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Discussion paper

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**Abstract**. Potential games and supermodular games are attractive games, especially because under certain conditions they possess pure Nash equilibria. Subclasses of games with a potential are considered which are also strategically equivalent to supermodular games. The focus is on two-person zero-sum games and two-person Cournot games.

**Keywords**: Pure Nash equilibrium, potential game, supermodular game, Cournot game, zero-sum game.

**JEL classification**: C72, C73.

## 1. Introduction

The aim of this paper is to investigate two interesting classes of games for which the existence of pure Nash equilibria is obtained under certain conditions, namely:

i) the class of potential games (Monderer and Shapley, 1996);

ii) the class of supermodular games (Topkis, 1998).

The question tackled here is whether there are games belonging to both classes. It turns out that two-person zero-sum supermodular games are potential games and

conversely that two-person zero-sum potential games can be transformed in a canonical way into supermodular games. Also Cournot games are, under special conditions, members of both classes of games.

A connection between ordinal potential games (Monderer and Shapley, 1996) and supermodular games is also established for certain Cournot games.

In Section 2 the definitions of potential games and of supermodular games are recalled together with some of their properties. In Section 3 the case of two-person zero-sum games is discussed and an example illustrating the connection between the two classes of games is given. Section 4 deals with Cournot duopoly competition and Cournot games. Section 5 contains some concluding remarks.

## 2. Preliminaries

Let $< A, B, K, L >$ be a *two-person game* with strategy space $A$ for player 1, strategy space $B$ for player 2, and $K \colon A \times B \mapsto \mathbb{R}$, $L \colon A \times B \mapsto \mathbb{R}$ the payoff function of player $1, 2$ respectively. If the players 1 and 2 choose $a \in A$ and $b \in B$ respectively, then player 1 obtains a payoff $K(a, b)$ and player 2 obtains $L(a, b)$.

A *Nash equilibrium* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \le K(\hat{a}, \hat{b})$ for each $a \in A$ and $L(\hat{a}, b) \le L(\hat{a}, \hat{b})$ for each $b \in B$.

Such a game is called a *potential game* (Monderer and Shapley, 1996) if there is a (*potential*) *function* $P \colon A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b), \quad \text{for all } a_1, a_2 \in A \text{ and for each } b \in B,$$

$$L(a, b_1) - L(a, b_2) = P(a, b_1) - P(a, b_2), \quad \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

Clearly, elements of $\text{argmax}(P)$ are Nash equilibria of the game.

The next lemma will be useful. It states that for a two-person potential game the payoff function of player 1 (player 2) can be written as the sum of a potential and a function on the Cartesian product of the strategy spaces, which only depends on the strategy choice of player 2 (player 1). This is a known result (Slade, 1994; Facchini et al., 1997); an alternative proof is given here.

**Lemma 1**. Let $< A, B, K, L >$ be a potential game with potential $P$. Then there exist functions $f \colon A \mapsto \mathbb{R}$ and $g \colon B \mapsto \mathbb{R}$ such that

$$K(a, b) = P(a, b) - 2g(b),$$

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$$L(a, b) = P(a, b) - 2f(a)$$

for each $a \in A$ and $b \in B$.

Proof. Take $a^* \in A$, $b^* \in B$ and define $f$ and $g$ as follows. For each $a \in A$ and $b \in B$, let

$$f(a) = 1/2\big(P(a, b^*) - L(a, b^*)\big),$$

$$g(b) = 1/2\big(P(a^*, b) - K(a^*, b)\big).$$

Since $P$ is a potential for the game $< A, B, K, L >$, we have

$$K(a, b) - K(a^*, b) = P(a, b) - P(a^*, b) \ \ \text{or} \ K(a, b) - P(a, b) = -2g(b),$$

and also

$$L(a, b) - L(a, b^*) = P(a, b) - P(a, b^*) \ \ \text{or} \ L(a, b) - P(a, b) = -2f(a)$$

for all $a \in A$ and $b \in B$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The game $< A, B, K, L >$ is called an *ordinal potential game* (Monderer and Shapley, 1996) if there is a ( potential) function $P \colon A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) > 0 \Longleftrightarrow P(a_2, b) - P(a_1, b) > 0, \ \text{for all } a_1, a_2 \in A \text{ and for each } b \in B$$

$$L(a, b_1) - L(a, b_2) > 0 \Longleftrightarrow P(a, b_1) - P(a, b_2) > 0, \ \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

