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Abstract

We present a definition of increasing uncertainty, independent of any notion of subjective probabilities, or of any particular model of preferences. Our notion of an elementary increase in the uncertainty of any act corresponds to the addition of an 'elementary bet' which increases consumption by a fixed amount in the (relatively) 'good' states and decreases consumption by a fixed (and possibly different) amount in the (relatively) 'bad' states. This definition naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty.

Keywords: uncertainty, ambiguity, risk, non-expected utility

JEL classifications: C72, D81

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1 Introduction

Most formal analysis of economic decisions under uncertainty has relied on concepts of subjective probability. Significant advances in the discussion of preferences in the absence of well-defined subjective probabilities, and in understanding the relationship between preferences and subjective probabilities, have been made by Schmeidler (1989), Machina and Schmeidler (1992), Epstein (1999) and Ghiradato and Marinacci (1999). A particularly important contribution is that of Epstein and Zhang (2001), who set out a range of desiderata for a definition of ambiguity, and provide a definition meeting most of these desiderata.

The analysis of economic decisions in the absence of well-defined subjective probabilities has often been referred to in terms of Knight's (1921) distinction between risk and uncertainty. However, Knight's discussion of the role of insurance companies and the Law of Large Numbers make it clear that his conception of risk was confined to cases where objective probabilities can be defined in frequentist terms, and where risk can effectively be eliminated through pooling and spreading. All other cases, including those where individuals possessed personal subjective probabilities, were effectively classed by Knight as involving uncertainty. The distinction now commonly drawn between 'risk' and 'uncertainty' could not be developed properly until the formulation of well-defined notions of subjective probability by de Finetti (1937) and Savage (1954).

The first writer to clearly identify cases where preferences were inconsistent with first-order stochastic dominance, relative to any possible probability distribution, was Ellsberg (1961) who distinguished between risk (subjective probabilities satisfying the Savage axioms) and ambiguity, leaving uncertainty as a comprehensive term. Therefore, consistent with the usage of Savage and Ellsberg, and with usage in the general economics literature, we will use the term uncertainty to encompass all decisions involving non-trivial state-contingent outcome vectors, whether or not the preferences and beliefs associated with these decisions can be characterized by well-defined subjective probabilities. Events for which subjective probabilities are (or are not) well-defined will be referred to as 'ambiguous' ('unambiguous') and problems involving acts measurable with respect to unambiguous events will be said to involve 'risk'. Our usage is consistent with Ghiradato and Marinacci (1999) and Epstein and Zhang (2001).

Epstein and Zhang (2001) provide a rigorous definition of ambiguous and unambiguous events, and lay the basis for an analysis of preferences under uncertainty, including both risk and ambiguity.[1] In light of this, the definition proposed by Epstein (1999) for a *comparative*

[1] To the best of our knowledge, the definition proposed by Epstein and Zhang (2001) is the only one that is based solely on preferences and hence model free. In other papers such as Gilboa and Schmeidler (1994), Mukerji (1997), Sarin and Wakker (1998), Ghiradato and Marinacci (1999), Nehring (2001) and

ambiguity aversion relation over preference relations can now be stated in a solely preference-based and model-free manner. However, questions of when one act is more uncertain or more ambiguous than another are not addressed in these analyses, except in the polar case where one act is ambiguous and the other is unambiguous. Ghiradato and Marinacci (1999) propose a *model-free* definition of *comparative uncertainty aversion*: one preference relation is more uncertainty averse than another, if whenever the latter relation expresses a weak preference for a constant act (that is, one that will yield the same outcome no matter what state of the world will obtain) over another act, then so must the former relation. They do not, however, consider the question of when one act is more uncertain than another except in the polar case where one of the acts yields a *certain* outcome.

By contrast, the concept of an increase in risk, and the economic consequences of increases in risk, have been analyzed extensively, beginning with the work of Hadar and Russell (1969), Hanoch and Levy (1969) and Rothschild and Stiglitz (1970). These authors independently derived and characterized the second-order stochastic dominance condition (in terms of mean-preserving spreads), under which all risk-averse expected utility maximizers will prefer one probability distribution to another. Quiggin (1992) introduced an alternative notion of *monotone (mean-preserving) increase in risk*, defined in terms of co-monotonic random variables instead of mean-preserving spreads. Landsberger and Meilijson (1994) pointed out that this notion of increase in risk coincides with the Bickel and Lehmann (1976) notion of *dispersion of random variables with equal means*. Yaari (1969) notes that since any lottery is by definition a 'mean-preserving spread' of its mean, the weakest notion of risk aversion simply requires that the mean of a lottery for sure is weakly preferred to the lottery itself. Subsequent studies examined a wide range of generalizations of these stochastic dominance conditions, typically associated with more restrictive conditions on utility functions. The concept of increasing risk has also been analyzed extensively for generalized expected utility models (Chew, Karni and Safra, 1987; Chateauneuf, Cohen and Meilijson, 1997; Quiggin, 1993; Safra and Zilcha, 1989).

