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Discussion paper

The Shapley value for partition function form games[1]

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Abstract: Different axiomatic systems for the Shapley value can be found in the literature. For games with a coalition structure, the Shapley value also has been axiomatized in several ways. In this paper, we discuss a generalization of the Shapley value to the class of partition function form games. The concepts and axioms, related to the Shapley value, have been extended and a characterization for the Shapley value has been provided. Finally, an application of the Shapley value is given.

Key words: partition function form game, coalition structure, Shapley value, unanimity game.

1 Introduction

One of the interests for those who plan to make up a cooperative relationship is how to share their joint profits. The Shapley value (Shapley, 1953) has been proven to be a useful solution concept for cooperative TU games as it provides a recommendation for the division of the joint profits of the grand coalition, which satisfies some reasonable properties. However, considering an economy with externalities one can not easily recommend a division of the joint profits in the same way. For example, in the context of a symmetric Cournot model with linear cost and demand, Selten (1973) showed that the connection between the number of competitors and the tendency to cooperate depends on specific institutional assumptions about possibilities of cooperation. If firms are free to form enforceable quota cartels then cartels (coalitions) may or may not include all firms in the market. The cooperative possibilities of a coalition are derived from equilibrium points of an associated non-cooperative game. In this situation, one can not employ the usual concept of a game in characteristic function form (TU game) to predict the outcome (solution) as the final profits depend on the coalition structure which has been formed. This feature, however, has been captured in the concept of *partition function form games* due to Thrall and Lucas (1963): a partition function assigns a value to each pair consisting of a coalition and a coalition structure which includes that coalition.

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Based on the axioms which characterize the Shapley value (Shapley, 1953) for cooperative TU games, there are apparently many ways to extend the Shapley value to games in partition function form (see, for example, Myerson (1977), Bolger (1989), Potter (2000)). Myerson (1977) derived an efficient value which is a natural extension of the Shapley value based on three simple axioms. Bolger (1989) derived an efficient value which assigns zero to dummies and assigns nonnegative values to players in monotone simple games, whereas Potter (2000) modified the regular concept of the dummy player which allows the dummy player to bring nonnegative worth to the game. All of them are in some way extensions of the Shapley value for cooperative TU games.
This paper studies another extension of the Shapley value for the class of partition function form games. The efficient value we define is different from previous authors. The key idea here is to construct a value which is the average of a collection of marginal vectors. We present a simple formula for calculating the Shapley value of partition function form games, using a decomposition in unanimity games.
The paper is organized as follows. We first briefly recall the main basic features of partition function form games in the next section. The Shapley value and unanimity games are introduced in section 3 and section 4. The properties of the solution concept are studied in section 5. Section 6 discusses an example, demonstrating how the Shapley value can be applied.

2 Preliminaries

Let $N = \{1, 2..., n\}$ be the finite set of players. Nonempty subsets of N are called coalitions. A *partition* κ of N, a so-called *coalition structure*, is a set of disjoint coalitions, $\kappa = \{S_1, ..., S_m\}$, so that their union is N. Let $\mathbb{P}(N)$ be the set of all partitions of N. For any subset $S \subseteq N$, the set of all partitions of S is denoted by $\mathbb{P}(S)$. A typical element of $\mathbb{P}(S)$ is denoted by κ_S.
A pair (S, κ) which consists of a coalition S and a partition κ of N to which S belongs is called an *embedded coalition.* Let $E(N)$ denote the set of embedded coalitions, i.e.

$$E(N) = \{(S, \kappa) \in 2^N \times \mathbb{P}(N) |\ S \in \kappa\}.$$

Definition 2.1 A mapping

$$w : E(N) \longrightarrow \mathbb{R}$$

that assigns a real value, $w(S, \kappa)$, to each embedded coalition (S, κ) is called a *partition function*. The ordered pair (N, w) is called a *partition function form game.* The set of partition function form games with player set N is denoted by $PFFG^N$.

The value $w(S, \kappa)$ represents the payoff of coalition S, given that coalition structure κ forms. For a given partition $\kappa = \{S_1, S_2, ..., S_m\}$ and a partition function w, let $\overline{w}(S_1, S_2, ..., S_m)$ denote the m-vector $(w(S_i, \kappa))_{i=1}^m$. It will be convenient to economize brackets and suppress the commas between elements of the

same coalition. Thus, we will write, for example, $w(\{ijk\},\{\{ijk\},\{lh\}\})$ as $w(ijk,\{ijk,lh\})$, and $\overline{w}(\{ikj\},\{lh\})$ as $\overline{w}(ijk,lh)$. For a partition $\kappa \in \mathbb{P}(N)$ and $i \in N$, we denote the coalition in κ to which player i belongs by $S(\kappa,i)$.
The typical partition which consists of *singleton coalitions* only, $\kappa = \{\{1\},\{2\},...,\{n\}\}$, is denoted by $[N]$, whereas the partition, which consists of the grand coalition only is denoted by $\{N\}$. For any subset $S \subseteq N$, let $[S]$ denote the typical partition which consists of the singleton elements of S, i.e.$[S] = \{\{j\}| j \in S\}$.