We will use the following

**Proposition 1** (Monderer and Shapley, 1996). Let $< A, B, K, L >$ be a two-person game. Let $A, B$ be intervals of real numbers and $K, L$ be twice continuously differentiable functions. Then $< A, B, K, L >$ is a potential game if and only if

$$\frac{\partial^2 K}{\partial a \partial b} = \frac{\partial^2 L}{\partial a \partial b} \ .$$

For more information on potential games see Voorneveld (1999) and Mallozzi, Tijs and Voorneveld (2000).

Let us now recall some definitions related to supermodular games. A *partially ordered set* is a set $X$ on which there is a binary relation $\preceq$ that is reflexive, anti-symmetric and transitive. Let us consider a partially ordered set $X$ and a subset $X'$ of $X$. If $x' \in X$ and $x' \preceq x$ for each $x \in X'$, then $x'$ is a *lower bound* for $X'$; if $x'' \in X$ and $x \preceq x''$ for each $x \in X'$, then $x''$ is an *upper bound* for $X'$. When the set

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of upper bounds of $X'$ has a least element, then this least upper bound of $X'$ is the *supremum* of $X'$ in $X$; when the set of lower bounds of $X'$ has a greatest element, then this greatest lower bound of $X'$ is the *infimum* of $X'$ in $X$.

If two elements $x_1$ and $x_2$ of a partially ordered set $X$ have a supremum in $X$, it is called the *meet* of $x_1$ and $x_2$ and is denoted by $x_1 \wedge x_2$; if $x_1$ and $x_2$ have an infimum in $X$, it is called the *join* of $x_1$ and $x_2$ and is denoted by $x_1 \vee x_2$. A partially ordered set that contains the join and the meet of each pair of its elements is a *lattice*. If a subset $X'$ of a lattice $X$ contains the join and the meet (with respect to $X$) of each pair of elements of $X'$, then $X'$ is a *sublattice* of $X$. The real line $\mathbb{R}$ with the natural ordering denoted by $\leq$ is a lattice with $x \vee y = max\{x, y\}$ and $x \wedge y = min\{x, y\}$ for $x, y \in \mathbb{R}$, and $\mathbb{R}^n$ ($n > 1$) with the natural partial ordering denoted by $\leq$ is a lattice with $x \vee y = (x_1 \vee y_1, ..., x_n \vee y_n)$ and $x \wedge y = (x_1 \wedge y_1, ..., x_n \wedge y_n)$ for $x, y \in \mathbb{R}^n$. Any subset of $\mathbb{R}$ is a sublattice of $\mathbb{R}$, and a subset $X$ of $\mathbb{R}^n$ is a sublattice of $\mathbb{R}^n$ if it has the property that $x, y \in X$ imply that $(max\{x_1, y_1\}, ..., max\{x_n, y_n\})$ and $(min\{x_1, y_1\}, ..., min\{x_n, y_n\})$ are in $X$.

The game $< A, B, K, L >$ is called a *supermodular game* (Topkis, 1998) if the following three properties are satisfied:

1) $A$ is a sublattice of $\mathbb{R}^{m_1}$ and $B$ is a sublattice of $\mathbb{R}^{m_2}$ for some $m_1 \in \mathbb{N}$, $m_2 \in \mathbb{N}$;