Most concepts of increasing risk that have been considered in the literature are inherently dependent on the existence of well-defined subjective probabilities. This is obviously true of mean-preserving increases in risk, since the mean depends on probabilities. Even notions such as that of a compensated increase in risk (Diamond and Stiglitz, 1974), which do not depend on mean values, incorporate probabilities in their definitions. Yet the intuitive concept of an increase in the uncertainty of a prospect does not seem to depend crucially on probabilities. To take a simple example, doubling the stakes of a bet surely increases the

Ryan (2001), the analysis focuses on a class of preference relations that admit a specific functional form. The criteria for what constitutes an ambiguous or unambiguous event is then defined in terms of a property or properties of the specific functional form representation that these preference relations all admit.

uncertainty associated with that bet, regardless of whether the parties have well-defined and common subjective probabilities regarding the event that is the subject of the bet.[2]

The main object of this paper is to present a definition of increasing uncertainty, independent of any notion of subjective probabilities, or of any particular model of preferences. This definition naturally gives rise to a dual definition of comparative aversion to uncertainty. We characterize this definition for a popular class of generalized models of choice under uncertainty.

An important objective of this work is to extend the economic applicability of the concepts developed by previous writers on this topic. Despite significant progress in characterizing preferences under uncertainty, without reliance on probability concepts, there has been relatively little analysis of the economic choices under uncertainty. One important problem is that comparative static analysis requires the adoption of some notion of an increase in uncertainty, and there is no generally agreed concept of an increase in uncertainty. We briefly outline the implications of our work for comparative static analysis. Proofs of the results, unless otherwise stated, appear in the appendix.

2 Preliminaries

Set-up and Notation. Denote by $\mathcal{S} = \{\dots, s, \dots\}$ a set of states and $\mathcal{E} = \{\dots, A, B \dots, E, \dots\}$ the set of events which is a given σ-field on $\mathcal{S}$. We take the set of outcomes to be the set of non-negative real numbers, or 'consumption levels'. An act is a (measurable) real-valued and bounded function $f : \mathcal{S} \to \mathrm{R}_+$. Let $f(\mathcal{S}) = \{f(s) \mid s \in \mathcal{S}\}$ be the outcome set associated with the act f, that is, the range of f. Let $\mathcal{F} = \{\dots, f, g, h, \dots\}$ denote the set of acts on $\mathcal{S}$; and let $\mathcal{F}_0$ denote the set of simple acts on $\mathcal{S}$; that is, those with finite outcome sets. We will abuse notation and use x to denote both the outcome x in R_+ and the constant act with $f(\mathcal{S}) = \{x\}$.

The following notation to describe an act will be convenient. For an event E in $\mathcal{E}$, and any two acts f and g in $\mathcal{F}$, let $f_E g$ be the act which gives, for each state s, the outcome $f(s)$ if s is in E and the outcome $g(s)$ if s is in the complement of E (denoted $\mathcal{S} \backslash E$).

In general, for any finite partition $\mathcal{P} := \{A^1, \dots, A^n\}$ of $\mathcal{S}$ and any list of n acts $(h^1, \dots, h^n)$, let $h^1_{A^1} h^2_{A^2} \dots h^{n-1}_{A^{n-1}} h^n$ be the act that yields $h^i(s)$ if s is in A^i.

Let $\succsim$ be a binary relation over ordered pairs of acts in $\mathcal{F}$, representing the individual's preferences. Let $\succ$ and $\sim$ correspond to strict preference and indifference, respectively.

[2] Indeed, assuming that parties are risk-averse, the acceptability of a bet to both parties depends on the fact that subjective probabilities differ. In the case where one party is a bookmaker, a bet may be acceptable because of other risks in the portfolios of the contracting parties. However, a bookmaker is merely an intermediary, and bookmaking is only feasible if there are differences in probabilities.

Given $\%$, for any act f in $\mathcal{F}$, we define the 'at least as good as f' set as the set $\%_f = \{g \in \mathcal{F} : g \ \% \ f\}$.

We say a sequence of acts f_n *converges point-wise* in the limit to f, written $f_n \to f$, if, for each s in $\mathcal{S}$, the sequence of real numbers, $f_n(s)$ converges to $f(s)$.