Definition 2.2 A *solution concept* on $PFFG^N$ is a function Ψ, which associates with each game (N,w) in $PFFG^N$ a vector $\Psi(N,w)$ of individual payoffs in $\mathbb{R}^n$, *i.e.* $\Psi(N,w) = (\Psi_i(N,w))_{i\in N} \in \mathbb{R}^n$.

3 The Shapley value

The aim of this section is to generalize the Shapley value to the class of partition function form games. In order to do so we first recall the definition of the Shapley value for the class of TU games. A *cooperative* TU *game* is a pair (N,v), where N is the finite set of players and $v(S) \in \mathbb{R}$ is the worth of coalition $S \subseteq N$, with the convention that $v(\phi) = 0$.
Let $\Pi(N)$ be the set of all *bijections* $\sigma : \{1,2,...,n\} \to N$ of N. For a given $\sigma \in \Pi(N)$ and $i \in \{1,2,...,n\}$ we define $S_i^\sigma = \{\sigma(1),\sigma(2),...,\sigma(i)\}$, and $S_0^\sigma = \emptyset$. We construct the vector $m^\sigma(v)$, which corresponds to the situation where the players enter a room one by one in order $\sigma(1),\sigma(2),...,\sigma(n)$ and where each player is given the marginal contribution he/she creates by entering. Formally, it is the vector in $\mathbb{R}^n$ defined by $m^\sigma_{\sigma(i)}(v) = v(S_i^\sigma) - v(S_{i-1}^\sigma)$, for any $i \in \{1,2,...,n\}$. The *Shapley value* $\phi(v)$ is equal to the average of the marginal vectors, i.e.

$$\phi(v) = (n!)^{-1} \sum_{\sigma\in\Pi(N)} m^\sigma(v). \tag{3.1}$$

In order to introduce the Shapley value for partition function form games we need some more notation. For a given $\sigma \in \Pi(N)$ and $i \in \{1,2,...,n\}$, we define the partition κ_i^σ associated with σ and i, by $\kappa_i^\sigma = \{S_i^\sigma\} \cup [N \backslash S_i^\sigma]$. So, in κ_i^σ the coalition S_i^σ has already formed, whereas all other players still form singleton coalitions. Furthermore, we define $\kappa_0^\sigma = [N]$.

For a game in partition function form we shall define the marginal vectors as follows. The marginal vector $m^\sigma(w)$ of a partition function form game (N,w) again corresponds to a situation, where the players enter a room one by one in the order $\sigma(1),\sigma(2),...,\sigma(n)$. The first player according to σ, i.e. $\sigma(1)$, receives $m^\sigma_{\sigma(1)}(w) = w(\{\sigma(1)\},[N]) = w(S_1^\sigma,\kappa_1^\sigma)$. If the second player, $\sigma(2)$, joins then the two players together can get $w(S_2^\sigma,\kappa_2^\sigma)$ and the marginal contribution of player $\sigma(2)$ to coalition S_2^σ is

$$m^\sigma_{\sigma(2)}(w) := w(S_2^\sigma,\kappa_2^\sigma) - w(S_1^\sigma,\kappa_1^\sigma).$$

Similarly, the marginal contribution of the k^{th} player $\sigma(k)$, with $k \geq 3$, to coalition S_k^σ is

$$m_{\sigma(k)}^\sigma(w) := w(S_k^\sigma, \kappa_k^\sigma) - w(S_{k-1}^\sigma, \kappa_{k-1}^\sigma).$$

Based on these marginal vectors $\{m^\sigma(w)\}_{\sigma\in\pi(N)}$, we define the *Shapley value* Φ of the partition function form game (N,w) as the average of the $n!$ marginal vectors,

$$\Phi(w) = \frac{1}{n!} \sum_{\sigma\in\Pi(N)} m^\sigma(w), \tag{3.2}$$

just like its counterpart for TU-games (c.f. Shapley, 1953).

