectangularity condition in Chen and Epstein (2002). Rectangularity grants dynamic consistency.

The Hamilton-Jacobi-Bellman (HJB) equation for Problem (3) is

$$\begin{aligned}&0=\max_{c,w}\min_h\left(u-\delta J+\tfrac{1}{dt}E_t^h(dJ)\right)\\&\qquad\text{s.t.}\\&\tfrac{1}{2}h'h\le\varphi f^2(X)\quad,\quad dY=\mu dt+\Lambda dZ.\end{aligned} \tag{4}$$

Under scenario h, the annualized expected change in the value function is

$$\frac{1}{dt}E_t^h(dJ)=\frac{1}{dt}E_t(dJ)+h'\Lambda' J_Y\quad,$$

where J_Y is J's gradient with respect to Y. The worst-case scenario comes from the constrained minimization with respect to the contaminating vector h.

Proposition 1 *The worst-case contaminating vector is*

$$h^* = -\frac{\sqrt{2\varphi}\,|f(X)|}{\left(J_Y' \Lambda\Lambda' J_Y\right)^{\frac{1}{2}}} \Lambda' J_Y. \tag{5}$$

The constraint is binding so that h^* is a vector of norm equal to the maximal allowed discrepancy $\sqrt{2\varphi}\,|f(X)|$ between scenarios and the reference model. $E_t^{h*}(\cdot)$ denotes the worst-case conditional expectation operator. h^* downweights the reference model expected change in the value function J:

$$\frac{1}{dt} E_t^{h^*}(dJ) = \frac{1}{dt} E_t(dJ) + h^{*\prime} \Lambda' J_Y = \frac{1}{dt} E_t(dJ) - \sqrt{2\varphi J_Y' \Lambda\Lambda' J_Y}\,|f(X)| \quad .$$

This occurs precisely when either the volatility $\sqrt{J_Y' \Lambda\Lambda' J_Y}$ of dJ is large or when a higher model uncertainty (that is a higher maximal allowed discrepancy $\sqrt{2\varphi}\,|f(X)|$) causes lower confidence in the reference model.

E. Model detection and amount of pessimism

There is more to constrained relative entropy than producing FORA pricing effects via Knightian max-min expected utility behavior. The constraint formulation does help in picking a reasonable and sharp magnitude for the pessimism parameter φ. This is important because it pre-empts ad hoc uses of the preference parameters.

φ can be determined via constrained relative entropy in the context of statistical model discrimination. To understand this point, set the function f constant to unity. The pessimistic agent is faced with two relevant models of the economy: the reference model and the worst-case contaminated model. Constrained pessimism implies that the worst case lies on the boundary of the neighborhood of contaminated models, so that $\frac{h^{*\prime} h^*}{2}$ equals the maximal level φ. If the agent has a sufficiently large set of past observations, the worst-case model implied by a large φ is not very realistic. It can be statistically told apart from the reference model, for it is too far from the benchmark and thus easy to detect. With a data set of size N, Anderson, Hansen, and Sargent (2000) show that the upper bound on the probability of failing to distinguish between the reference model and the worst-case model is:

$$\text{probability bound } = \frac{1}{2} \exp\left(-N \frac{h^{*\prime} h^*}{8}\right).$$

This makes the probability bound a function of the pessimism parameter φ:

$$\text{probability bound } = \frac{1}{2}\exp\left(-N\frac{\varphi}{4}\right).$$

Imagine that the agent has access to quarterly century-long data. A prudent choice of the level of economy-wide pessimism originates from $N = 400$ to yield a probability bound of 11%. The corresponding level of φ is 0.015. To summarize,

$$0.11 = \frac{1}{2}\exp\left(-400 \cdot \frac{0.015}{4}\right).$$

In our calibration of Section 5 we make use of a 400-strong set of quarterly century-long observations to pin the pessimism parameter level down to 0.015. This is the interpretation: The agent starts generating the 20th century of observations with a small amount of pessimism corresponding to the steady-state model uncertainty. Steady-state model uncertainty is identified with the residual model indistinguishability associated to an ex-post look at the 20th century. This implies a rather conservative statistical calibration of the degree of pessimism in the economy.

3 Pessimistic exchange equilibrium

Homogeneity of the HJB problem (4) and power utility in (2) lead to the following educated guess on J's functional form,

$$J(X, W) = \frac{1}{\delta}\frac{\left(e^{g\left(\gamma,\sqrt{\varphi},X\right)}W\right)^{\gamma} - 1}{\gamma} \quad . \tag{6}$$

The function g expresses how the agent's welfare is affected by time variation of her reference model (the X-driven opportunity set) as well as by time variation of her confidence in such model (confidence depends on $\sqrt{2\varphi}\,|f(X)|$). Thus, g's derivatives with respect to X determine agent's intertemporal hedging policies. For convenience, we use the shortened notation

$$\frac{\partial}{\partial X}g\left(\gamma,\sqrt{\varphi},X\right) = g' \quad , \quad \frac{\partial^2}{\partial^2 X}g\left(\gamma,\sqrt{\varphi},X\right) = g'' \quad .$$

Closed forms for g are typically 'wishful thinking'. Also, market clearing transforms g, for it makes risk / return profile (α_P, σ_P) and correlation ρ_P endogenous. This adds complexity. We give full and general description of the equilibrium g in terms of the economy primitives. This permits exact numerical solution of g for any parametric specification of the reference model dynamics. However, analytical solutions deliver a broader analysis of pessimism economics in the whole relevant domain of the preference parameters γ and φ. Therefore, we use a perturbative way around analytical intractability and expand g in terms of the preference parameter γ in Section 4. This supports the analytical description of pessimistic equilibria. Closed-form solutions are obtained for the pessimistic log utility case ($\gamma \to 0$). Asymptotics based on the pessimistic log utility solution are obtained for the pessimistic power utility case ($\gamma \neq 0$).

We first qualify the pessimistic exchange economy in general equilibrium, where FORA effects are directly visible in the optimal demand for equity.

A. FORA effects

Supply of equity is standardized to 1 share, so that the equilibrium price $\widehat{P}$ of the risky asset coincides with the aggregate wealth of the economy. 'Hat' symbols indicate the equilibrium values of the relevant variables.

Definition 2 *A pessimistic exchange equilibrium is a vector process* $\left(\widehat{P}, \widehat{r}, \widehat{w}, \widehat{c}\right)'$ *such that:*

1. *Representative agent's portfolio and consumption rules,* $\widehat{w}$ *and* $\widehat{c}$*, are optimal, i.e. they solve (4).*
2. *Financial and good markets clear, i.e.* $\widehat{w} = 1$ *and* $\widehat{c}\widehat{P} = e$*.*

Given a solution g in the pessimistic value function (6), the optimal consumption and investment policies follow.

Proposition 3 *The optimal policies to the HJB Problem (4) are:*

$$c^* = \left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}, \tag{7}$$

$$w^* = \frac{1}{1-\left(\gamma - \sqrt{\frac{2\varphi}{G(w^*)}}\,|f|\right)} \cdot \left(\frac{\alpha_P - r}{\sigma_P^2} + \left(\gamma - \sqrt{\frac{2\varphi}{G(w^*)}}\,|f|\right) g' \frac{\rho_P \xi}{\widehat{\sigma}_P}\right), \tag{8}$$

where

$$G(w) = \sigma_P^2 w^2 + \xi^2 (g')^2 + 2w\rho_P\sigma_P\xi g' \quad . \tag{9}$$

Examination of the optimal policies uncovers the acts of Knightian uncertainty aversion. Firstly, pessimism has only an indirect impact on optimum consumption via g, whereas it has direct impact on both the myopic and the hedging demands for equity: φ enters c^* only via

g, whereas it is directly present in w^*. Secondly, impact on optimum equity demand is state-dependent in a non-standard way. The resulting demand is akin to a Merton's (1971) demand originated by the state-dependent effective relative risk aversion

$$1-\left(\gamma-\sqrt{\frac{2\varphi}{G\left(w^*\left(X\right)\right)}}\left|f\left(X\right)\right|\right).$$

Such effective risk aversion penalizes states where model misspecification can strongly reduce portfolio performance. This yields either procyclical or countercyclical portfolio behaviors, depending on the assumptions on X and $f(X)$. When equity risk barely pays off (the Sharpe ratio $(\alpha_P - r)/\sigma_P$ goes to zero) and there is scarce need for hedging ($\xi g'$ goes to zero) or equity is an inadequate hedging tool (ρ_P goes to zero), Knightian pessimism generates the FORA effects. In such circumstances, desired equity holdings are small even in the absence of pessimism. Standard equity risk exposure is small. Pessimism reinforces the convergence of equity risk exposure to zero by strongly propelling effective risk aversion via the exposure-dependent $G(w)$. Effective risk aversion corrections depend on the state X also through $|f(X)|$. The largest portfolio corrections occur when $|f(X)|$ tends to infinity, that is, when the state of the economy destroys the agent's confidence. Total distrust in the reference model comes about so that myopic demand for equity is squeezed down to zero. Expected returns on equity become totally uncertain and they kill any speculative incentive. Hedging demand for equity converges to $-g'\frac{\rho_P\xi}{\sigma_P}$. The instantaneous covariance between the state of the economy and tradables is not affected by model misspecification so that hedging demand for equity remains alive even with an unbounded $|f(X)|$.

B. Equilibrium cumulative returns on equity

In this section we describe the equilibrium structure of cumulative returns on equity based on the equilibrium function $\widehat{g}$. The next proposition shows that pessimism indirectly affects equilibrium equity returns.

Proposition 4 *The pessimistic exchange equilibrium implies conditional expectations of cumu-*

lative returns on equity, conditional variances and correlations given by

$$\widehat{\alpha}_P = \alpha_e + \frac{\gamma}{1-\gamma}\left(\widehat{g}'\left(\zeta + \xi\rho_e\sigma_e\right) + \frac{\xi^2}{2}\left(\frac{\gamma}{1-\gamma}\left(\widehat{g}'\right)^2 + \frac{\partial^2 \widehat{g}}{\partial^2 X}\right)\right) + \left(\frac{e^{\gamma\widehat{g}}}{\delta}\right)^{\frac{1}{\gamma-1}} \quad , \tag{10}$$

$$\widehat{\sigma}_P^2 = \sigma_e^2 + \frac{\gamma}{1-\gamma}2\rho_e\sigma_e\xi\widehat{g}' + \left(\frac{\gamma}{1-\gamma}\xi\right)^2\left(\widehat{g}'\right)^2 \quad , \tag{11}$$

$$\widehat{\rho}_P = \frac{\sigma_e\rho_e + \frac{\gamma}{1-\gamma}\widehat{g}'\xi}{\sqrt{\sigma_e^2 + \frac{\gamma}{1-\gamma}2\rho_e\sigma_e\xi\widehat{g}' + \left(\frac{\gamma}{1-\gamma}\xi\right)^2\left(\widehat{g}'\right)^2}} \quad , \tag{12}$$

respectively.

Pessimism has no direct impact on the equilibrium equity price dynamics. Indeed, the φ-dependent $\widehat{g}$ enters equity returns only via premultiplication with γ. Therefore, for power utility with $\gamma \neq 0$, there is no direct presence of the pessimism parameter φ: Pessimism can affect equilibrium equity expected returns, volatilites and correlations only through the joint interplay of risk aversion and model uncertainty aversion. Log utility provides the chief example. Log utility arises when $\gamma \to 0$ so that for any $\varphi \geq 0$

$$\widehat{\alpha}_P = \alpha_e \quad , \quad \widehat{\sigma}_P^2 = \sigma_e^2 \quad , \quad \widehat{\rho}_P = \rho_e \quad ,$$

i.e., pessimism is completely absent from log-utility equity returns.