2) $K, L$ have increasing differences on $A \times B$, i.e. for all $(a_1, a_2) \in A^2$ and for all $(b_1, b_2) \in B^2$ such that $a_1 \geq a_2$ and $b_1 \geq b_2$,

$$K(a_1, b_1) - K(a_1, b_2) \geq K(a_2, b_1) - K(a_2, b_2),$$

$$L(a_1, b_1) - L(a_2, b_1) \geq L(a_1, b_2) - L(a_2, b_2);$$

3) $K$ is supermodular in the first coordinate and $L$ is supermodular in the second coordinate, i.e. for each $b \in B$, for all $a_1, a_2 \in A$ we have

$$K(a_1, b) + K(a_2, b) \leq K(a_1 \vee a_2, b) + K(a_1 \wedge a_2, b)$$

and for each $a \in A$, for all $b_1, b_2 \in B$ we have

$$L(a, b_1) + L(a, b_2) \leq L(a, b_1 \vee b_2) + L(a, b_1 \wedge b_2).$$

We recall the following propositions:

**Proposition 2** (Topkis, 1998). Let $f$ be a differentiable function on $\mathbb{R}^n$, then $f$ has increasing differences on $\mathbb{R}^n$ if and only if $\dfrac{\partial f}{\partial x_i}$ is increasing in $x_j$ for all distinct $i$ and $j$ and all $x$.

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**Proposition 3** (Topkis, 1998). Let $f$ be a twice differentiable function on $\mathbb{R}^n$, then $f$ has increasing differences on $\mathbb{R}^n$ if and only if $\dfrac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$, for all distinct $i$ and $j$.

The following two examples show that the classes of potential games and supermodular games do not coincide. So the study of special subclasses becomes interesting.

**Example 1.** Let $A = B = [0,1]$ and $K(a,b) = 2ab$, $L(a,b) = a + b$ for all $a, b \in [0,1]$. Then the game $< A, B, K, L >$ is a supermodular game because $A$ and $B$ are sublattices of $\mathbb{R}$, $K, L$ have increasing differences on $[0,1] \times [0,1]$, and $K$ is supermodular in the first coordinate and $L$ in the second coordinate. This game is not an exact potential game because the condition in Proposition 1 is not satisfied since $\dfrac{\partial^2 K}{\partial a \partial b} = 2 \neq \dfrac{\partial^2 L}{\partial a \partial b} = 0$. Let us remark that the game is an ordinal potential game with potential function $P$ given by $P(a,b) = a + b$ for all $a, b \in [0,1]$.

On the other hand there are games that are exact potential games and not supermodular games:

**Example 2.** Let $A = B = [0,1]$ and $K(a,b) = a^2 - 2a(b - \frac{1}{2})^2 + b$, $L(a,b) = -2a(b-\frac{1}{2})^2$ for all $a, b \in [0,1]$. Then the game $< A, B, K, L >$ is a potential game with potential function $P$ given by $P(a,b) = a^2 - 2a(b-\frac{1}{2})^2$ for all $a, b \in [0,1]$ but it is not a supermodular game in view of Proposition 3 because $\dfrac{\partial^2 K}{\partial a \partial b} = -4(b - \frac{1}{2}) < 0$ if $b > \frac{1}{2}$.

## 3. Zero-sum potential games and supermodular games

A two-person game of the form $< A, B, K, -K >$ is called a *zero-sum game*. Such a game will be denoted by $< A, B, K >$. In a zero-sum game one player pays the other. A *saddle point* for such a game is a point $(\hat{a}, \hat{b}) \in A \times B$ such that $K(a, \hat{b}) \leq K(\hat{a}, \hat{b}) \leq K(\hat{a}, b)$ for each $a \in A$ and $b \in B$. We denote by $S(A, B, K)$ the set of all saddle points of $< A, B, K >$. Note that $< A, B, K >$ is a potential game if there is a (potential) function $P: A \times B \mapsto \mathbb{R}$ such that

$$K(a_2, b) - K(a_1, b) = P(a_2, b) - P(a_1, b), \quad \text{for all } a_1, a_2 \in A \text{ and for each } b \in B,$$

$$-K(a, b_1) + K(a, b_2) = P(a, b_1) - P(a, b_2), \quad \text{for each } a \in A \text{ and for all } b_1, b_2 \in B.$$

Clearly, elements of $\text{argmax}(P)$ are saddle-points of the game. Also the converse turns out to hold as we see in Remark 2.