Example 3.1 Consider the partition function form game (N, w) defined by $\overline{w}(1,2,3) = (0,0,0)$, $\overline{w}(12,3) = (2,0)$, $\overline{w}(23,1) = (3,2)$, $\overline{w}(13,2) = (2,1)$, $\overline{w}(123) = 10$.

The marginal vectors are:

$$\begin{array}{llcl}
\text{if} & \sigma_1 & = & (1,2,3) \text{ then } m^{\sigma_1}(w) = (0,2,8) \\
\text{if} & \sigma_2 & = & (2,1,3) \text{ then } m^{\sigma_2}(w) = (2,0,8) \\
\text{if} & \sigma_3 & = & (1,3,2) \text{ then } m^{\sigma_3}(w) = (0,8,2) \\
\text{if} & \sigma_4 & = & (2,3,1) \text{ then } m^{\sigma_4}(w) = (7,0,3) \\
\text{if} & \sigma_5 & = & (3,1,2) \text{ then } m^{\sigma_5}(w) = (2,8,0) \\
\text{if} & \sigma_6 & = & (3,2,1) \text{ then } m^{\sigma_6}(w) = (7,3,0).
\end{array}$$

So, the Shapley value $\Phi(w) = (3, 3.5, 3.5)$. One can verify that the value, introduced by Potter (2000) as well as the value introduced by Bolger (1989), yields the vector $(3.25, 3.5, 3.25)$ for this game.[2] The difference between our value and Bolger's value stems from the fact that Bolger was considering a different collection of marginal vectors. The value, introduced by Potter, is obtained by considering the sum of an "average worth" of coalitions.

4 Unanimity games

In this section we will introduce unanimity games for the class of partition function form games as a generalization of unanimity games for the class of TU games. We establish a decomposition theorem, which states that every partition function form game can be written in a unique way as a linear combination of unanimity games. First, we recall the corresponding concepts for TU games. For $S \subseteq N$ the *unanimity game* (N, u_S) is defined by

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T \\ 0, & \text{otherwise} \end{cases}$$

[2] For $n = 3$, the value introduced by Potter coincides with Bolger's value (Potter, 2000).

for all $T \subseteq N$.

The unanimity games $\{(N, u_S) |\ S \in 2^N \backslash \{\phi\}\}$ form a basis for the class of all TU games with player set N. The unique linear expansion of a characteristic function v in terms of unanimity games is given by

$$v = \sum_{S \in 2^N \backslash \{\phi\}} c_S u_S, \text{ where } c_S = \sum_{T : T \subseteq S} (-1)^{|S|-|T|} v(T).$$

We will now extend the various notions for TU games to partition function form games.

Let $\tau = (S, \kappa)$ and $\tau' = (S', \kappa')$ be two embedded coalitions of N. We say that τ is a *generalized subset* of τ', denoted by $\tau \sqsubseteq \tau'$, if the two following conditions hold

(i) $S \subseteq S'$
(ii) for every two players $i, j \in N \backslash S'$, $S(\kappa, i) \neq S(\kappa, j)$ if and only if $S(\kappa', i) \neq S(\kappa', j)$.

So, an embedded coalition $\tau = (S, \kappa)$ is a generalized subset of $\tau' = (S', \kappa')$ if $S \subseteq S'$ and if κ' is the partition which results from partition κ by merging the players in $S' \backslash S$ with S.

Example 4.1 Let $N = \{1, 2, 3, 4, 5, 6\}$, and $\tau = (123, \{123, 45, 6\}) \in E(N)$. Then τ is a generalized subset of $\tau' = (1234, \{1234, 5, 6\})$ but τ is not a generalized subset of $\tau'' = (1234, \{1234, 56\})$.

Now we will define *unanimity games* for partition function form games.

Definition 4.1 Let $\tau = (S, \kappa) \in E(N)$ be an embedded coalition. The *unanimity game* w_τ, *corresponding to* τ, is defined by

$$w_\tau(\tau') = \begin{cases} 1, & \text{if } \tau \sqsubseteq \tau' \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

for every $\tau' \in E(N)$.

Example 4.2 Let $N = \{1, 2, 3\}$. Let $\kappa_1 = [N], \kappa_2 = \{12, 3\}, \kappa_3 = \{13, 2\}$, $\kappa_4 = \{23, 1\}$, $\kappa_5 = \{N\}$, and let $\tau_1 = (1, \kappa_1)$, $\tau_2 = (2, \kappa_1)$, $\tau_3 = (3, \kappa_1)$, $\tau_4 = (12, \kappa_2)$, $\tau_5 = (3, \kappa_2)$, $\tau_6 = (13, \kappa_3)$, $\tau_7 = (2, \kappa_3)$, $\tau_8 = (23, \kappa_4)$, $\tau_9 = (1, \kappa_4)$, $\tau_{10} = (123, \kappa_3)$. The Table 1 gives the values of $w_\tau(\tau')$ for all embedded coalitions τ and τ'.