C. Equilibrium equity premia

Pessimism has only an indirect impact on equity price dynamics, but one expects to see direct FORA effects on the riskfree rate and, thus, on equity premia. It is so indeed. FORA portfolio behavior implies a strong demand for bonds in those states where equity model uncertainty yields highly unreliable equity returns. This has a clear direct pricing impact on the reference model equity premia.

Corollary 5 *The pessimistic worst-case model and reference model equilibrium equity premia are given by*

$$\left(\widehat{\alpha}_P - r\right)_{\widehat{h}^*} = \widehat{\alpha}_P - \widehat{r} + \widehat{\sigma}_P^2\left[\widehat{\rho}_P\widehat{h}^{X*} + \sqrt{1-\widehat{\rho}_P^2}\,\widehat{h}^{P*}\right] = \widehat{\sigma}_P^2 - \gamma\left(\widehat{\sigma}_P^2 + \widehat{\rho}_P\xi\widehat{\sigma}_P\widehat{g}'\right) \quad , \tag{13}$$

and

$$\widehat{\alpha}_P - \widehat{r} = \left(\widehat{\alpha}_P - r\right)_{\widehat{h}^*} + \sqrt{\frac{2\varphi}{\widehat{\sigma}_P^2 + \frac{2\widehat{\rho}_P\widehat{\sigma}_P\xi}{1-\gamma}\widehat{g}' + \left(\frac{\xi}{1-\gamma}\widehat{g}'\right)^2}}\,\left|f\left(X\right)\right|\left(\widehat{\sigma}_P^2 + \widehat{\rho}_P\xi\widehat{\sigma}_P\widehat{g}'\right) \quad , \tag{14}$$

respectively.

Corollary 5 states that reference model equity premia are directly affected by time varying pessimism. They are higher in the presence of model uncertainty if and only if

$$\widehat{\sigma}_P^2 + \widehat{\rho}_P \xi \widehat{\sigma}_P \widehat{g}' > 0.$$

This requires that the sum of the standard and pessimistic speculative demands for equity dominates over the corresponding intertemporal hedging demand. The size of the FORA pricing impact becomes visible: The pessimism-driven component in the reference equity premium generates a quantity of the same order as equity returns volatility $\widehat{\sigma}_P$.

By contrast, worst-case model equity premia are affected only indirectly by pessimism. Its impact is felt only through the equilibrium function $\widehat{g}$ and the equilibrium volatility and correlation, $\widehat{\sigma}_P$ and $\widehat{\rho}_P$. The log utility case renders $(\widehat{\alpha}_P - r)_{\widehat{h}^*} = \widehat{\sigma}_e^2$, that is, worst-case premia are precisely the equity premia of a non-pessimistic log utility economy. Only the joint interplay of risk aversion (when $\gamma \neq 0$) and model uncertainty aversion (when $\varphi > 0$) determines equilibrium equity premia in the worst-case model, as it was for the equilibrium equity price dynamics.

D. Equilibrium value function

Knowledge of the equilibrium function $\widehat{g}$ is the key to a sharper characterisation of the equilibrium quantities in the presence of pessimism. The issue is tackled in this section.

In partial equilibrium, the value function of the pessimistic consumption investment problem cannot be written explicitly as the solution of the relevant HJB equation, because optimal equity risk exposure w^* depends on itself via the $G(w^*)$ in Equations (8) and (9). Thus, FORA creates a difficult fixed point problem, which adds to the closed-form analysis extra intractability in excess of the standard difficulty in describing the optimal hedge portfolio. However, in general equilibrium the value function solution admits explicit description, which is a remarkable fact. This is because market clearing in our representative exchange economy pinpoints equity exposure to one. The next theorem characterises the equilibrium solution to the HJB Equation (4) by laying the differential equation that makes the general equilibrium $\widehat{g}$ a function of the economy primitives.

Theorem 6 *In general equilibrium the value function of a pessimistic representative agent is*

$$\widehat{J}\left(\widehat{P}, X\right) = \frac{1}{\delta} \frac{\left(e^{\widehat{g}(X)}\widehat{P}\right)^{\gamma} - 1}{\gamma} \quad ,$$

where $\widehat{g}$ is the solution to the ordinary differential equation

$$
\begin{aligned}
0 &= \frac{1}{\gamma}\left(\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}-\delta\right)+\alpha_e+\frac{\gamma-1}{2}\sigma_e^2+\frac{1}{1-\gamma}\left(\zeta+\gamma\xi\rho_e\sigma_e\right)g'+\frac{\gamma}{(1-\gamma)^2}\frac{\xi^2}{2}\left(g'\right)^2 \\
&\quad +\frac{1}{1-\gamma}\frac{\xi^2}{2}g''-\frac{\sqrt{2\varphi}}{(1-\gamma)}\left|f\right|\left((1-\gamma)^2\sigma_e^2+2\left(1-\gamma\right)\rho_e\sigma_e\xi g'+\left(\xi g'\right)^2\right)^{\frac{1}{2}}. \qquad (15)
\end{aligned}
$$

Equation (15) is the foundation of our description of equilibrium asset returns.

4 Closed-form equilibrium analysis

More detailed characterisation of FORA effects calls for a more explicit equilibrium analysis. We achieve this by resorting to an analytic approach based on perturbation theory. First, we study a more tractable problem, that is, the pessimistic log utility equilibrium. Log utility is more tractable because it cancels the hedging motives driven by standard risk aversion. We calculate log utility closed forms in a few concrete examples. Then, we turn to the less tractable problems associated to pessimistic power utility equilibria. For them, we calculate asymptotic closed forms in the same concrete examples. These asymptotics come from the γ-first order perturbation of the pessimistic log utility function $\widehat{g}$.

A. Log utility

$\widehat{g}_{\log,\varphi}$ denotes the solution of Equation (15) for the log utility case $\gamma \to 0$. The corresponding value function is

$$
\widehat{J}_{\log,\varphi}\left(\widehat{P},X\right)=\frac{1}{\delta}\left(\ln\left(\widehat{P}\right)+\widehat{g}_{\log,\varphi}(X)\right) \quad .
$$

For $\gamma \to 0$ and $\varphi = 0$, Equation (15) characterises $\widehat{g}_{\log,0}$ in a non-pessimistic log-utility economy. Kogan and Uppal (2001) perturb $\widehat{g}_{\log,0}$ to analytically study non-pessimistic equilibria, whereas we characterise $\widehat{g}_{\log,\varphi}$ and perturbe it to analytically study pessimistic equilibria. For settings where $\widehat{g}_{\log,\varphi}$ has analytic solution, it is possible to give equilibrium asymptotics that depend only on the risk aversion parameter γ. On the other hand, if a closed-form solution is known only for the special case $\gamma \to 0$ and $\varphi = 0$, it will be necessary to expand in both risk aversion and pessimism parameters, γ and $\sqrt{\varphi}$. Here are three examples. The first two admit closed forms for $\widehat{g}_{\log,\varphi}$. The third example allows closed-form expressions of $\widehat{g}_{\log,0}$ only.

Example 7 *Geometric Ornstein-Uhlenbeck (GOU) state dynamics for expected dividend growth and dividend growth volatility:*

$$\begin{aligned} dX &= -\lambda\left(X-\overline{X}\right)dt+\xi X dZ^X, \\ \frac{de}{e} &= \left(X+\frac{\sigma_e^2}{2}X^2\right)dt+\sigma_e X\left[\rho_e dZ^X+\sqrt{1-\rho_e^2}dZ^e\right], \\ f(X) &= 1 \quad , \end{aligned}$$

where λ, $\overline{X}$, ξ, $\sigma_e>0$ *and* $\rho_e\in[-1,1]$. *Equation (15) with* $\gamma\to 0$ *admits solution*

$$\widehat{g}_{\log,\varphi}=a(\varphi)+b(\varphi)X \quad , \tag{16}$$

where

$$a(\varphi)=\frac{\lambda\overline{X}b(\varphi)}{\delta}+\ln\delta \quad , \tag{17}$$

and $b=b(\varphi)$ *is a root of the quadratic equation*

$$0=\left((\lambda+\delta)^2-2\varphi\xi^2\right)b^2-2(\lambda+\delta+2\varphi\xi\rho_e\sigma_e)b+1-2\varphi\sigma_e^2 \quad , \tag{18}$$

such that

$$b(\varphi)\leq b(0)=\frac{1}{\lambda+\delta} \quad .$$

Example 7 assumes a constant Knightian uncertainty aversion. An example with a time-varying maximal distrust function $f(X)$ is the following.

Example 8 *Cox Ingersoll Ross (CIR) state dynamics for dividend growth volatility and pessimistic maximal distrust function proportional to dividend growth volatility:*

$$\begin{aligned} dX &= -\lambda\left(X-\overline{X}\right)dt+\xi\sqrt{X}dZ^X, \\ \frac{de}{e} &= \alpha_e dt+\sigma_e\sqrt{X}\left[\rho_e dZ^X+\sqrt{1-\rho_e^2}dZ^e\right], \\ f(X) &= \frac{\sqrt{X}}{\sqrt{\overline{X}}} \quad . \end{aligned}$$

where λ, $\overline{X}$, ξ, α_e, $\sigma_e>0$ *and* $\rho_e\in[-1,1]$. *Equation (15) with* $\gamma\to 0$ *admits solution*

$$\widehat{g}_{\log,\varphi}(X)=a(\varphi)+b(\varphi)X \quad , \tag{19}$$

where

$$a(\varphi)=\ln\delta+\frac{\alpha_e}{\delta}+\frac{\lambda\overline{X}b(\varphi)}{\delta} \quad , \tag{20}$$

and $b=b(\varphi)$ *is a root of the quadratic equation*

$$b^2\left((\delta+\lambda)^2-\frac{2\varphi\xi^2}{\overline{X}}\right)+\sigma_e\left((\delta+\lambda)\sigma_e-\frac{4\varphi\rho_e\xi}{\overline{X}}\right)b+\sigma_e^2\left(\frac{\sigma_e^2}{4}-\frac{2\varphi}{\overline{X}}\right)=0 \quad , \tag{21}$$

such that

$$b(\varphi) \leq b(0) = -\frac{\sigma_e^2}{2(\lambda+\delta)} \quad .$$

Via the time-varying distrust function $\sqrt{X}/\sqrt{\overline{X}}$, Example 8 expresses the link between the degree of distrust / uncertainty and the state of the economy. Higher conditional volatility for the dividend growth is associated with less confidence in the reference model. Such formulation of $f(X)$ captures the empirical evidence that investors tend to be more uncertain about the future growth of the economy during recessions. For example, Veronesi (1999) reports that economists' forecasts on the future real output growth (taken from the US Livingston survey) are more dispersed - that is, they have greater cross-sectional standard deviation - when the economy is contracting. In particular, Example 8 represents weak states of the economy as states where the conditional volatility of dividend growth is high, so that the speed of mean reversion in volatility governs the dynamics of contractions and expansions in the economy.

Pessimistic log utility cannot always find explicit solution, even if it has one for the special case $\varphi = 0$. An example of such a situation is the following.