Useful will be the following

**Theorem 1**. Let $< A, B, K >$ be a two-person zero-sum game. Then the following assertions are equivalent:

$(1_i)$ $< A, B, K >$ is a potential game;

$(1_{ii})$ there exists a pair of functions $(f, g)$ with $f: A \mapsto \mathbb{R}$ and $g: B \mapsto \mathbb{R}$ such that $K(a, b) = f(a) - g(b)$ for all $a \in A$, $b \in B$ (separation property).

<u>Proof</u>. That $(1_{ii})$ implies $(1_i)$ follows by taking the potential $P$ defined by

$$P(a, b) = f(a) + g(b) \text{ for all } a \in A \text{ and } b \in B.$$

Conversely, suppose $(1_i)$. Then by Lemma 1, there are functions $f: A \mapsto \mathbb{R}$ and $g: B \mapsto \mathbb{R}$ such that for each $a \in A$ and $b \in B$

$$K(a, b) = P(a, b) - 2g(b), \quad -K(a, b) = P(a, b) - 2f(a).$$

So $K(a, b) = f(a) - g(b)$ for all $(a, b) \in A \times B$. $\qquad\qquad\square$

**Remark 1**. This theorem is also proved in Potters, Raghavan and Tijs (1999), in an alternative way. In that paper it was also observed that for 2x2-subgames of a two-person zero-sum potential game the "diagonal property" holds. This is

$$K(a_1, b_1) + K(a_2, b_2) = K(a_1, b_2) + K(a_2, b_1)$$

for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$. This property follows easily from $(1_{ii})$ in Theorem 1. Conversely, it was proved in Potters, Raghavan and Tijs (1999) that the diagonal property for two-person zero-sum games implies also that the game is a potential game.

**Remark 2**. A pair $(f, g)$ as in $(1_{ii})$ of Theorem 1 is called a *separating pair* for the potential game $< A, B, K >$. For a potential $P$ of this game we have $P(a, b) = c + f(a) + g(b)$ for each $a \in A$, $b \in B$ and some $c \in \mathbb{R}$. Clearly, $(\hat{a}, \hat{b})$ is a saddle point of $< A, B, K >$ if and only if $\hat{a} \in \underset{a \in A}{\operatorname{argmax}} f(a)$, $\hat{b} \in \underset{b \in B}{\operatorname{argmax}} g(b)$ if and only if $(\hat{a}, \hat{b}) \in \operatorname{argmax}(P)$.

Theorem 1 gives us the possibility to connect a two-person zero-sum potential game with a related game where the strategy spaces are ordered subsets of $\mathbb{R}$ and the payoff function satisfies monotonicity conditions.

Given $< A, B, K >$ with potential function $P$ and separating pair $(f, g)$ such that $P(a, b) = f(a) + g(b)$ for all $a \in A$, $b \in B$, define $< \bar{A}, \bar{B}, \bar{K} >$ as follows. Take $\bar{A} = f(A)$, $\bar{B} = g(B)$ and for $(\bar{a}, \bar{b}) \in \bar{A} \times \bar{B}$ let $\bar{K}(\bar{a}, \bar{b}) = \bar{a} - \bar{b}$.

So we use the real valued functions $f \colon A \mapsto \mathbb{R}$ and $g \colon B \mapsto \mathbb{R}$ to find a game $<\bar{A}, \bar{B}, \bar{K}>$ with strategy spaces in $\mathbb{R}$, which is strategically equivalent to $< A, B, K >$ because

$$K(a, b) = \bar{K}(f(a), g(b)) \quad \text{for all } (a, b) \in A \times B,$$

$$\bar{K}(c, d) = K(a, b) \quad \text{for all } a \in f^{-1}(c), \ b \in g^{-1}(d).$$

From this follows

$$(a, b) \in S(A, B, K) \Longrightarrow (f(a), g(b)) \in S(\bar{A}, \bar{B}, \bar{K}),$$

$$(c, d) \in S(\bar{A}, \bar{B}, \bar{K}) \Longrightarrow (a, b) \in S(A, B, K) \text{ for all } a \in f^{-1}(c), \ b \in g^{-1}(d).$$

The strategy space $\bar{A}$ can be smaller than $A$ because two strategies $a_1$ and $a_2$ in $A$ which are equivalent in the sense that

$$K(a_1, b) = K(a_2, b) \ \text{ for all } b \in B$$

are mapped into the same point $f(a_1) = f(a_2) \in \bar{A}$.