The only maintained assumptions we make on this preference relation is that it is a (point-wise) continuous preference ordering and satisfies a weak form of monotonicity.

Axiom 1 *The preference relation* $\%$ *is a continuous ordering: that is, it is transitive and complete and, for any of sequences acts* $\langle f_n \rangle$ *and* $\langle g_n \rangle$*, such that* $f_n \to f$ *and* $g_n \to g$*, if* $f_n \ \% \ g_n$ *for all* n*, then* $f \ \% \ g$*.*

Axiom 2 *The preference relation* $\%$ *is monotonic. That is, if for any pair of acts,* f *and* g *in* $\mathcal{F}$*,* $f(s) > g(s)$ *for all* s *in* Ω*, then* $f \succ g$*.*

We observe that any preference relation $\%$ satisfying the axioms above may be characterized by a unique certainty equivalent of the form

$$m(f) = sup\{x \in \mathsf{R} : f \ \% \ x\}.$$

2.1 An elementary increase in uncertainty

Our notion of increasing the uncertainty of an act is based on the idea of adding an 'elementary bet' to that act which increases consumption by a fixed amount in the (relatively) 'good' states and decreases consumption by a fixed (and possibly different) amount in the (relatively) 'bad' states. We refer to the addition of such a comonotonic elementary bet as an *elementary increase in uncertainty*.

Definition 1 *Fix a pair of acts* $f, g \in \mathcal{F}$*. The act* g *represents an* elementary increase in uncertainty *of the act* f*, denoted* gUf *if there exists a pair of positive numbers* α *and* β*, and an event* $E^+ \in \mathcal{E} \setminus \{\mathcal{S}, \emptyset\}$ *such that: (i) for all* s *in* E^+*,* $g(s) - f(s) = \alpha$*; (ii) for all* s *in* $\mathcal{S} \setminus E^+$*,* $f(s) - g(s) = \beta$*; and (iii)* $\sup\{f(s) : s \in \mathcal{S} \setminus E^+\} \leq \inf\{f(s) : s \in E^+\}$*.*

Correspondingly, we define a notion of comparative uncertainty aversion:

Definition 2 *Fix* $\%$ *and* $\widehat{\%}$*. The preference relation* $\%$ *is at least as* uncertainty averse *at* f *as* $\widehat{\%}$ *if for any* gUf*,* $f \ \widehat{\%} \ g$ *implies* $f \ \% \ g$*. The preference relation* $\%$ *is* everywhere *at least as* uncertainty averse *as* $\widehat{\%}$ *if for all* f*,* $\%$ *is as least as uncertainty averse at* f *as* $\widehat{\%}$*.*

Before we explore in more detail the implications of this definition, let us first evaluate it against the desiderata set out in Epstein and Zhang (2001). They argue that a definition should be:

D1. Behavioral or expressed in terms of preferences: that is, verifiable in principle given suitable data on behavior.

D2. Model-free: since concepts of uncertainty and ambiguity are more basic than specifications of preferences, a definition should not be tied to any particular model.

D3. Explicit and constructive: Given an event, it should be possible to check whether or not it is ambiguous.

D4. Consistent with probabilistic sophistication on unambiguous events.

Since, wherever relevant, we employ the Epstein and Zhang characterization of ambiguous and unambiguous events, our definition of increasing uncertainty automatically inherits property D4. Also, as with Epstein and Zhang, the degree to which D1 is satisfied is limited by the assumption of an objectively given state-space over which acts are defined as mappings from that state-space to an outcome space. Our main concern, therefore relates to the properties D2 and D3, and particularly with D2, the requirement that the characterization of increasing uncertainty should be model-free. Subject only to the assumption of an objectively given state-space, our definition of an elementary increase in uncertainty is entirely model-free and this model-free status carries over to the definition of comparative uncertainty aversion.

3 Increases in uncertainty and uncertainty aversion

Our first observation about the definition of an elementary increase in uncertainty is that, no matter what assessment an individual attaches to any event (that may incorporate his or her belief and/or decision weight), an elementary increase in uncertainty always reduces consumption in the worst event and increases consumption in the best event. Furthermore, if gUf then g, f and $g-f$ are pairwise *co-monotonic* functions, that is, for every pair of states $s,t \in \mathcal{S}$,

$$(g(s)-g(t))(f(s)-f(t)) \geq 0$$

$$(g(s)-f(s)-g(t)+f(t))(f(s)-f(t)) \geq 0$$

$$(g(s)-g(t))(g(s)-f(s)-g(t)+f(t)) \geq 0$$

which can be more succinctly expressed as

$$(g(s) - g(t))^2 \geq (g(s) - g(t))(f(s) - f(t)) \geq (f(s) - f(t))^2 . \tag{1}$$