Example 9 *Geometric Ornstein Uhlenbeck state dynamics for dividend growth volatility:*

$$\begin{aligned} dX &= -\lambda\left(X-\overline{X}\right)dt + \xi X dZ^X, \\ \frac{de}{e} &= \alpha_e dt + \sigma_e X\left[\rho_e dZ^X + \sqrt{1-\rho_e^2}dZ^e\right], \\ f_1(X) &= 1 \quad ; \quad f_2(X) = \frac{X}{\overline{X}} \quad , \end{aligned}$$

where λ, $\overline{X}$, ξ, α_e, $\sigma_e > 0$ *and* $\rho_e \in [-1,1]$. *Equation (15) with* $\gamma \to 0$ *admits no closed-form solution when* $\varphi > 0$ *even if for* $\varphi = 0$ *we have*

$$\widehat{g}_{\log,0}(X) = a_0 + b_0 X + c_0 X^2 \quad ,$$

where

$$\begin{aligned} a_0 &= \ln(\delta) + \frac{\alpha_e}{\delta} + \frac{\lambda\overline{X}}{\delta}b_0 \quad , \\ b_0 &= \frac{2\lambda\overline{X}}{(\delta+\delta)}c_0 \quad , \\ c_0 &= -\frac{\sigma_e^2}{2\left(2\lambda+\delta-\xi^2\right)} \quad . \end{aligned}$$

In Section 5, we make use of Examples 7 and 8 to empirically calibrate our pessimistic exchange economy.

B. Power utility

For power utility with $\gamma \neq 0$, we make a $\gamma-$first order expansion that serves the case whenever pessimistic log utility is tractable. It achieves a better analysis of the interaction between risk aversion and pessimism in determining asset prices:

$$\widehat{g} = \widehat{g}_{\log,\varphi} + \gamma \widehat{g}_1 + O\left(\gamma^2\right) \quad . \tag{22}$$

Expansion (22) assists in a $O\left(\gamma^3\right)-$description[1] of equilibrium asset returns as $\widehat{g}$ enters them only premultiplied by γ in Proposition 4 and Corollary 5. The same accuracy is achieved in describing the equilibrium earnings/price ratio, $\frac{e}{P} = \left(\frac{e^{\gamma \widehat{g}}}{\delta}\right)^{\frac{1}{\gamma-1}}$. The next proposition shows how to determine $\widehat{g}_1$.

Proposition 10 *The first order function $\widehat{g}_1$ in (22) is the solution of the differential equation*

$$0 = \mathbf{D}_1\left(\widehat{g}_{\log,\varphi}, \widehat{g}_1\right) + \mathbf{R}_1\left(\widehat{g}_{\log,\varphi}\right) \quad , \tag{23}$$

where operators $\mathbf{D}_1(\cdot,\cdot)$ and $\mathbf{R}_1(\cdot)$ are respectively

$$\begin{aligned}
\mathbf{D}_1(g,q) &= -\delta q + \zeta \frac{\partial q}{\partial X} + \frac{\xi^2}{2}\frac{\partial^2 q}{\partial^2 X} - \sqrt{2\varphi}\,|f|\,\frac{\xi g' + \rho_e \sigma_e}{\left(\sigma_e^2 + 2\rho_e\sigma_e \xi g' + (\xi g')^2\right)^{\frac{1}{2}}}\xi \frac{\partial q}{\partial X} \quad , \\
\mathbf{R}_1(g) &= \delta\left(\left(\frac{g}{2} - 1 - \ln(\delta)\right) g + \ln(\delta)\left(1 + \frac{\ln(\delta)}{2}\right)\right) \\
&\quad + \frac{\sigma_e^2}{2} + \left(\zeta + \xi\rho_e\sigma_e\right)\frac{\partial g}{\partial X} + \frac{\xi^2}{2}\left(\frac{\partial g}{\partial X}\right)^2 + \frac{\xi^2}{2}\frac{\partial^2 g}{\partial^2 X} \quad .
\end{aligned}$$

The first order term $\widehat{g}_1$ is the solution of a second order linear differential equation with homogenous equation $\mathbf{D}_1\left(\widehat{g}_{\log,\varphi}, g\right) = 0$ and inhomogeneity $\mathbf{R}_1\left(\widehat{g}_{\log,\varphi}\right)$. Given a system of fundamental solutions to the homogenous equation $\mathbf{D}_1\left(\widehat{g}_{\log,\varphi}, g\right) = 0$, $\widehat{g}_1$ can be in principle computed by the variation of constant method. Unfortunately, this may imply very involved expressions. The fundamental system of solutions can be made of two non trivial basis functions and the final solution will also contain linear combinations of integrals of these functions. However, some important model settings allow for a simple polynomial solution. Such settings correspond to Examples 7 and 8.

Corollary 11 *With the explicit GOU dynamics of Example 7 the function $\widehat{g}_1$ is quadratic in X:*

$$\widehat{g}_1(X) = \alpha(\varphi) + \beta(\varphi) X + \varepsilon(\varphi) X^2 \quad , \tag{24}$$

[1] Higher order equilibrium characterisations are very rare. An example of an infinite order partial equilibrium asymptotic for a constant opportunity set pessimistic economy is given in Trojani and Vanini (2001b).

where

$$\varepsilon(\varphi) = \frac{\sigma_e^2 + 2\rho_e\sigma_e\xi b(\varphi) + \left(\delta + \xi^2\right) b(\varphi)^2}{2\left(\delta + 2\left(\lambda + \sqrt{2\varphi}\xi\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}\right) - \xi^2\right)} \tag{25}$$

$$\beta(\varphi) = \frac{2\lambda\overline{X}}{\delta + \lambda + \sqrt{2\varphi}\xi\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}}\varepsilon(\varphi) + \frac{\delta\left(\left(\frac{1}{2}a(\varphi) - 1 - \ln\delta\right) b(\varphi) + \frac{1}{2}b(\varphi)a(\varphi)\right) - \lambda b(\varphi)}{\delta + \lambda + \sqrt{2\varphi}\xi\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}} \tag{26}$$

$$\alpha(\varphi) = \frac{\lambda\overline{X}}{\delta}\beta(\varphi) + \frac{\lambda\overline{X}}{\delta}b(\varphi) + \left(\left(\frac{1}{2}a(\varphi) - 1 - \ln\delta\right)a(\varphi) + (\ln\delta)\left(1 + \frac{1}{2}\ln\delta\right)\right) \tag{27}$$

and where $a(\varphi)$ *and* $b(\varphi)$ *are given in (17) and (18), respectively.*

Corollary 12 *With the explicit CIR dynamics of Example 8 the function* $\widehat{g}_1$ *is quadratic in* X*:*

$$\widehat{g}_1(X) = \alpha(\varphi) + \beta(\varphi)X + \varepsilon(\varphi)X^2 \quad , \tag{28}$$

where

$$\varepsilon(\varphi) = \frac{\delta b(\varphi)^2}{2\left(\delta + 2\left(\lambda + \sqrt{2\varphi}\frac{\xi}{\sqrt{\overline{X}}}\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}\right)\right)} \quad , \tag{29}$$

$$\beta(\varphi) = \frac{2\lambda\overline{X} + \xi^2}{\lambda + \delta + \sqrt{2\varphi}\frac{\xi}{\sqrt{\overline{X}}}\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}}\varepsilon(\varphi) + \frac{\delta(a(\varphi) - 1 - \ln\delta)b(\varphi) + (\xi\rho_e\sigma_e - \lambda)b(\varphi) + \frac{1}{2}\left(\sigma_e^2 + \xi^2 b(\varphi)^2\right)}{\lambda + \delta + \sqrt{2\varphi}\frac{\xi}{\sqrt{\overline{X}}}\frac{\xi b(\varphi)+\rho_e\sigma_e}{\left(\sigma_e^2+2\rho_e\sigma_e\xi b(\varphi)+(\xi b(\varphi))^2\right)^{\frac{1}{2}}}} \quad , \tag{30}$$

$$\alpha(\varphi) = \frac{\lambda\overline{X}}{\delta}\beta(\varphi) + \frac{\lambda\overline{X}}{\delta}b(\varphi) + \left(\frac{a(\varphi)}{2} - 1\right)a(\varphi) + \ln\delta\left(1 + \frac{1}{2}\ln\delta - a(\varphi)\right) \quad . \tag{31}$$

and where $a(\varphi)$ *and* $b(\varphi)$ *are given in (20) and (21), respectively.*

5 Calibration

We calibrate Examples 7 and 8 to the last century of US data. The goal is firstly to quantify the impact of Knightian pessimism in some concrete calibrations and secondly to perform a quality check of our asymptotic equilibrium computations. We compare them with the exact equilibrium quantities implied by a numerical solution of the differential equation in Theorem 6.

The calibration employs the unconditional moments of US historical consumption growth. In choosing model parameters we use historical per capita consumption over the 1889 - 1994 period. Consumption data are real with annual frequency. We choose the drift and diffusion of the dividend process (e_t) to match the corresponding values of per capita aggregate consumption. This puts restrictions on the parameters governing the state process X as well. We take the estimates of the unconditional moments of the consumption growth from Table 8.1, p. 308, of Campbell, Lo, and MacKinlay (1997). They define consumption growth as the change in log real consumption of non-durables and services.

Variable	Mean	Standard deviation
Consumption growth	0.0172	0.0328

Because consumption growth of nondurables and services is a smooth series, equity looks like a small bet in this exchange economy. The long-run variance of dividend growth is only 0.11%. This is where FORA pricing impacts are expected to show their strength, because FORA equity premia are proportional to standard deviations with long-run level of 3.28%.

We generate a century-long sample of weekly data from the processes that drive the economy. This produces 4800 candidate observations. 400 of them are sampled quarterly and used as actual observations. The above long-run estimates are based on annual frequency data. The theoretical counterpart they come from is $\int_0^1 d\ln e$. We take the following liberty in tuning model parameters. We force the unconditional mean and standard deviation of $\int_0^1 d\ln e$ to equal the unconditional mean and standard deviation of the annualized weekly changes in the log-dividends, $\frac{1}{\Delta t}\Delta \ln e$ with $\Delta t = 1/48$.

The calibration of the GOU dynamics in Example 7 is

$$\begin{aligned} \Delta X &= -3\,(X - 0.0172)\,\Delta t + 1 \cdot X \Delta Z^X, \\ \Delta \ln e &= X\Delta t + 1.9070 \cdot X \left[0.1 \Delta Z^X + \sqrt{1 - 0.1^2}\Delta Z^e\right]. \end{aligned}$$

X acts both as conditional mean and conditional volatility of the log-dividend changes. We set the long-run mean of (X_t) to $\overline{X} = 0.0172$ and the long run volatility of $\frac{1}{\Delta t}\Delta \ln e$ to $\sigma_e \overline{X} = 0.0328$. The adjustment-speed parameter λ is 3. Positivity of $\overline{X}$ and λ assures stationarity. We privilege X's interpretation as consumption growth forecast, so that we set ρ_e to 0.1. Forecasts of high short term consumption growth are associated with good short-term news on earnings.

The calibration of the CIR dynamics in Example 8 is

$$\begin{aligned} \Delta X &= -3\left(X - (0.0328)^2\right)\Delta t + 0.0783 \cdot \sqrt{X}\Delta Z^X, \\ \Delta \ln e &= \left(0.0177 - 0.5 \cdot 1^2 \cdot X\right)\Delta t + 1 \cdot \sqrt{X}\left[-0.25 \cdot \Delta Z^X + \sqrt{1 - (-0.25)^2}\Delta Z^e\right]. \end{aligned}$$

X acts only as conditional variance of the log-dividend changes. We set the long run mean of (X_t) to $\overline{X} = (0.0328)^2$. The long run mean of $\frac{1}{\Delta t}\Delta \ln e$ is set to $\alpha_e - 0.5 \cdot \sigma_e^2 \cdot \overline{X} = 0.0172$. The adjustment-speed parameter λ is 3. The coefficient ξ^2 is a 95% fraction of $2\lambda\overline{X}$, to avoid absorption at zero and to grant ergodicity. The correlation coefficient ρ_e is set to -0.25. High conditional variances of short term consumption growth are signals of jitters associated with bad short-term news on earnings.

A. Quality of asymptotics

For Examples 7 and 8, we compare the calibrated analytical approximation $\widetilde{g} = \widehat{g}_{\log,\varphi} + \gamma\widehat{g}_1$ to the exact numerical solution[2] $\widehat{g}$ of Equation (15). The corresponding equilibrium levels of equity return volatility, expected return on equity, and earnings/price ratio are also confronted.