Relations between $< A, B, K >$ and $< \bar{A}, \bar{B}, \bar{K} >$ are described in

**Theorem 2**. Let $< A, B, K >$ a game with potential $P$ and let $< \bar{A}, \bar{B}, \bar{K} >$ be as above. Then

$(2_i)$ $< \bar{A}, \bar{B}, \bar{K} >$ is a potential game with potential $\bar{P} \colon \bar{A} \times \bar{B} \mapsto \mathbb{R}$ such that $\bar{P}(\bar{a}, \bar{b}) = \bar{a} + \bar{b}$ for all $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$;

$(2_{ii})$ $\max(\bar{A}) \times \max(\bar{B}) = \operatorname{argmax}(\bar{P}) = S(\bar{A}, \bar{B}, \bar{K})$;

$(2_{iii})$ $(a, b) \in S(A, B, K) \Longleftrightarrow f(a) = \max(\bar{A}), \ g(b) = \max(\bar{B})$.

Note that $S(\bar{A}, \bar{B}, \bar{K})$ has cardinality 0 or 1.

**Example 3**. Consider the matrix game

|   | L | R | E |
|---|---|---|---|
| T | 8 | 13 | 13 |
| M | 5 | 10 | 10 |
| F | 10 | 15 | 15 |

corresponding to the two-person zero-sum game $< A, B, K >$ where $A = \{T, M, F\}$, $B = \{L, R, E\}$ and $K(T, L) = 8$, $K(T, R) = K(T, E) = 13$, $K(M, L) = 5$, $K(M, R) =$

$K(M, E) = K(F, L) = 10$, $K(F, R) = K(F, E) = 15$. If we take $f : \{T, M, F\} \mapsto \mathbb{R}$ and $g : \{L, R, E\} \mapsto \mathbb{R}$ as follows: $f(T) = 5$, $f(M) = 2$, $f(F) = 7$, $g(L) = -3$, $g(R) = g(E) = -8$, then $K(a, b) = f(a) - g(b)$ for all $a \in A$, $b \in B$ and $P : A \times B \mapsto \mathbb{R}$ with $P(a, b) = f(a) + g(b)$ for all $a \in A$, $b \in B$ is a potential for this matrix game.

Transforming this game to $< \bar{A}, \bar{B}, \bar{K} >$ with the aid of $(f, g)$ results in $\bar{A} = \{2, 5, 7\}$, $\bar{B} = \{-8, -3\}$ and $\bar{K}(\bar{a}, \bar{b}) = \bar{a} - \bar{b}$ or the "monotonic" matrix game

|     | $-8$ | $-3$ |
| --- | --- | --- |
| 2 | 10 | 5 |
| 5 | 13 | 8 |
| 7 | 15 | 10 |

with the unique saddle point in $(7, -3)$ corresponding to maximum 4 of the potential $\bar{P}$ which can be written in matrix form as follows

|     | $-8$ | $-3$ |
| --- | --- | --- |
| 2 | $-6$ | $-1$ |
| 5 | $-3$ | $-2$ |
| 7 | $-1$ | 4 |

Note that $7 = \max(\bar{A})$, $-3 = \max(\bar{B})$.

**Remark 3**. If $max(\bar{A})$ (or $max(\bar{B})$) does not exists, then there are no saddle points. If $K$ is bounded, then there are $\varepsilon$-saddle points for each $\varepsilon > 0$ corresponding to points $(a', b')$ with $P(a', b') \geq supP(a, b) - \varepsilon$.