Table 1. The values of unianimity games w_τ.

$\tau \setminus \tau'$	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6	τ_7	τ_8	τ_9	τ_{10}
τ_1	1	0	0	1	0	1	0	0	0	1
τ_2	0	1	0	1	0	0	0	1	0	1
τ_3	0	0	1	0	0	1	0	1	0	1
τ_4	0	0	0	1	0	0	0	0	0	1
τ_5	0	0	0	0	1	1	0	1	0	1
τ_6	0	0	0	0	0	1	0	0	0	1
τ_7	0	0	0	1	0	0	1	1	0	1
τ_8	0	0	0	0	0	0	0	1	0	1
τ_9	0	0	0	1	0	1	0	0	1	1
τ_{10}	0	0	0	0	0	0	0	0	0	1

We can now prove, similarly to the case of TU games, that the unanimity games form a basis for the class of partition function form games.

Lemma 4.1 *If (N, w) is a partition function form game, then there exist uniquely determined real numbers μ_τ, $\tau \in E(N)$, such that*

$$w = \sum_{\tau \in E(N)} \mu_\tau w_\tau. \tag{4.2}$$

These numbers are given by $\mu_\tau = \sum_{\eta:\eta \sqsubseteq \tau} (-1)^{|\tau|-|\eta|} w(\eta)$, where $|\tau|$ denotes the cardinality of coalition T in an embedded coalition $\tau = (T, \kappa)$.

Proof. It is suffices to show for the μ_τ, specified in the lemma, that $w = \sum_{\tau \in E(N)} \mu_\tau w_\tau$. Let $\tau' = (S', \kappa') \in E(N)$. We have

$$\begin{aligned}
\sum_{\tau \in E(N)} \mu_\tau w_\tau(\tau') &= \sum_{\tau:\tau \sqsubseteq \tau'} \mu_\tau \\
&= \sum_{\tau:\tau \sqsubseteq \tau'} \Big(\sum_{\eta:\eta \sqsubseteq \tau} (-1)^{|\tau|-|\eta|} w(\eta) \Big) \\
&= \sum_{\eta:\eta \sqsubseteq \tau'} \Big(\sum_{\tau:\eta \sqsubseteq \tau \sqsubseteq \tau'} (-1)^{|\tau|-|\eta|} \Big) w(\eta).
\end{aligned}$$

Now, let $\eta = (S, \kappa)$ be such that $\eta \sqsubseteq \tau' (= (S', \kappa'))$ and consider the expression $\sum_{\tau:\eta \sqsubseteq \tau \sqsubseteq \tau'} (-1)^{|\tau|-|\eta|}$. Note that for every S'' with $S \subseteq S'' \subseteq S'$ there is precisely one partition $\kappa'' \in \mathbb{P}(N)$ such that for $\tau = (S'', \kappa'')$ we have $\eta \sqsubseteq \tau \sqsubseteq \tau'$.
So,

$$\begin{aligned}
\sum_{\tau:\eta \sqsubseteq \tau \sqsubseteq \tau'} (-1)^{|\tau|-|\eta|} &= \sum_{S'':S \subseteq S'' \subseteq S'} (-1)^{|S''|-|S|} \\
&= \sum_{k=0}^{|S' \setminus S|} \binom{|S' \setminus S|}{k} (-1)^k.
\end{aligned}$$

If $\eta = \tau'$ we clearly have $\sum_{\tau:\eta\sqsubseteq\tau\sqsubseteq\tau'}(-1)^{|\tau|-|\eta|} = 1$.
If $\eta \neq \tau'$, then $S \subseteq S', S \neq S'$. Hence,

$$\sum_{\tau:\eta\sqsubseteq\tau\sqsubseteq\tau'}(-1)^{|\tau|-|\eta|} = (1-1)^{|S'\setminus S|} = 0.$$

Therefore, we can conclude that $\sum_{\tau\in E(N)}\mu_\tau w_\tau(\tau') = w(\tau')$ for all $\tau' \in E(N)$, which finishes the proof. ■

The following example shows the linear expansion of a partition function form game (N, w) with respect to the unanimity games w_τ.