In Figure **1** (Figure **4**), the top-left panel compares the proxied $\widehat{g}$ with the exact $\widehat{g}$ given the relevant range of the state variable X and for $\gamma = -1$ (that is, a CRRA of two). X's domain is broader than the empirical support of the 400 observations generated by the calibrated dynamics. X has minimum 0.008 (0) and maximum 0.026 (0.0025) for the GOU (CIR) setting. A similar graphical comparison has been carrried out for $\gamma = -0.5, 0.5$. As expected, the proxy quantities are very sharp and even more accurate than the displayed case $\gamma = -1$. The evidence for $\gamma = -1$ is remarkable: The maximal relative error of approximation across both examples is 1.2%. Moreover, the stochastics left out in the $O\left(\gamma^3\right)-$approximation of equilibrium asset returns are very small even in parameter regions that no longer belong to log utility neighborhoods.

Insert Figures 1 and 4 about here

The exact $\widehat{g}$ is virtually linear in X on the given domain and the difference $\widehat{g} - \widetilde{g}$ is a small constant. In fact, the values of $\widehat{g}'$ and $\widetilde{g}'$ are nearly indistinguishable. This implies virtually perfect proxies for the equilibrium equity returns volatility and correlation, $\widehat{\sigma}_P$ and $\widehat{\rho}_P$, since they are entered by $\widehat{g}'$ only. The top-right panel of Figures **1** and **4** graphically confirms this for the equity returns volatility case, with a 0.7% maximal relative error of approximation

[2] The exact numerical solution of $\widehat{g}$ is obtained using built-in MATLAB routines for non-linear ordinary differential equations. The two initial conditions are in the level (Equation (15) at $X = 0$) and in the first derivative (equality with $\widetilde{g}'$ at $X = 0$). Solution accuracy is good. The maximum absolute value of the ordinary differential equation (Equation (15)) residuals across all γ levels is $4.43 \cdot 10^{-6}$.

across both examples. However, the expected return $\widehat{\alpha}_P$ is entered by $\widehat{g}$'s level via the term $\left(\frac{e^{\gamma\widehat{g}}}{\delta}\right)^{\frac{1}{\gamma-1}}$, that is, the equilibrium earnings/price ratio $\frac{e}{P}$. The bottom-right panel of Figures **1** and **4** shows that the asymptotics of the ratio are a good proxy of the exact ratio, with a 1.5% maximal relative error of approximation across both examples. $\frac{e}{P}$ remains stable in X's values. The ratio slightly increases in the GOU setting. Increases in the forecast of short term earnings growth are matched by lower equity prices, for earnings volatility and equity return volatility rise. This is confirmed by high expected returns on equity when X is high. The ratio slightly decreases in the CIR setting. Current high levels of earnings volatility imply low future levels in the medium run, given X's ergodic mean reversion. This lifts current equity prices and lowers expected returns on equity. Good asymptotics for $\frac{e}{P}$ means good asymptotics for $\widehat{\alpha}_P$. The bottom-left panel of Figures **1** and **4** shows that, across both examples, the maximal relative error of approximation for $\widehat{\alpha}_P$ is 2% only.

B. Asset returns

We examine the time series of the calibrated equilibrium asset returns and their dependence on the preference parameters. We consider a small log utility neighborood ($\gamma = -0.10$) and focus on the equilibrium impact of pessimism on equity premia. We have selected a level of 0.015 for the maximal distrust parameter φ. Log utility and pessimism in tiny amounts highlight the strength of the FORA impact on equity premia, for log utility equilibrium equity returns do not feel any pressure from pessimism. The time preference parameter δ is set to 0.053.

Figure **2** focuses on GOU equity premia and worst-case equity premia.

Insert Figures 2 and 3 about here.

In the bottom left panel of Figure **2**, conditional equity premia with pessimism are more than five times larger than those without pessimism. This illustrates the amplitude of FORA effects on excess expected returns already for a CRRA close to log utility. Pessimistic premia are more volatile. This occurs even if $f(X)$ is constant because FORA adds a premium component proportional to the dividend growth volatility X. The implied unconditional equity premium[3] with ($\gamma = -0.10, \varphi = 0.015$) has a spread of 55 basis points upon the case ($\gamma = -0.10, \varphi = 0$). The trade-off between risk aversion and pessimism in determining conditional equity premia is

[3] 1000 paths of conditional equity premia are simulated. We measure the unconditional premium as the cross-path mean of the 1000 conditional premia at the 400th quarter.

illustrated in the top left panel of Figure **2**, where isoquants of asymptotic equity premia are plotted on a relevant region of preference parameters. This makes FORA even more evident. A small decrease in the pessimism parameter φ requires a substantial increase in the risk aversion parameter γ in order to remain on the same premium level curve. Our equilibrium asymptotics are the key to obtain a thorough contour plot analysis in the preference parameter space (γ, φ).

The top right panel of Figure **2** shows how worst-case premium level curves are nearly unelastic to the pessimism parameter. The indirect impact of pessimism on worst-case premia is quantitatively very small. This holds even for $\gamma \approx -1$. The time series of worst-case premia is indistinguishable from those of non-pessimistic premia (see the bottom-right panel). Figure **3** shows that the indirect impact of pessimism on the equity price dynamics is also negligible. Expected equity returns, volatilities and correlations are nearly unaffected. As expected, the pessimistic $\widehat{g}$ becomes more negative (cf. the bottom right panel in Figure **3**): The value function is reduced by max-min behavior.

Figure **5** focuses on CIR equity premia and worst-case equity premia.

Insert Figures 5 and 6 about here.

In this setting also, via FORA, conditional equity premia with pessimism are more than five times larger than those without pessimism. Pessimistic premia are substantially volatile and cyclical. FORA adds a premium component proportional to the dividend growth volatility $\sqrt{X}$ and the time-varying distrust $f(X)$ creates medium-term countercycles in the effective risk aversion via X's ergodic mean reversion. The unconditional equity premium with $(\gamma = -0.10, \varphi = 0.015)$ has a spread of 56 basis points upon the case $(\gamma = -0.10, \varphi = 0)$. The trade-off between risk aversion and pessimism in determining conditional equity premia reflects FORA, with a particular pattern at high levels of pessimism. For high levels of φ, a higher γ stimulates a higher φ in order to maintain the same premium level. Higher aversion to risk and to model uncertainty lifts equity premia. In expected utility, CRRA is inversely related to the elasticity of intertemporal substitution. Higher risk aversion implies a lower desire to substitute consumption intertemporally, that is, it implies higher riskfree rates. In the CIR example, this second effect is compensated only via a higher degree of pessimism. The larger the risk aversion, the larger this second effect. However, notice that the pattern emerges only for φs well above the 'realistic' 0.015 threeshold. Worst-case premium level curves remain unelastic to the pessimism parameter, so that their time series coincides with the series of non-pessimistic premia. The indirect impact of pessimism on worst-case premia is again quantitatively not significant even when injecting substantial levels of CRRA. Figure **6** shows again that equity price dynamics has light reaction to pessimism and that the value function is reduced by max-min behavior.

The time series of the GOU and CIR conditional riskfree rates appear in Figure **7**. Knightian FORA triggers a strong spurt in the demand for bonds whenever equity looks like a small bet with scarce hedging potential, because Knightian uncertainty over the expected reward of such a bet kills any appetite for it. The unconditional premium spread is mainly due to reduction in the riskfree rates. GOU conditional riskfree rates with pessimism seem less volatile than non-pessimistic riskfree rates, even if FORA injects a component proportional to the dividend growth volatility X.

6 Conclusions

We presented a continuous-time intertemporal setting for a representative agent exchange economy, in which the equilibrium pricing impact of state-dependent Knightian pessimism can find analytical description. We apply pessimism to genuine model uncertainty. Allowed alternative models are generated via local non-parametric contaminations of the reference model and satisfy a relative entropy constraint with respect to the reference model. Knightian pessimism is max-min expected utility behavior over constrained relative entropy models. Relative entropy is essential in the objective choice of a realistic amount of pessimism given the sample data the researcher uses in the calibration.

Exchange economies force equity to be a claim on consumption growth. An enduring macro-finance puzzle is that US consumption growth is too smooth. In states where equity is a small bet with scarse hedging ability, Knightian pessimism yields a strong asset substitution between equity and bonds by emboldening effective risk aversion. This is the FORA effect. It produces lower equilibrium riskfree rates and significantly higher equity premia. In our setting, a constant and low local curvature of the instantaneous utility function is compatible with time-varying effective FORA, sizable equity premia, and realistic amount of pessimism. By contrast, in Campbell and Cochrane (1999) and Chan and Kogan (2002), time-varying and high local curvatures are necessary to yield sizable equity premia via time-varying SORA.

These results seem promising for future research. In the spirit of Campbell and Cochrane (1999) and Chan and Kogan (2002), one would like to elaborate a consumption-based setting that explains a wide variety of dynamic asset pricing phenomena, including the procyclical variation of equity prices and the long-horizon predictability of excess equity returns. To this purpose, such a setting could mix time-varying entropy-based Knightian pessimism with non-standard CRRA instantaneous utility, agents' heterogeneity, or jump risk within the reference model.

7 Appendix

In this appendix we provide the proofs for the Examples, Theorems, Propositions, and Corollaries in the paper.

Proof of Proposition 1. The minimization of the objective $u - \delta J + \frac{1}{dt} E_t^h (dJ)$ with respect to the contaminating vector h under the binding constraint on h's size yields

$$h^* = -\frac{1}{\lambda} \Lambda' J_Y, \quad \frac{1}{2} h^{*\prime} h^* = \varphi f^2,$$

where λ is the Lagrangean multiplier. The binding constraint fixes λ and this completes the proof. ■

Proof of Proposition 3. The quadratic form $J_Y' \Lambda \Lambda' J_Y$ reads explicitly

$$W^2 J_W^2 \left(\sigma_P^2 w^2 + \frac{\xi^2 J_X^2}{W^2 J_W^2} + 2 w \rho_P \sigma_P \xi \frac{J_X}{W J_W} \right) = W^2 J_W^2 G(w).$$

Thus, the Hamilton-Jacobi-Bellman (HJB) equation implied by the worst-case scenario in Proposition 1 is

$$\begin{aligned} 0 &= \max_{c,w} \left\{ \frac{(cW)^\gamma - 1}{\gamma} - \delta J + (wW(\alpha_P - r) + (rW - cW)) J_W + \frac{1}{2} w^2 W^2 \sigma_P^2 J_{WW} \right. \\ &\quad \left. + \zeta J_X + \frac{1}{2} \xi^2 J_{XX} + w W \rho_P \sigma_P \xi J_{XW} - \sqrt{2\varphi} |f| \left[W^2 J_W^2 G(w) \right]^{\frac{1}{2}} \right\}. \end{aligned}$$

The First Order Conditions express the optimal policies in terms of J's partial derivatives:

$$\begin{aligned} c^* &= \frac{(J_W)^{\frac{1}{\gamma-1}}}{W}, \\ w^* &= \frac{\frac{\alpha_P - r}{\sigma_P^2} + \left(\frac{J_{WX}}{J_W} - \sqrt{2\varphi} |f| \left(W^2 J_W^2 G(w^*) \right)^{-\frac{1}{2}} J_X \right) \frac{\rho_P \xi}{\sigma_P}}{\left(-\frac{W J_{WW}}{J_W} + \sqrt{2\varphi} |f| \left(G(w^*) \right)^{-\frac{1}{2}} \right)}. \end{aligned}$$