It is straightforward to check that for any triple of acts f, g, and h, if hUg and gUf then

$$\begin{aligned}(h(s) - h(t))^2 &\geq (h(s) - h(t))(g(s) - g(t)) \geq (g(s) - g(t))^2 \\ &\geq (g(s) - g(t))(f(s) - f(t)) \geq (f(s) - f(t))^2\end{aligned}$$

and, hence,

$$(h(s) - h(t))^2 \geq (h(s) - h(t))(f(s) - f(t)) \geq (f(s) - f(t))^2 .$$

That is, we have hUf. As nothing in the inequalities in (1) requires the acts in question to be simple, we shall adopt these inequalities to define the *more uncertain* relation between *any* pair of acts.

Definition 3 *Fix a pair of acts $f, g \in \mathcal{F}$. The act g is* more uncertain *than the act f, denoted $g\overline{U}f$, if*

$$\inf_{s\in\mathcal{S}} g(s) < \inf_{s\in\mathcal{S}} f(s), \ \sup_{s\in\mathcal{S}} g(s) > \sup_{s\in\mathcal{S}} f(s),$$

and for every pair of states $s, t \in \mathcal{S}$, if $g(s) = g(t)$ then $f(s) = f(t)$ else

$$1 \geq \frac{(f(s) - f(t))}{(g(s) - g(t))} \geq 0. \tag{2}$$

Our main result in this section is that the relation $\overline{U}$ is simply the transitive continuous closure of the relation U.

Proposition 3 *Fix a pair of acts $f, g \in \mathcal{F}$. If $g\overline{U}f$ then there exist sequences of simple acts, $\langle f_n \rangle$ and $\langle g_n \rangle$, such that $f_n \to f$ and $g_n \to g$, and for each n there exists a finite sequence of simple acts $\langle h_m^n \rangle_{m=1}^{M^n}$, such that $h_1^n = f_n$, $h_{M^n}^n = g_n$ and $h_{m+1}^n U h_m^n$, $m = 1, \ldots M^n - 1$.*

The following is an immediate corollary of Proposition (3).

Corollary 4 *Fix $\%$ and $\widehat{\%}$. The preference relation $\%$ is* everywhere *at least as* uncertainty averse *as $\widehat{\%}$, if and only if,*

$$f \widehat{\%} g \textit{ implies } f \% g \textit{ for all } g\overline{U}f.$$

Also, we obtain

Corollary 5 *Any act f is more uncertain than its certainty equivalent $m(f)$.*

Corollary 6 *If $\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$, then for any f*

$$m(f) \leq \hat{m}(f).$$

3.1 Special cases

The definitions of comparative uncertainty, and of comparative uncertainty-aversion presented above, are model-free. It is of interest, however, to consider the case when preferences may be represented by some specific model, to characterize the relationship '$\succsim$ is everywhere at least as uncertainty averse as $\widehat{\succsim}$' in terms of the parameters of that model, and, where appropriate, to compare that characterization to existing results on comparative risk aversion. We beginning by demonstrating that the usual characterization of comparative risk aversion for subjective expected utility is consistent with our definition. This reflects the fact that our approach satisfied the Epstein and Zhang desideratum D4. More substantively, we analyze the case of cumulative utility preferences, incorporating such important special cases as Choquet Expected Utility (Schmeidler 1989), rank-dependent expected utility under risk (Quiggin 1993) and the dual model of Yaari (1987).

3.1.1 Subjective Expected Utility

Let us consider the case when $\succsim$ and $\widehat{\succsim}$ satisfy the assumptions of Savage's theory of subjective expected utility (SEU). That is, assume both preference relations can be represented by certainty equivalent functionals $m, \widehat{m}$ of the form

$$m(f) = u^{-1}\left(\int_s u(f(s))\pi\,(ds)\right) \text{ and } \widehat{m}(f) = \widehat{u}^{-1}\left(\int_s \widehat{u}(f(s))\widehat{\pi}\,(ds)\right),$$

where π and $\widehat{\pi}$ are countably-additive and convex-ranged probability measures defined over $\mathcal{E}$ (which in Savage's theory is the power set of $\mathcal{S}$), and u and $\widehat{u}$ are von Neumann-Morgenstern utility functions defined over $\mathcal{X}$.[3]

The same set of necessary and sufficient conditions that are required for one preference relation to be at least as risk averse (in the sense of Rothschild and Stiglitz, 1970) as another are also necessary and sufficient for one to be at least as uncertainty averse as another.