Example 4.3 Consider the partition function form game (N, w) in Example 3.1. Calculating the numbers of μ_τ, we have $\mu_{\tau_i} = 0$ for $i = 1, 2, 3, 4, 6$, $\mu_{\tau_j} = 1$ for $j = 5, 7, 8, 9$, and $\mu_{\tau_{10}} = 6$. So, the decomposition of w is given by

$$w = w_{\tau_5} + w_{\tau_7} + w_{\tau_8} + w_{\tau_9} + 6w_{\tau_{10}}.$$

5 The characterization

In this section we characterize the Shapley value for partition function form games, that we introduced in the previous section, by efficiency, additivity, symmetry and the null player property.
For $S \subseteq N$, $i, j \notin S$ and $k \in S$, we denote $S_{+i} = S \cup \{i\}$, $S_{+i,j} = S \cup \{i, j\}$, and $S_{-k} = S\setminus\{k\}$.

Definition 5.1 Let (N, w) be a partition function form game and $i \in N$. We say that player i is a *null player* if for all $\kappa_{N\setminus\{i\}} \in \mathbb{P}(N\setminus\{i\})$ and $S \in \kappa_{N\setminus\{i\}}$,

$$w(S, \kappa_{N\setminus\{i\}} \cup \{\{i\}\}) = w(S_{+i}, (\kappa_{N\setminus\{i\}}\setminus\{S\}) \cup \{S_{+i}\}).$$

Definition 5.2 Given a partition function form game $(N, w) \in PFFG$, we say that two players i and j are *symmetric* if for all $\kappa_{N\setminus\{i,j\}} \in \mathbb{P}(N\setminus\{i, j\})$ and $S \in \kappa_{N\setminus\{i,j\}}$,

$$w(S_{+i}, (\kappa_{N\setminus\{i,j\}}\setminus S) \cup \{\{j\}\} \cup \{S_{+i}\}) = w(S_{+j}, (\kappa_{N\setminus\{i,j\}}\setminus S) \cup \{\{i\}\} \cup \{S_{+j}\}).$$

Let $\Psi : PFFG^N \to \mathbb{R}^n$ be a solution for $PFFG^N$. The solution concept Ψ
(i) is called *efficient (EFF)* if

$$\sum_{i=1}^{n}\Psi_i(w) = w(N, \{N\}) \text{ for all } w \in PFFG^N \;;$$

(ii) is called *symmetric (SYM)* if for all $w \in PFFG^N$, and for all symmetric players i, j in (N, w), we have $\Psi_i(w) = \Psi_j(w)$;

(iii) satisfies the *null player property (NP)* if for all $w \in PFFG^N$, and for

all $i \in N$ such that player i is a null player in (N, w), we have $\Psi_i(w) = 0$;

(iv) satisfies *additivity (ADD)* if for any two games (N, w_1) and (N, w_2) in $PFFG^N$ we have $\Psi(w_1 + w_2) = \Psi(w_1) + \Psi(w_2)$. Here $w_1 + w_2$ is defined by $(w_1 + w_2)(S, \kappa) = w_1(S, \kappa) + w_2(S, \kappa)$ for every $(S, \kappa) \in E(N)$.

Theorem 1 *The Shapley value satisfies EFF, SYM, ADD, and NP.*

Proof. (i) *EFF:* Let $w \in PFFG^N$, then we have

$$\begin{aligned}
\sum_{i=1}^{n} \Psi_i(w) &= \sum_{i=1}^{n} \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m_i^{\sigma}(w) \\
&= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} \sum_{i=1}^{n} m_i^{\sigma}(w) \\
&= \frac{1}{n!} \sum_{\sigma \in \Pi(N)} w(N, \{N\}) = w(N, \{N\}).
\end{aligned}$$

(ii) *SYM*: Let $w \in PFFG^N$ and let i, j be symmetric players in *(N,w)*. Let $\sigma \in \Pi(N)$ and let $\sigma_{ij} \in \Pi(N)$ be the permutation which is obtained by interchanging in σ the positions of i and j. Since i and j are symmetric one easily finds that $m_i^{\sigma}(w) = m_j^{\sigma_{ij}}(w)$. Since σ_{ij} ranges over all permutations if σ does, and the Shapley value is the average of all marginal vectors, we get $\Psi_i(w) = \Psi_j(w)$.
(iii) *ADD*: Obvious.
(iv) *NP*: Obtained from the fact that if player k is a null player in (N, w) then $m_k^{\sigma}(w) = 0$ for every $\sigma \in \Pi(N)$. ■