Use of J's functional form (6) and algebra complete the proof. ■

Proof of Theorem 4. Market clearing implies

$$P = \frac{e}{c^*} \quad , \quad \frac{dP}{P} = \frac{de}{e} - \frac{dc^*}{c^*} - \frac{dc^*}{c^*} \frac{de}{e} + \frac{dc^*}{c^*} \frac{dc^*}{c^*} \quad .$$

From Proposition 3, we have

$$\frac{dc^*}{c^*} = \left(\frac{e^{\gamma g}}{\delta} \right)^{\frac{-1}{\gamma-1}} \left[\frac{\partial}{\partial X} \left(\frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{\gamma-1}} dX + \frac{1}{2} \frac{\partial^2}{\partial^2 X} \left(\frac{e^{\gamma g}}{\delta} \right)^{\frac{1}{\gamma-1}} dX^2 \right].$$

This gives

$$\frac{dc^*}{c^*}\frac{de}{e}=\frac{\gamma}{\gamma-1}g'\xi\rho_e\sigma_e dt \quad , \quad \frac{dc^*}{c^*}\frac{dc^*}{c^*}=\left(\frac{\gamma}{\gamma-1}\xi\right)^2 (g')^2\,dt.$$

The cumulative return dynamics are then

$$\begin{aligned}\frac{dP+edt}{P} &= \left[\alpha_e+\frac{\gamma}{1-\gamma}\left(g'\left(\zeta+\xi\rho_e\sigma_e\right)+\frac{1}{2}\left(\frac{\gamma}{1-\gamma}\left(g'\right)^2+g''\right)\xi^2\right)+\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}\right]dt \\ &\quad+\left(\sigma_e\rho_e+\frac{\gamma}{1-\gamma}g'\xi\right)dZ^X+\sigma_e\sqrt{1-\rho_e^2}dZ^e\end{aligned}$$

The two equations,

$$\sigma_P\rho_P=\sigma_e\rho_e+\frac{\gamma}{1-\gamma}g'\xi \quad , \quad \sigma_P\sqrt{1-\rho_P^2}=\sigma_e\sqrt{1-\rho_e^2} \quad ,$$

imply

$$\sigma_P^2=\sigma_e^2+\frac{\gamma}{1-\gamma}2\rho_e\sigma_e\xi g'+\left(\frac{\gamma}{1-\gamma}\xi\right)^2\left(g'\right)^2 \quad .$$

This concludes the proof. ■

Proof of Corollary 5. With financial market clearing $w=1$, optimality implies

$$\widehat{\alpha}_P-\widehat{r}=\widehat{\sigma}_P^2-\left(\gamma-\sqrt{\frac{2\varphi}{\widehat{G}(1)}}\left|f\right|\right)\left(\widehat{\sigma}_P^2+\widehat{\rho}_P\xi\widehat{\sigma}_P\widehat{g}'\right) \quad .$$

The worst-case drift contamination is $-\sqrt{\frac{2\varphi}{\widehat{G}(1)}}\left|f(X)\right|\left(\widehat{\sigma}_P^2+\widehat{\rho}_P\xi\widehat{\sigma}_P\widehat{g}'\right)$. Thus, the worst-case model equity premium is

$$\left(\widehat{\alpha}_P-\widehat{r}\right)_{h^*}=\widehat{\sigma}_P^2-\gamma\left(\widehat{\sigma}_P^2+\widehat{\rho}_P\xi\widehat{\sigma}_P\widehat{g}'\right) \quad .$$

This concludes the proof. ■

Proof of Theorem 6 . With financial market clearing $w=1$, the HJB equation becomes

$$\begin{aligned}0 &= \max_c\left\{\frac{(cW)^\gamma-1}{\gamma}-\delta J+W\left(\alpha_P-c\right)J_W+\frac{1}{2}W^2\sigma_P^2J_{WW}+\zeta J_X+\frac{1}{2}\xi^2J_{XX}+W\rho_P\sigma_P\xi J_{XW}\right. \\ &\quad\left.-\sqrt{2\varphi}\left|f\right|\left[W^2J_W^2G(1)\right]^{\frac{1}{2}}\right\},\end{aligned}$$

We substitute for optimal consumption $c^*=\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}$ in order to express the HJB equation in terms of g and its derivatives:

$$\begin{aligned}0 &= \frac{1}{\gamma}\left(\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}-\delta\right)+\alpha_P-\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}+\frac{\gamma-1}{2}\sigma_P^2+\left(\zeta+\gamma\rho_P\sigma_P\xi\right)g'+\frac{1}{2}\xi^2\left(\gamma\left(g'\right)^2+g''\right) \\ &\quad-\sqrt{2\varphi}\left|f\right|\left(\sigma_P^2+\xi^2\left(g'\right)^2+2\rho_P\sigma_P\xi g'\right)^{\frac{1}{2}} \quad .\end{aligned}$$

By inserting the equilibrium structure of equity returns (see Equations (10), (11), and (12)), algebra leads to Equation (15). For the log utility case $\gamma \to 0$, it follows

$$0=\delta\left(\ln\left(\delta\right)-g\right)+\alpha_e-\frac{\sigma_e^2}{2}+\zeta g'+\frac{\xi^2}{2}g''-\sqrt{2\varphi}\left|f\right|\left(\sigma_e^2+\xi^2(g')^2+2\xi g'\rho_e\sigma_e\right)^{\frac{1}{2}}. \tag{32}$$

This concludes the proof. ■

Proof of Example 7. Equation (32) becomes

$$0=\delta\left(\ln\left(\delta\right)-\widehat{g}_{\log,\varphi}\right)+X-\lambda\left(X-\overline{X}\right)\widehat{g}'_{\log,\varphi}+\frac{\left(\xi X\right)^2}{2}\widehat{g}''_{\log,\varphi}-X\sqrt{2\varphi}\left(\sigma_e^2+\xi^2(\widehat{g}'_{\log,\varphi})^2+2\xi\widehat{g}'_{\log,\varphi}\rho_e\sigma_e\right)^{\frac{1}{2}}.$$

The solution is obtained by formulating the linear guess $\widehat{g}_{\log,\varphi}=a+bX$ and by matching the resulting coefficients. ■

Proof of Example 8. The proof uses the same arguments of the proof for Example 7. ■

Proof of Example 9. The proof uses the same arguments of the proof for Example 7. A polynomial guess for $\widehat{g}_{\log,0}$ is employed. ■

Proof of Proposition 10. Define the operator $\mathbf{F}\left(\gamma,g\right)$ to be

$$\begin{aligned}\mathbf{F}\left(\gamma,g\right) &= \frac{1}{\gamma}\left(\left(\frac{e^{\gamma g}}{\delta}\right)^{\frac{1}{\gamma-1}}-\delta\right)+\alpha_e+\frac{\gamma-1}{2}\sigma_e^2+\frac{1}{1-\gamma}\left(\zeta+\gamma\xi\rho_e\sigma_e\right)\frac{\partial g}{\partial X}+\frac{\gamma}{\left(1-\gamma\right)^2}\frac{\xi^2}{2}\left(\frac{\partial g}{\partial X}\right)^2\\ &+\frac{1}{1-\gamma}\frac{\xi^2}{2}\frac{\partial^2 g}{\partial^2 X}-\frac{\sqrt{2\varphi}}{\left(1-\gamma\right)}\left|f\right|\left(\left(1-\gamma\right)^2\sigma_e^2+2\left(1-\gamma\right)\rho_e\sigma_e\xi\frac{\partial g}{\partial X}+\left(\xi\frac{\partial g}{\partial X}\right)^2\right)^{\frac{1}{2}}.\end{aligned}$$

$\mathbf{F}\left(\gamma,g\right)$ corresponds to the right-hand side of Equation (15). We perform a γ-first order expansion of it around log utility:

$$\begin{aligned}\mathbf{F}\left(\gamma,g\right) &= \mathbf{F}\left(0,g\right)+\gamma\mathbf{F}_1\left(g\right)+O\left(\gamma^2\right),\\ \mathbf{F}\left(0,g\right) &= \delta\left(\ln\left(\delta\right)-g\right)+\alpha_e-\frac{\sigma_e^2}{2}+\zeta\frac{\partial g}{\partial X}+\frac{\xi^2}{2}\frac{\partial^2 g}{\partial^2 X}-\mathbf{R}_2\left(g\right),\\ \mathbf{R}_2\left(g\right) &= -\sqrt{2\varphi}\left|f\right|\left(\sigma_e^2+2\rho_e\sigma_e\xi\frac{\partial g}{\partial X}+\left(\xi\frac{\partial g}{\partial X}\right)^2\right)^{\frac{1}{2}},\\ \mathbf{F}_1\left(g\right) &= \delta\left(\left(\frac{\widehat{g}}{2}-1-\ln\left(\delta\right)\right)g+\ln\delta\left(1+\frac{\ln\delta}{2}\right)\right)\\ &+\frac{\sigma_e^2}{2}+\left(\zeta+\xi\rho_e\sigma_e\right)\frac{\partial g}{\partial X}+\frac{\xi^2}{2}\left(\frac{\partial g}{\partial X}\right)^2+\frac{\xi^2}{2}\frac{\partial^2 g}{\partial^2 X}\\ &= \mathbf{R}_1\left(g\right).\end{aligned}$$

We also expand the candidate solution $\widehat{g}$ of Equation (15) to γ-first order: $\widehat{g}=\widehat{g}_{\log,\varphi}+\gamma\widehat{g}_1+O\left(\gamma^2\right)$. Although the operator $\mathbf{R}_2\left(\cdot\right)$ does not directly depend on γ, $\widehat{g}$'s expansion leads to the following γ-first order decomposition of the operator $\mathbf{R}_2(\cdot)$ when applied to $\widehat{g}$:

$$\mathbf{R}_2(\widehat{g}) = \mathbf{R}_2(\widehat{g}_{\log,\varphi}) + \gamma \left\{ 2\varphi \left[\mathbf{R}_2(\widehat{g}_{\log,\varphi})\right]^{-1} \left(\rho_e \sigma_e + \xi \frac{\partial \widehat{g}_{\log,\varphi}}{\partial X}\right) \xi \frac{\partial \widehat{g}_1}{\partial X} \right\} + O\left(\gamma^2\right) \quad .$$

The pessimistic log utility HJB problem in Equation (32) is solved by $\widehat{g}_{\log,\varphi}$ and Equation (15) states that $\mathbf{F}(\gamma, \widehat{g})$ must equal zero. These imply Equation (23) and conclude the proof. ■

Proof of Corollary 11. We plug in Equation (23) $\widehat{g}_{\log,\varphi}$'s explicit form coming from Example 7, where for brevity $a = a(\varphi)$ and $b = b(\varphi)$. This yields the following differential equation:

$$\begin{aligned} 0 \quad = \quad & -\delta \widehat{g}_1 + \left(-\lambda\left(X - \overline{X}\right) - X\sqrt{2\varphi}\xi \frac{\xi b + \rho_e \sigma_e}{\left(\sigma_e^2 + 2\rho_e \sigma_e \xi b + (\xi b)^2\right)^{\frac{1}{2}}} \right) \widehat{g}_1' + \frac{(\xi X)^2}{2} \widehat{g}_1'' \\ & + \delta \left(\left(\frac{a + bX}{2} - 1 - \ln(\delta) \right)(a + bX) + \ln(\delta)\left(1 + \frac{\ln(\delta)}{2}\right) \right) \\ & + \frac{\sigma_e^2 X^2}{2} + \left(\lambda\left(\overline{X} - X\right) + \xi \rho_e \sigma_e X^2\right) b + \frac{(\xi X b)^2}{2} \quad . \end{aligned}$$

The solution is obtained by formulating the polynomial guess $\widehat{g}_1 = \alpha + \beta X + \varepsilon X^2$ and by matching the resulting coefficients. ■

Proof of Corollary 12. The proof uses the same arguments of the proof of Corollary 11.