**Theorem 3**. The game $< \bar{A}, \bar{B}, \bar{K} >$ with $\bar{K}(a, b) = a - b$ for each $a \in \bar{A}$ and $b \in \bar{B}$ is a supermodular game.

<u>Proof</u>. The subsets $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$. For each $b \in \bar{B}$ the function $a \mapsto \bar{K}(a, b)$ is supermodular on $A$ and also $b \mapsto -\bar{K}(a, b)$ is supermodular on $B$ for each $a \in A$. We have finished the proof if we show that for each $a_1, a_2 \in A$, $b_1, b_2 \in B$ the functions

$$a \mapsto \bar{K}(a, b_1) - \bar{K}(a, b_2) \quad (a \in \bar{A})$$

$$b \mapsto -\bar{K}(a_1, b) + \bar{K}(a_2, b) \quad (b \in \bar{B})$$

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are monotonic. This is true because these functions are in fact constant:

$$\bar{K}(a, b_1) - \bar{K}(a, b_2) = -b_1 + b_2,$$

$$-\bar{K}(a_1, b) + \bar{K}(a_2, b) = -a_1 + a_2.$$

□

We have seen in Theorem 3 that two-person zero-sum potential games can be embedded in the family of supermodular games. The converse is treated in

**Theorem 4**. Let $< A, B, K >$ be a two-person zero-sum game with $A \subset \mathbb{R}$, $B \subset \mathbb{R}$, which is supermodular. Then $< A, B, K >$ is a potential game.

<u>Proof</u>. The supermodularity implies that for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $a_1 < a_2, b_1 < b_2$ we have

$$K(a_2, b_2) - K(a_2, b_1) \geq K(a_1, b_2) - K(a_1, b_1)$$

$$-K(a_2, b_2) + K(a_1, b_2) \geq -K(a_2, b_1) + K(a_1, b_1)$$

From these two inequalities follows the diagonal property. Then, according to Remark 1, $< A, B, K >$ is a potential game.

□

**Example 4**. Let $< A, B, K, L >$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(i, j) = i + j$ for all $i \in A$ and $j \in B$, and $L(1, 1) = 4$, $L(1, 2) = 7$, $L(2, 1) = 5$ and $L(2, 2) = 9$. Then this game is a supermodular game but not a potential game.

**Example 5**. Let $< A, B, K, L >$ be the non-zero sum game with $A = \{1, 2\}$, $B = \{1, 2\}$; $K(1, 1) = 3$, $K(1, 2) = 1$, $K(2, 1) = 5$, $K(2, 2) = 2$ and $L(1, 1) = 3$, $L(1, 2) = 8$, $L(2, 1) = 6$, $L(2, 2) = 10$. Then the game is a potential game but not a supermodular game.

**Remark 4**. A subclass of general two-person potential games can be embedded into the class of supermodular games in a similar way as we embedded two-person zero-sum potential games. These are games of the form $< A, B, K, L >$ with separable payoff functions i. e. $K$ and $L$ can be written in the form

$$K(a, b) = f(a) + g(b), \; L(a, b) = h(a) + k(b)$$

for all $a \in A$, $b \in B$, and where $f, h$ are real valued functions on $A$ such that $f$ is injective, and $g, k$ are real valued functions on $B$ such that $k$ is injective. A

potential is then $P$ given by $P(a, b) = f(a) + k(b)$ for each $a \in A$ and $b \in B$. A strategically equivalent supermodular game is the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ where $\bar{A} = f(A)$, $\bar{B} = k(B)$ and where for all $c \in \bar{A}$, $d \in \bar{B}$:

$$\bar{K}(c, d) = K(f^{-1}(c), k^{-1}(d)),$$

$$\bar{L}(c, d) = L(f^{-1}(c), k^{-1}(d)).$$

## 4. Cournot games

Consider Cournot's model of duopoly where the demand arises from a competitive market of a single homogeneous commodity.