Theorem 2 *There is a unique solution on PFFG^N satisfying EFF, ADD, SYM and NP. This solution is the Shapley value.*

Proof. (i) From Theorem 1, it follows that the Shapley value satisfies *EFF, ADD, SYM* and *NP.*
(ii) Conversely, suppose ψ satisfies the four properties. We have to show that $\psi = \Phi$.
Let w be a partition function form game on N. Then

$$w = \sum_{\tau} \mu_{\tau} w_{\tau} \text{ with } \mu_{\tau} = \sum_{\eta : \eta \sqsubseteq \tau} (-1)^{|\tau| - |\eta|} w(\eta).$$

By the additivity property we have

$$\psi(w) = \sum_{\tau} \psi(\mu_{\tau} w_{\tau}) \text{ and } \Phi(w) = \sum_{\tau} \Phi(\mu_{\tau} w_{\tau}).$$

So, it suffices to show that for all τ and $\mu_{\tau} \in \mathbb{R}$ we have $\psi(\mu_{\tau} w_{\tau}) = \Phi(\mu_{\tau} w_{\tau})$. Let $\tau = (S, \kappa) \in E(N)$ and $\mu_{\tau} \in \mathbb{R}$. For $i \notin S$, let $\kappa_{N \setminus \{i\}} \in \mathbb{P}(N \setminus \{i\})$ and $T \in \kappa_{N \setminus \{i\}}$. Let τ' denote the embedded coalition $(T, \kappa_{N \setminus \{i\}} \cup \{\{i\}\})$ and

τ''denote the embedded coalition $(T_{+i}, (\kappa_{N\backslash\{i\}}\backslash T) \cup \{T_{+i}\})$. One easily verifies that $\tau \sqsubseteq \tau'$ if and only if $\tau \sqsubseteq \tau''$, so $\mu_\tau w_\tau(\tau') = \mu_\tau w_\tau(\tau'')$. Hence, i is a null player of $(N,\mu_\tau w_\tau)$. Therefore, by the NP property, we have

$$\psi_i(\mu_\tau w_\tau) = \Phi_i(\mu_\tau w_\tau) = 0 \text{ for all } i \notin S. \tag{5.1}$$

For any two players $i, j \in S$, $i \neq j$, let $\kappa_{N\backslash\{i,j\}} \in \mathbb{P}(N\backslash\{i,j\})$, and $T \in \kappa_{N\backslash\{i,j\}}$. Denote by τ' the embedded coalition $(T_{+i}, (\kappa_{N\backslash\{i,j\}}\backslash T) \cup \{\{j\}\} \cup \{T_{+i}\}$ and by τ'' the embedded coalition $(T_{+j}, (\kappa_{N\backslash\{i,j\}}\backslash T) \cup \{\{i\}\} \cup \{T_{+j}\}$. One can see that τ is not a generalized subset of τ' and not a generalized subset of τ''. So, $\mu_\tau w_\tau(\tau') = 0 = \mu_\tau w_\tau(\tau'')$. Therefore, i and j are symmetric players in $(N,\mu_\tau w_\tau)$. Thus, by *SYM,*

$$\Phi_i(\mu_\tau w_\tau) = \Phi_j(\mu_\tau w_\tau) \text{ for all } i, j \in S \tag{5.2}$$

and similarly

$$\psi_i(\mu_\tau w_\tau) = \psi_j(\mu_\tau w_\tau) \text{ for all } \ i, j \in S \tag{5.3}$$

Therefore, *EFF* and (5.1)-(5.3) imply that

$$\Phi_i(\mu_\tau w_\tau) = f_i(\mu_\tau w_\tau) = |S|^{-1}\mu_\tau \text{ for all } i \in S. \ \blacksquare$$

As a corollary of Theorem 2 we get an alternative description of the Shapley value for partition function form games.

Corollary 5.1 *The Shapley value of a partition function form game (N, w) can be written as*

$$\Phi_i(w) = \sum_{\tau=(S,\kappa):i\in S} |S|^{-1}\mu_\tau \textit{ for all } i \in N.$$