■

References

[1] Abel, B. (2002) . An Exploration of the Effects of Pessimism and Doubt on Asset Returns, Journal of Economic Dynamics and Control, 26, 7-8, 1075-1092.

[2] Anderson, E.W., L.P. Hansen, and T.J. Sargent (1998) . Risk and robustness in general equilibrium, http:// www.stanford.edu/ sargent/ research.html.

[3] Anderson, E. W., L. P. Hansen, and T. J. Sargent (2000) . Robustness, Detection and the Price of Risk (version with statistical model detection theory), http:// www.stanford.edu/ sargent/ research.html.

[4] Campbell, Y. C., and J. H. Cochrane, (1999) . By Force of Habit: a Consumption-Based Explanation of Aggregate Stock Market Behavior, Journal of Political Economy, vol 107, 2, 205-251.

[5] Chan, L., C. and L. Kogan, (2001) . Catching up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices, forthcoming, Journal of Political Economy, http://web.mit.edu/ lkogan2/ www/.

[6] Chen, Z. and L. G. Epstein, (2002) . Ambiguity, risk and asset returns in continuous time, Econometrica 70, 1403-1443.

[7] Cochrane, J. (2001) . Asset Pricing, Princeton University Press.

[8] Dow, J., and R. Werlang (1992) . Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio, Econometrica 60, 197-204.

[9] Ellsberg, D.,(1961) . Risk, Ambiguity and the Savage Axioms, Quarterly Journal of Economics, 25, 643-669.

[10] Epstein, L., and M. Schneider (2002) . Learning under ambiguity, Working Paper, http:// www.econ.rochester.edu/ Faculty/ Epstein.html

[11] Epstein, L, and S. Zin (1989) . Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework, Econometrica 57 (1989), 937-69.

[12] Gilboa, I. and D. Schmeidler, (1989) . Max-min Expected Utility with Non-unique Prior, Journal of Mathematical Economics, 18, 141-153.

[13] Knight, F., (1921) . Risk, Uncertainty and Profit. Boston, Mass: Houghton Mifflin. Reprint, London: London School of Economics, 1946.

[14] Kogan, L. and R. Uppal, (2000) . Risk Aversion and Optimal Policies in Partial and General Equilibrium Economies, Working Paper, http:// www.finance.commerce.ubc.ca/ ~uppal/.

[15] Kogan, L., and T. Wang (2002) . A Simple Theory of Asset Pricing under Model Uncertainty, Working Paper, http://web.mit.edu/ lkogan2/ www/.

[16] Lei, C.I., (2001) . Why Don't Investors Have Large Positions in Stocks? A Robustness Perspective, Mimeo, Department of Economics, University of Chicago, http:// www.chonlei.com/ research.htm.

[17] Liu, J., J. Pan, and T. Wang (2002) . An Equilibrium Model of Rare Event Premia, Working Paper, http://web.mit.edu/junpan/www/.

[18] Marinacci, M. (1999) . Limit Laws for Non-Additive Probabilities, and Their Frequentist Interpretation, Journal of Economic Theory 84, 145-195.

[19] Marinacci, M. (2002) . Probabilistic Sophistication and Multiple Priors, Econometrica 70, 755-764, 2002.

[20] Maenhout, P., (1999) . Robust Portfolio Rules and Asset Pricing, http:// www.insead.edu/ facultyresearch/ finance/ maenhout/.

[21] Merton, R., (1971) . Optimum Consumption and Portfolio Rules in a Continuous-Time Model, Journal of Economic Theory, 3, 373–413.

[22] Trojani, F., and P. Vanini (2001a) . Risk, Robustness and Knightian Uncertainty in Continuous Time, Heterogeneous Agents, Financial Equilibria, Working Paper, University of Southern Switzerland, Lugano, Switzerland. http:// papers.ssrn.com/ sol3/ cf_dev/ AbsByAuth.cfm? per_id=290962.

[23] Trojani, F., and P. Vanini (2001b) . Perturbative Solutions of Hamilton Jacobi Bellman Equations in Non-Homothetic Robust Decision Making, Working Paper, University of Southern Switzerland, Lugano, Switzerland. http:// papers.ssrn.com/ sol3/ cf_dev/ AbsByAuth.cfm? per_id=290962.

[24] Trojani, F., and P. Vanini, (2002) . A Note on Robustness in Merton's Model of Intertemporal Consumption and Portfolio Choice, Journal of Economic Dynamics and Control, 26/3, p. 423-435.

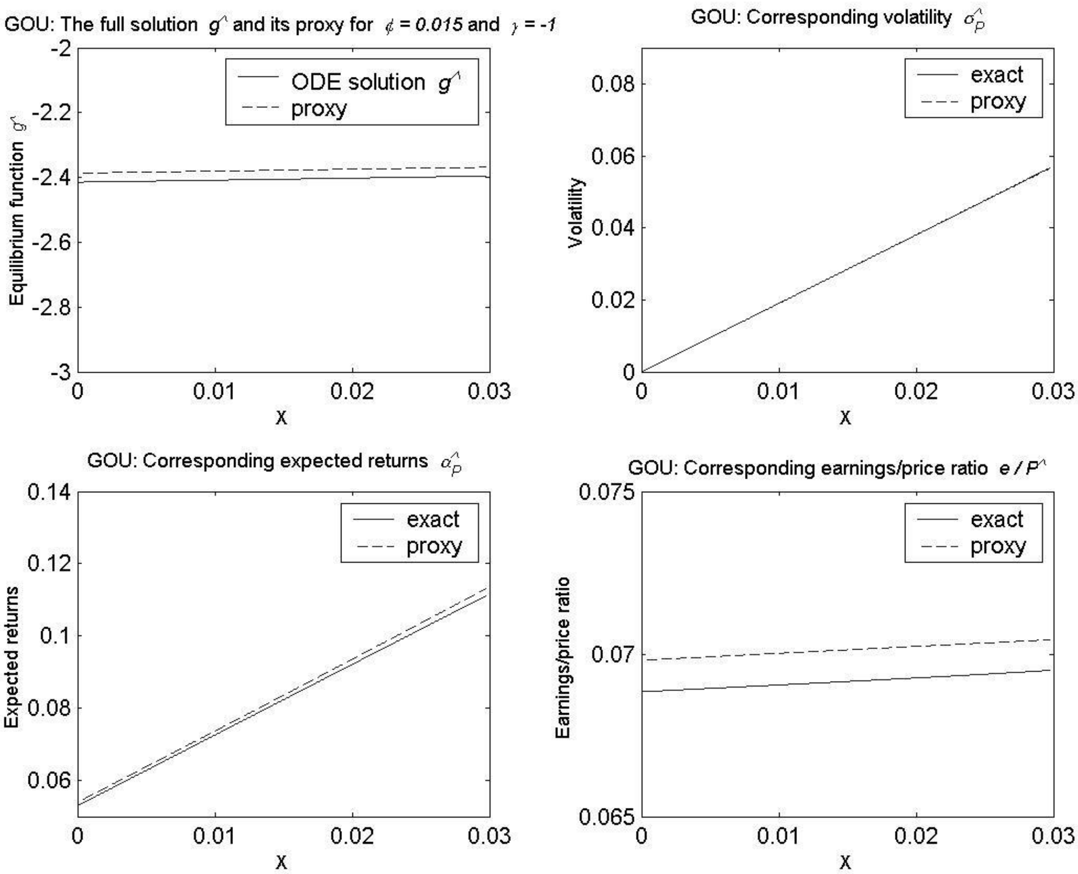
[25] Uppal, R., and T. Wang (2000) . Model misspecification and under-diversification, Working paper, http:// www.london.edu/ faculty/ ruppal.

[26] Veronesi, P., (1999) . Stock Market Overreaction To Bad News In Good Times: A Rational Expectation Equilibrium Model, Review of Financial Studies, 12, 975-1007.

[27] Veronesi, P., (2000) . How Does Information Quality Affect Stock Returns?, Journal of Finance, 55, 807-837.

[28] Veldkamp, L., (2001) . Slow Boom, Sudden Crash, INSEAD working paper #2002/27/EPS, http:// faculty.insead.fr/ veldkamp/.

Figure **1**: **GOU Example: Quality of asymptotics**

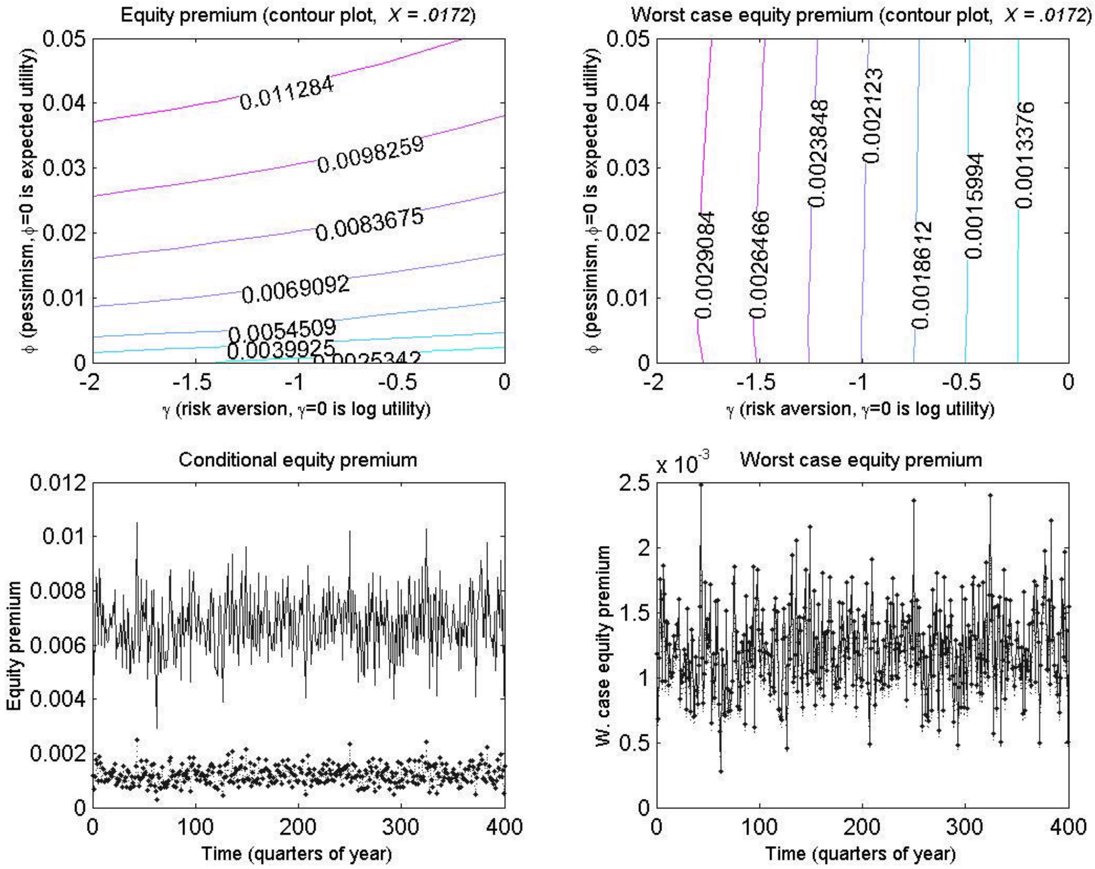


The figure compares the equilibrium $O\left(\gamma^2\right)$ $-$asymptotic proxy $\widehat{g}_{\log,\varphi}+\gamma\widehat{g}_1$ to the exact numerical solution $\widehat{g}$ of the Ordinary Differential Equation (ODE) in Theorem 6. $\widehat{g}$ is a key component of the equilibrium value function of the pessimistic representative agent, $\widehat{J}\left(\widehat{P},X\right)=(1/\delta)\left(\left(e^{\widehat{g}(X)}\widehat{P}\right)^{\gamma}-1\right)/\gamma$. The analytical equilibrium $O\left(\gamma^3\right)$ $-$asymptotics of equity returns volatility $\widehat{\sigma}_P$, expected return $\widehat{\alpha}_P$, and earnings price ratio $\frac{e}{P}$ are also compared to the corresponding exact equilibrium quantities. The economy has Geometric Ornstein Uhlenbeck (GOU) state dynamics for the conditional mean and volatility of dividend growth, and it exhibits constant pessimism, $f(X)=1$. Such dynamics is calibrated to US annual data on changes in the log real consumption. The calibrated dynamics is:

$$
\begin{aligned}
\Delta X &= -3\left(X-0.0172\right)\Delta t+1\cdot X\Delta Z^X,\\
\Delta\ln e &= X\Delta t+1.9070\cdot X\left[0.1\Delta Z^X+\sqrt{1-0.1^2}\Delta Z^e\right].
\end{aligned}
$$