Suppose that firm $i$, $i = 1, 2$, can supply the single homogeneous product in any non negative bounded quantity $q_i \in [0, q_i^0]$ with production cost $c_i(q_i)$. The price of the single homogeneous commodity is given by the inverse demand function $Q(q_1, q_2)$ which is assumed to be twice continuously differentiable function. We suppose that firm $i$'s cost $c_i(q_i)$, $i = 1, 2$, is differentiable.

Given the output level selected by the other firm, the objective of firm $i$ is to maximize its profit
$$\Pi_i(q_1, q_2) = q_i Q(q_1, q_2) - c_i(q_i)$$
by the choice of its output $q_i$, where $q_i Q(q_1, q_2)$ expresses the revenue (return) of firm $i$. We assume that the marginal revenue of firm $i$ (i.e. $Q(q_1, q_2) + q_i \dfrac{\partial Q(q_1, q_2)}{\partial q_i}$) is decreasing with respect to $q_j$ $(j \neq i)$.

A *Cournot game* is a game of the form $< A, B, K, L >$ where $A = [0, q_1^0]$, $B = [0, q_2^0]$ and
$$K(a, b) = aQ(a, b) - c_1(a),$$

$$L(a, b) = bQ(a, b) - c_2(b)$$

for all $a \in A$ and $b \in B$. If the inverse demand function $Q$ is linear in $a + b$, then the corresponding Cournot duopoly game is also called a *quasi Cournot game*.

Now we put $\bar{a} = a$ and $\bar{b} = -b$ for each $a \in A$ and $b \in B$ and consider the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ where $\bar{A} = A$, $\bar{B} = -B = [-q_2^0, 0]$ and

$$\bar{K}(\bar{a}, \bar{b}) = K(\bar{a}, -\bar{b}), \;\; \bar{L}(\bar{a}, \bar{b}) = L(\bar{a}, -\bar{b})$$

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for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. So

$$\bar{K}(\bar{a},\bar{b}) = \bar{a}Q(\bar{a},-\bar{b}) - c_1(\bar{a}), \quad \bar{L}(\bar{a},\bar{b}) = -\bar{b}Q(\bar{a},-\bar{b}) - c_2(-\bar{b}).$$

The game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is strategically equivalent to $< A, B, K, L >$ because $\bar{K}(\bar{a},\bar{b}) = K(a,b)$ and $\bar{L}(\bar{a},\bar{b}) = L(a,b)$ for all $a \in A, b \in B$. We will denote by $NE(A, B, K, L)$ the set of all Nash equilibria of the game $< A, B, K, L >$. Note that $(a,b) \in NE(A, B, K, L)$ if and only if $(a,-b) \in NE(\bar{A}, \bar{B}, \bar{K}, \bar{L})$. Moreover if $< A, B, K, L >$ is a Cournot potential game with potential function $P$, then the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ as above is also a potential game with potential $\bar{P}$ given by $\bar{P}(\bar{a},\bar{b}) = P(\bar{a},-\bar{b})$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$.

**Theorem 5**. Let $< A, B, K, L >$ be a Cournot game and consider $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ as above. Then

$(5_i)$ if the cost functions $c_i$ are of the form $c_i(q_i) = cq_i$, for $i = 1, 2$, then $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is an ordinal potential game and also a supermodular game;

$(5_{ii})$ if the inverse demand function $Q$ is linear in the aggregate output level, given by $Q(a,b) = \alpha - \beta(a+b)$, $\alpha, \beta > 0$ (i.e. $< A, B, K, L >$ is a quasi Cournot game), then $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is a potential game and also a supermodular game.

Proof.

$(5_i)$ The Cournot duopoly with cost functions $c_i$, $i = 1, 2$ is an ordinal potential game with potential function $P$ given by $P(a,b) = ab[Q(a,b) - c]$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is also an ordinal potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a},\bar{b}) = P(a,b)$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. Moreover $\bar{K}(\bar{a},\bar{b}) = \bar{a}[Q(\bar{a},-\bar{b}) - c]$ and $\bar{L}(\bar{a},\bar{b}) = -\bar{b}[Q(\bar{a},-\bar{b}) - c]$ satisfy the increasing differences property because by Proposition 3 we have $\frac{\partial^2 \bar{K}}{\partial \bar{a} \partial \bar{b}} = -\frac{\partial}{\partial b}[Q + a\frac{\partial Q}{\partial a}] \geq 0$ and $\frac{\partial^2 \bar{L}}{\partial \bar{a} \partial \bar{b}} = -\frac{\partial}{\partial a}[Q + b\frac{\partial Q}{\partial b}] \geq 0$, since we assumed that the marginal revenue is decreasing. Moreover the transformed strategy spaces $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate and $\bar{L}$ is supermodular in the second coordinate. Then the Cournot game is a supermodular game.