6 An illustrative example

In this section we will apply the Shapley value to oligopoly games in partition function form. Particularly, we focus our attention on a linear oligopoly market of a homogeneous good with asymmetric costs, no fixed costs and no capacity constraints. Such an oligopoly is defined by the vector $(b; c) \in \mathbb{R}_+^{n+1}$, where $b > 0$ is the intercept of the inverse demand function, $c = (c_1, c_2, ..., c_n) \geq 0$ is the marginal cost vector. Without loss of generality, assume $c_1 \leq c_2 \leq ... \leq c_n$. We also assume that an equilibrium price always exceeds the largest marginal cost, i.e. $\frac{b+\sum_{j=1}^n c_j}{n+1} > c_n$.[3] For each supply (input) vector $x = (x_1, x_2, ..x_n)$, the price is $p(x) = b - \sum_{i=1}^n x_i$, whereas player i's cost and profit (payoff) are $C_i(x_i) = c_i x_i$ and

$$\pi_i(x) = p(x)x_i - C_i(x_i) = (b - \sum_{i=1}^n x_i)x_i - c_i x_i. \tag{6.1}$$

[3] This assumption is equivalent to the requirement of positive market shares at the equilibrium for all players (Zhao, 2001).

Player i's reaction curve is implicitly defined by the first order condition:

$$\frac{\partial \pi_i(x)}{\partial x_i} = p(x) - c_i - x_i = 0, \text{ or } 2x_i = b - c_i - \sum_{j \neq i} x_j. \tag{6.2}$$

A Cournot-Nash equilibrium is a vector such that each player's action x_i is a best response to the complementary choice $x_{-i} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. This equilibrium is graphically the intersection point of all reaction curves and algebraically the solution of the system of equations (6.2). The unique equilibrium, $x^* = (x_1^*, x_2^*, ..., x_n^*)$, is determined by

$$x_i^* = \frac{b - (n+1)c_i + \sum_{j=1}^{n} c_j}{n+1}$$

and the payoff of player i at this equilibrium is

$$\pi_i(x^*) = (x_i^*)^2 = \frac{[b - (n+1)c_i + \sum_{j=1}^{n} c_j]^2}{(n+1)^2}. \tag{6.3}$$

Now suppose that after sufficient communication, some players may agree to cooperate (for example, players intend to adjust negative externalities which are caused by decreasing returns to inputs). In such a situation a coalition structure might form, in which, however, the payoff of coalition S depends on the behaviour of the players outside S. Notice that the payoff for coalition S under one coalition structure is different from that under another coalition structure if the number of coalitions is different. Assume that the marginal cost of coalition S is $c_S = \min_{i \in S} c_i$ (i.e. a coalition's most efficient technology can be costlessly adopted by all players in S). Moreover, if a coalition structure $\kappa = \{S_1, S_2, ..., S_k\}$ is formed, then in equilibrium each coalition S in κ will choose the total (input) quantity levels to maximize the sum of its members' profits, given the total inputs of the other coalitions in κ.
Let $x_{S_j} = \sum_{i \in S_j} x_i$ denote the total input level for a coalition S_j and $\pi_{S_j}(x)$ denote the profit of coalition S_j under structure κ,

$$\pi_{S_j}(x) = p(x)x_{S_j} - C_S(x_{S_j}) = (b - \sum_{i=1}^{k} x_{S_i})x_{S_j} - c_{S_j}x_{S_j}.$$

Coalition S_j's reaction curve under coalition structure κ is also implicitly defined by the first order condition:

$$\frac{\partial \pi_{S_j}(x)}{\partial x_{S_j}} = p(x) - c_{S_j} - x_{S_j} = 0, \text{ or } 2x_{S_j} = b - c_{S_j} - \sum_{i \neq j} x_{S_i}.$$

The unique equilibrium under the structure κ, $(x_{S_1}^*, x_{S_2}^* ..., x_{S_k}^*)$, and the equilibrium profit $\pi_{S_j}(x^*)$ of coalition S_j are defined by

$$x_{S_j}^* = \frac{b - (k+1)c_{S_j} + \sum_{i=1}^{k} c_{S_i}}{k+1},$$

and

$$\pi_{S_j}(x^*) = \frac{[b - (k+1)c_{S_j} + \sum_{i=1}^{k} c_{S_i}]^2}{(k+1)^2}.$$

The oligopoly game in partition function form (N, w) is determined for every (S_j, κ) by $w(S_j, \kappa) = \pi_{S_j}(x^*)$, where x^* is the equilibrium vector under structure κ.
To get further illustration of how the Shapley value can be used we specify the 3-person oligopoly game in partition function form (N, w). The partition function form game is given by $\overline{w}(1,2,3) = (\alpha, \beta, \gamma)$, $\overline{w}(12,3) = (a_1, b_1)$, $\overline{w}(13,2) = (a_2, b_2)$, $\overline{w}(23,1) = (b_2, a_2)$, $\overline{w}(123) = g$, where

$$\begin{aligned} \alpha &= \frac{1}{16}(b - 3c_1 + c_2 + c_3)^2, \\ \beta &= \frac{1}{16}(b - 3c_2 + c_1 + c_3)^2, \\ \gamma &= \frac{1}{16}(b - 3c_3 + c_1 + c_2)^2, \\ a_1 &= \frac{1}{9}(b - 2c_1 + c_3)^2, \ b_1 = \frac{1}{9}(b - 2c_3 + c_1)^2 \\ a_2 &= \frac{1}{9}(b - 2c_1 + c_2)^2, \ b_2 = \frac{1}{9}(b - 2c_2 + c_1)^2 \\ g &= \frac{1}{4}(b - c_1)^2. \end{aligned} \tag{6.4}$$