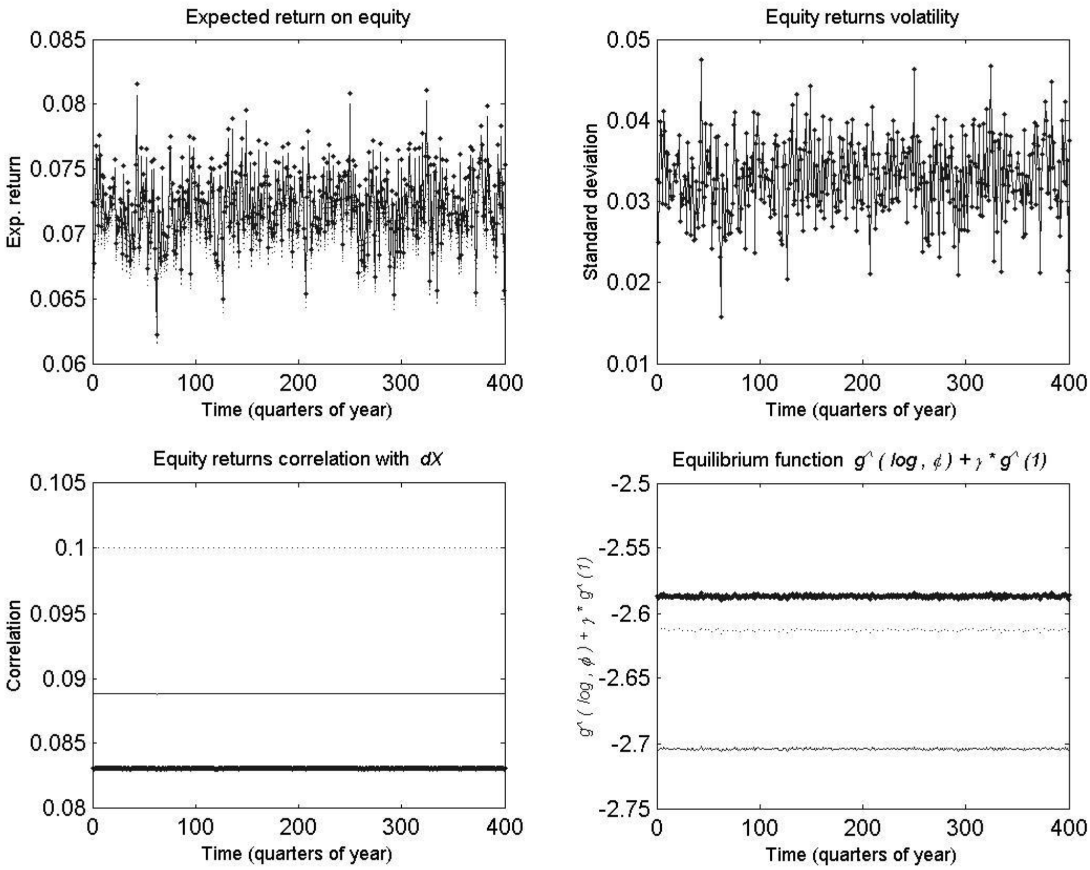
Δt is the weekly frequency, $1/48$. The time preference parameter δ is 0.053.

Figure **2**: **GOU Example: Conditional equity premia**



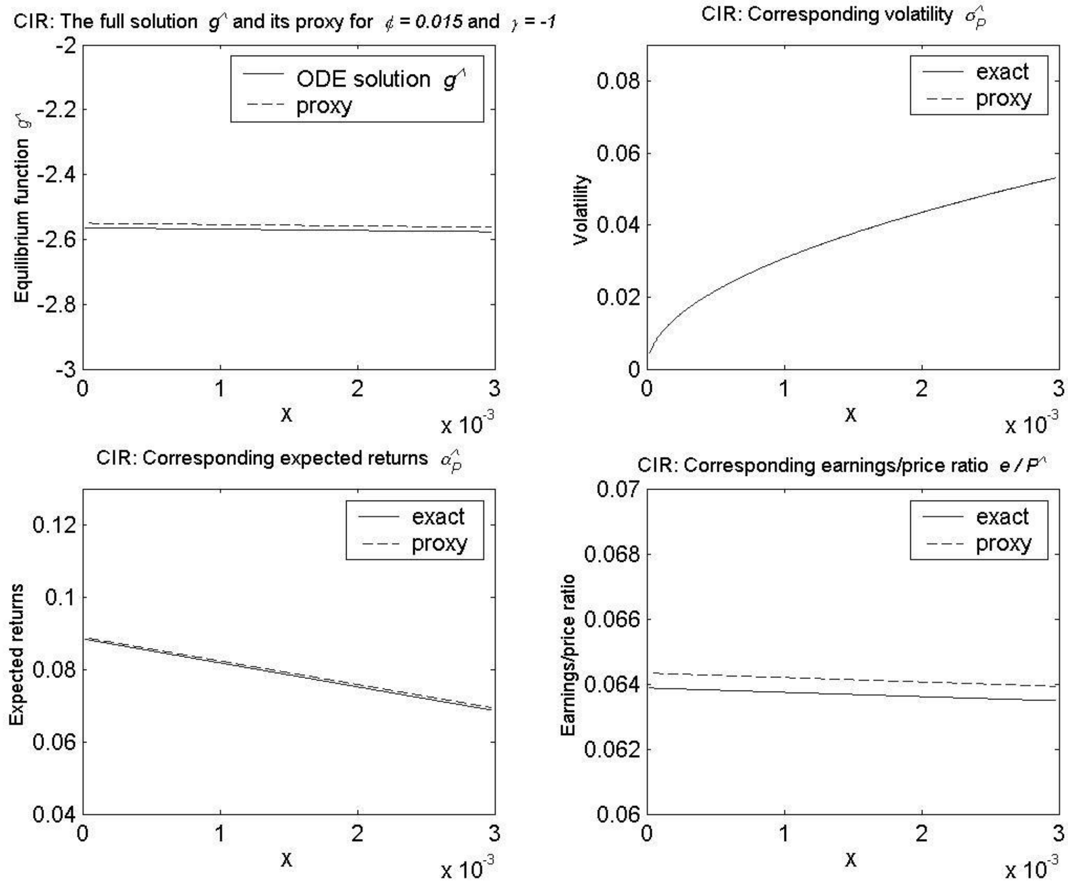
The analytical $O\left(\gamma^3\right)$ −equilibrium asymptotics of equity premium, $\widehat{\alpha}_P - r$, and worst-case equity premium, $(\widehat{\alpha}_P - r)_{\widehat{h}^*}$, are shown. The economy has Geometric Ornstein Uhlenbeck (GOU) state dynamics for the conditional mean and volatility of dividend growth, and it exhibits constant pessimism, $f(X) = 1$. The calibrated dynamics is the same of Figure **1**. Δt is the weekly frequency, $1/48$. 400 quarterly observations are selected from a century-long time series. The time preference parameter δ is 0.053. The dotted lines in the graphs correspond to the case $(\gamma = 0, \varphi = 0)$, the black-diamond lines to the case $(\gamma = -0.1, \varphi = 0)$, and the straight lines to the case $(\gamma = -0.1, \varphi = 0.015)$.

Figure **3**: **GOU Example: Equity returns and function** $\widehat{g}$



The analytical $O\left(\gamma^3\right)-$ asymptotics of the conditional moments of equity returns, $\frac{d\widehat{P}+edt}{\widehat{P}}$, and the analytical $O\left(\gamma^2\right)-$ asymptotics of the equilibrium function $\widehat{g}$ are shown. $\widehat{g}$ is a key component of the equilibrium value function of the pessimistic representative agent, $\widehat{J}\left(\widehat{P},X\right)=(1/\delta)\left(\left(e^{\widehat{g}(X)}\widehat{P}\right)^{\gamma}-1\right)/\gamma$. The time preference parameter δ is 0.053. The economy has GOU state dynamics for the conditional mean and volatility of dividend growth, and it exhibits constant pessimism, $f(X)=1$. The calibrated dynamics is the same of Figure **1**. The dotted lines in the graphs correspond to the case $(\gamma=0,\varphi=0)$, the black-diamond lines to the case $(\gamma=-0.1,\varphi=0)$, and the straight lines to the case $(\gamma=-0.1,\varphi=0.015)$.