$(5_{ii})$ The quasi Cournot competition is a potential game with potential function $P$ given by $P(a,b) = \alpha(a+b) - \beta(a^2 + b^2) - \beta ab - c_1(a) - c_2(b)$ for all $a \in [0, q_1^0]$ and $b \in [0, q_2^0]$ (Monderer and Shapley, 1996), so the game $< \bar{A}, \bar{B}, \bar{K}, \bar{L} >$ is also a potential game with the potential $\bar{P}$ given by $\bar{P}(\bar{a},\bar{b}) = P(a,b)$ for all $\bar{a} \in \bar{A}$, $\bar{b} \in \bar{B}$. Moreover $\bar{K}(\bar{a},\bar{b}) = \bar{a}[\alpha - \beta(\bar{a} - \bar{b})] - c_1(\bar{a})$ and $\bar{L}(\bar{a},\bar{b}) = -\bar{b}[\alpha - \beta(\bar{a} - \bar{b})] - c_2(-\bar{b})$ satisfy the increasing differences property because by Proposition 3 we have $\frac{\partial^2 \bar{K}}{\partial \bar{a} \partial \bar{b}} =$

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$\frac{\partial^2 \bar{L}}{\partial \bar{a} \partial \bar{b}} = \beta > 0$. As in the previous case, $\bar{A}$ and $\bar{B}$ are sublattices of $\mathbb{R}$, $\bar{K}$ is supermodular in the first coordinate, $\bar{L}$ is supermodular in the second coordinate and the quasi Cournot game is a supermodular game. $\qquad \square$

## 5. Concluding remarks

Let us first summarize the main results we obtained:

i) a supermodular two-person zero-sum game is a potential game (Theorem 4). Conversely, if a two-person zero-sum game is a potential game then it is strategically equivalent to a supermodular game (Theorems 2 and 3), which is monotonic and has at most one saddle point; the set of pure saddle points of a two-person zero-sum potential game turns out to coincide with the potential maximizers (Remark 2);

ii) two subclasses of Cournot games are described, which are strategically equivalent to supermodular games and which are simultaneously (ordinal or exact) potential games (Theorem 5).

In Remark 4 we discussed a subclass of general two-person potential games which can be embedded in the class of supermodular games. This result holds for a similar subclass of general $n$-person strategic games with separable payoff functions.

A game of the form $< A_1, ..., A_n, K_1, ..., K_n >$ where $K_i(a_i, a_{-i}) = f_i(a_i) + g_i(a_{-i})$ for all $a_i \in A_i$ and $a_{-i} \in \Pi_{j \in \mathbb{N} - \{i\}} A_j$ is a potential game and it is strategically equivalent to a supermodular game if $f_1, ..., f_n$ are injective functions. A potential is given by

$$P(a) = \sum_{i=1}^{n} f_i(a_i)$$

and the (strategically equivalent) supermodular game is defined as follows:

- for each $i \in N = \{1, ..., n\}$, $\bar{A}_i = f_i(A_i)$;

- for all $b_1 \in \bar{A}_1, ..., b_n \in \bar{A}_n$ and all $i \in N$

$$\bar{K}_i(b_1, ..., b_n) = K_i(f_1^{-1}(b_1), ..., f_n^{-1}(b_n)).$$

Also duopoly results in Section 4 can be extended to multimarket oligopoly (Topkis, 1998). It is interesting to find other economic situations leading to strategic games which are potential games and also supermodular games.

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