Given the ordering of marginal costs, one can easily see that $\alpha \geq \beta \geq \gamma$, and $a_1 \geq a_2 \geq b_2 \geq b_1$. The Shapley value of this game, $\Phi(w) = (\Phi_i(w))_{i=1,2,3}$, can be computed as follows:

$$\begin{aligned} \Phi_1(w) &= \frac{g}{3} + 2g_1 + g_2 \\ \Phi_2(w) &= \frac{g}{3} - g_1 + g_2 \\ \Phi_3(w) &= \frac{g}{3} - g_1 - 2g_2 \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} g_1 &= \frac{1}{6}(a_2 - b_2 + \alpha - \beta) \geq 0, \\ g_2 &= \frac{1}{6}(a_1 - a_2 + \beta - \gamma) \geq 0. \end{aligned} \tag{6.6}$$

From (6.4) one can see that if players have identical costs, then $\alpha = \beta = \gamma$, $a_1 = a_2 = b_1 = b_2$ so $g_1 = g_2 = 0$. The Shapley value gives an equal payoff to all players, i.e. $\Phi_i(w) = \frac{g}{3}$. Now increase the marginal costs of players 2 and 3 by the same amount, i.e. $c_1 \leq c_2 = c_3$. Then $a_1 = a_2$, $\beta = \gamma$ so $g_2 = 0$. Thus, due to this increase of costs, the Shapley value reduces the payoff for player 2 and 3 by g_1, whereas the payoff of player 1 is increased by $2g_1$. A further increase of the marginal cost of player 3 alone reduces, according to the Shapley value, the payoff of player 3 by $2g_2$, whereas the payoffs of players 1 and 2 are increased by g_2.

Example 6.1 The game in partition function form (N, w) associated with a linear oligopoly market $(b; c)$, where $b = 20, c = (1, 3, 4)$, given by
$\overline{w}(1, 2, 3) = (36, 16, 9)$, $\overline{w}(12, 3) = (53.78, 18.78)$,
$\overline{w}(13, 2) = (49, 25)$, $\overline{w}(23, 1) = (25, 49)$, $\overline{w}(123) = (90.25)$.

The Shapley value for this game is $\Phi(w) = (46.70, 24.71, 18.83)$. This value indicates the different payoffs due to the different costs of players. If players have identical costs, i.e. $c = (1, 1, 1)$, then $\Phi(w) = (30.08, 30.08, 30.08)$. If the cost of players 2 and 3 increase by 2 units, i.e. $c = (1, 3, 3)$, then $\Phi(w) = (40.42, 24.92, 24.92)$. Hereby, $g = 90.25$, $g_1 = 7.33$, $g_2 = 1.96$.

For n-person oligopoly games in partition function form, the generalization of the observations above is straightforward.

References

[1] Bolger, E.M.(1989) A set of axioms for a value for partition function games. *International Journal of Game Theory* **18:** 37-44.

[2] Myerson, R. (1977) Values for games in partition function form. *International Journal of Game Theory* **6:** 23-31

[3] Potter, A. (2000) *A value for partition function form games*. Working paper. Dept. of Mathematics, Hardin-Simmons University Abilene, Texas, USA.

[4] Selten, R. (1973) A simple model of imperfect competition, where 4 are few and 6 are many. *International Journal of Game Theory* **2**: 141-201.

[5] Shapley, L (1953) A value for n-person games, in *Contributions to the Theory of Games II*, H.W. Kuhn and A.W.Tucker (eds), Annals of Mathematics Studies, vol.28, 307-312, Princeton University Press.

[6] Thrall, R.M. and Lucas, W.F (1963) n-person games in partition function form. *Naval Research Logistic Quarterly* **10**: 281-293

[7] Zhao, J. (2001) A characterization for the negative welfare effects of cost reduction in Cournot oligopoly. *International Journal of Industrial Organization* **19**: 455-469.