Figure 4: **CIR Example: Quality of asymptotics**

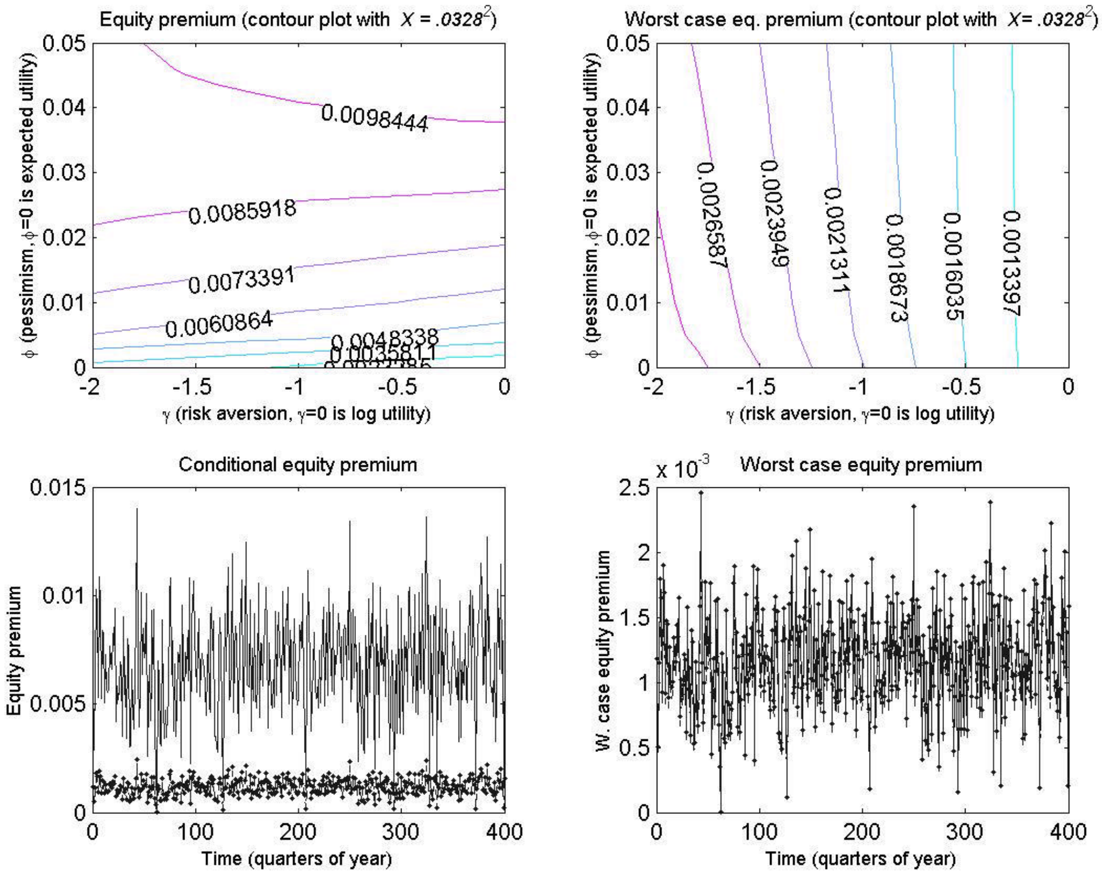


The figure compares the equilibrium $O\left(\gamma^2\right)-$asymptotic proxy $\widehat{g}_{\log,\varphi}+\gamma\widehat{g}_1$ to the exact numerical solution $\widehat{g}$ of the Ordinary Differential Equation (ODE) in Theorem 6. $\widehat{g}$ is a key component of the equilibrium value function of the pessimistic representative agent, $\widehat{J}\left(\widehat{P},X\right)=(1/\delta)\left(\left(e^{\widehat{g}(X)}\widehat{P}\right)^{\gamma}-1\right)/\gamma$. The analytical equilibrium $O\left(\gamma^3\right)-$asymptotics of equity returns volatility $\widehat{\sigma}_P$, expected return $\widehat{\alpha}_P$, and earnings price ratio $\frac{e}{P}$ are also compared to the corresponding exact equilibrium quantities. The economy has Cox Ingersoll Ross (CIR) state dynamics for for the conditional variance of dividend growth, and it exhibits time-varying pessimism, $f(X)=\sqrt{\frac{X}{\bar{X}}}$. Such dynamics is calibrated to US annual data on changes in the log real consumption. The calibrated dynamics is:

$$\begin{aligned}\Delta X &= -3\left(X-(0.0328)^2\right)\Delta t+0.0783\cdot\sqrt{X}\Delta Z^X,\\ \Delta\ln e &= \left(0.0177-0.5\cdot 1^2\cdot X\right)\Delta t+1\cdot\sqrt{X}\left[-0.25\cdot\Delta Z^X+\sqrt{1-(-0.25)^2}\Delta Z^e\right].\end{aligned}$$

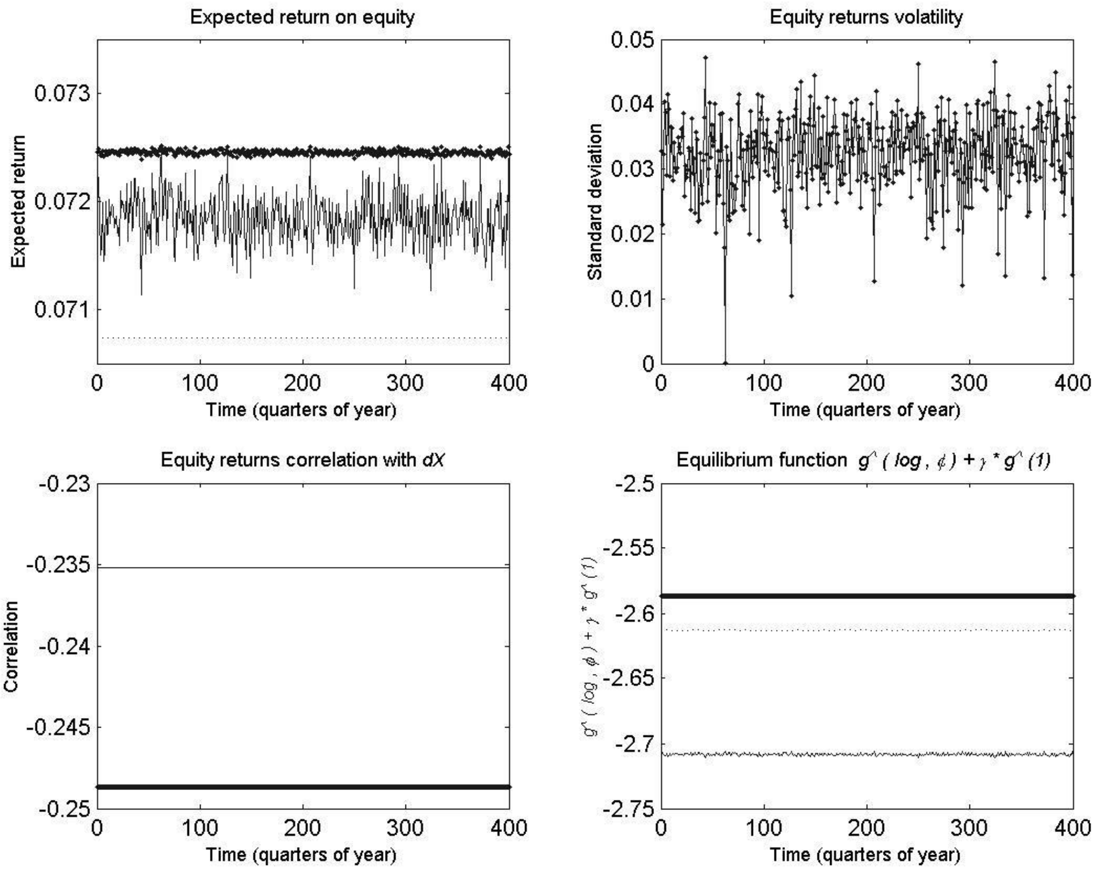
Δt is the weekly frequency, $1/48$. The time preference parameter δ is 0.053.

Figure **5**: **CIR Example: Conditional equity premia**



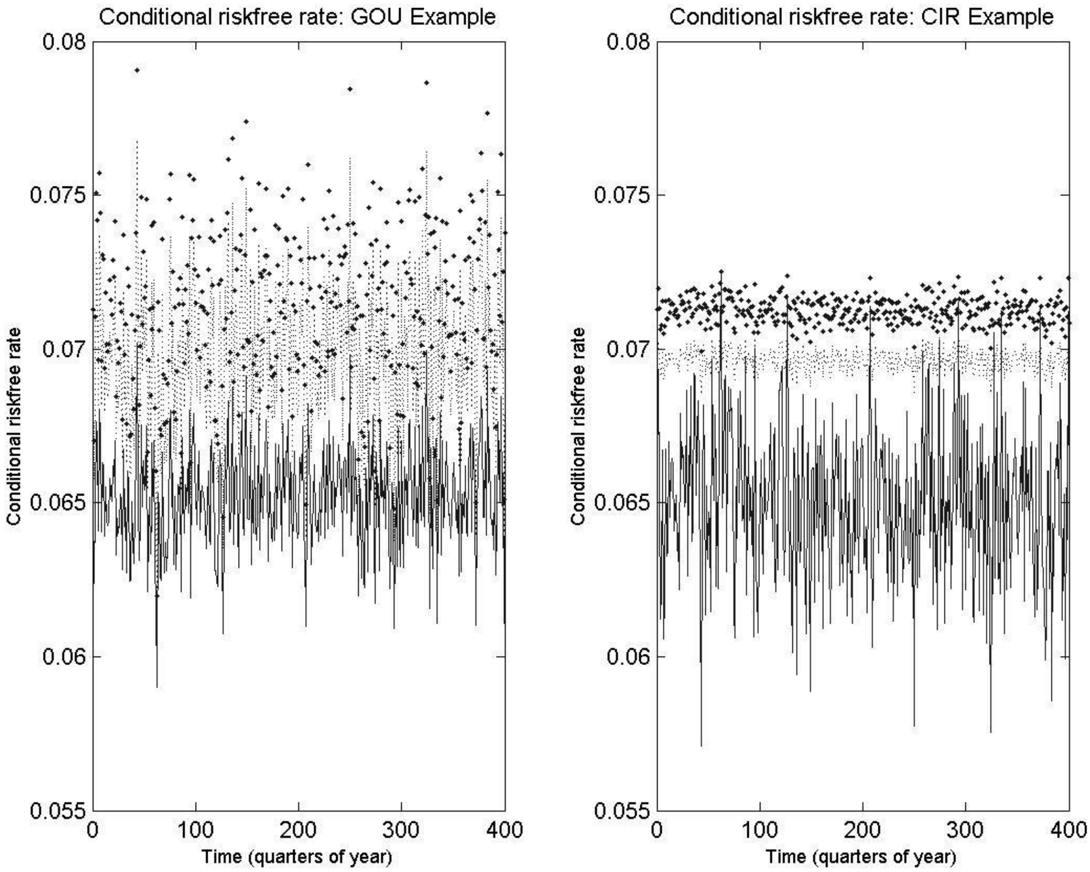
The analytical $O\left(\gamma^3\right)$ −asymptotics of equity premium, $\widehat{\alpha}_P - r$, and worst-case equity premium, $(\widehat{\alpha}_P - r)_{\widehat{h}^*}$, are shown. The economy has Cox Ingersoll Ross (CIR) state dynamics for for the conditional variance of dividend growth, and it exhibits time-varying pessimism, $f(X) = \sqrt{\frac{X}{\overline{X}}}$. The calibrated dynamics are the ones of Figure **4**. Δt is the weekly frequency, $1/48$. 400 quarterly observations are selected from a century-long time series. The time preference parameter δ is 0.053. The dotted lines in the graphs correspond to the case $(\gamma = 0, \varphi = 0)$, the black-diamond lines to the case $(\gamma = -0.1, \varphi = 0)$, and the straight lines to the case $(\gamma = -0.1, \varphi = 0.015)$.

Figure **6**: **CIR Example: Equity returns and the function $\widehat{g}$**



The analytical $O\left(\gamma^3\right)-$ asymptotics of the conditional moments of equity returns, $\frac{d\widehat{P}+edt}{\widehat{P}}$, and the analytical $O\left(\gamma^2\right)-$ asymptotics of the equilibrium function $\widehat{g}$ are shown. $\widehat{g}$ is a key component of the equilibrium value function of the pessimistic representative agent, $\widehat{J}\left(\widehat{P},X\right)=(1/\delta)\left(\left(e^{\widehat{g}(X)}\widehat{P}\right)^{\gamma}-1\right)/\gamma$. The time preference parameter δ is 0.053. The economy has CIR state dynamics for the conditional variance of dividend growth and time varying pessimism, $f\left(X\right)=\sqrt{\frac{X}{\overline{X}}}$. The calibrated dynamics are the ones of Figure **4**. The dotted lines in the graphs correspond to the case $(\gamma=0,\varphi=0)$, the black-diamond lines to the case $(\gamma=-0.1,\varphi=0)$, and the straight lines to the case $(\gamma=-0.1,\varphi=0.015)$.

Figure **7**: **Conditional riskfree rates**



The analytical $O\left(\gamma^3\right)-$ asymptotics of the conditional riskfree rates, $\widehat{r}$, are shown. The time preference parameter δ is 0.053. In the left panel, the economy has GOU state dynamics for the conditional mean and volatility of dividend growth, and it exhibits constant pessimism, $f\left(X\right)=1$. In the right panel, the economy has CIR state dynamics for the conditional variance of dividend growth and time varying pessimism, $f\left(X\right)=\sqrt{\frac{X}{\overline{\overline{X}}}}$. The calibrated dynamics are the ones of Figures **1** and **4**, respectively. The dotted lines in the graphs correspond to the case $\left(\gamma=0,\varphi=0\right)$, the black-diamond lines to the case $\left(\gamma=-0.1,\varphi=0\right)$, and the straight lines to the case $\left(\gamma=-0.1,\varphi=0.015\right)$.