[3] Strictly speaking Savage's axiomatization only guarantees that π is finitely-additive, but since we have already assumed that the set of events $\mathcal{E}$ is a $\sigma-$algebra set of subsets of $\mathcal{S}$, we shall also impose the additional continuity necessary to ensure that the probability measure representing beliefs is countably additive.

Proposition 7 *Suppose* $\succsim$ *and* $\widehat{\succsim}$ *both admit SEU certainty equivalent representations* $m(.)$ *and* $\widehat{m}(.)$*, with associated probability measure and utility function pairs,* (π, u) *and* $(\widehat{\pi}, \widehat{u})$*, respectively. Then,* $\succsim$ *is* everywhere *at least as* uncertainty averse *as* $\widehat{\succsim}$ *if and only if* $\pi(A) = \tilde{\pi}(A)$ *for all* $A \in \mathcal{E}$*, and* u *is a concave transform of* $\widehat{u}$*.*

If we identify an SEU-maximizer with a *linear* utility index as being risk neutral (with respect to π), then an immediate corollary of Proposition (7) is that a necessary and sufficient condition for an SEU-maximizer to be averse to *monotone* mean-preserving spreads (with respect to π) is that his utility function is concave. And without requiring any other restrictions, we also know that his preference relation would agree with the partial ordering of second-order stochastic dominance (or equivalently, he is averse to *all* mean-preserving spreads).[4] These results are not surprising since it is well-known that under the expected utility model for decision making under risk (with exogenously specified probabilities) a decision maker is risk-averse in the weakest sense of always (weakly) preferring the mean of a lottery for sure to the lottery itself if and only if his utility index is concave. Such a coincidence of conditions necessary and sufficient for these three distinct notions of risk aversion (and their uncertainty analogs) does not hold in general for non-EU models of decision making under risk and non-SEU models of decision making under uncertainty. And it is to one of the most widely studied classes of non-SEU models that we turn our focus in the next subsection.

3.1.2 Cumulative Utility, Choquet Expected Utility and Aversions to Uncertainty, Ambiguity and Risk

One of the main directions for generalizing subjective expected utility has been so-called rank-dependent theories. As Chew and Wakker (1996) observe, all of these models may be viewed as underpinned by a weakening of Savage's sure-thing principle that they dub *comonotonic independence*. Comonotonic independence essentially restricts Savage's sure-thing principle to comonotonic acts, that is, acts that induce the same ordering over the states of nature according to their outcomes. They characterized comonotonic independence by means of a functional form called *cumulative utility* (CU).

For ease of exposition let us restrict attention to preference relations defined over the set of simple acts. The following notation draws heavily on Chew and Wakker. Let $\mathcal{P} = (A^1, \ldots, A^n)$ denote a finite partition of the state space $\mathcal{S}$. For each $i = 1, \ldots, n$, let $C^i_{\mathcal{P}}$ denote the cumulative set $\bigcup_{j=1}^{i} A^j$ and set $C^0_{\mathcal{P}} := \emptyset$. $\mathcal{F}_{\mathcal{P}}$ is the set of acts of the form

[4] Formal definitions in the Savage-act framework of monotone mean-preserving spreads and the partial ordering of second-order stochastic dominance are provided in the next subsection.

$x^1_{A^1}x^2_{A^2}\ldots x^{n-1}_{A^{n-1}}x^n$. $\mathcal{F}_{\mathcal{P}\downarrow}$ is the subset of $\mathcal{F}_{\mathcal{P}}$ for which $x^1 \geq \ldots \geq x^n$. Notice that the set of simple acts, $\mathcal{F}_0$ is the union of $\mathcal{F}_{\mathcal{P}}$ over all partitions $\mathcal{P}$.

Fixing a preference relation %, we say an event A is *inessential* if $f \sim g$, for all simple acts f and g that coincide on $\mathcal{S}\backslash A$. By convention we shall consider the empty set to be a member of the set of inessential events. We refer to all other events as *essential.* We call % *jointly monotonic* on $\mathcal{F}_0$ if, for all simple acts f and essential events A,

$$y > x \text{ implies } y_A f \succ x_A f.$$

In CU, joint monotonicity implies monotonicity with respect to outcome as well as with respect to inclusion of essential events.

A preference relation admits a CU representation if there exists an *outcome dependent jointly monotonic capacity*, that is, a function $\Psi : \mathsf{R}_+ \times \mathcal{E} \to \mathsf{R}_+$ satisfying (i) for all A in $\mathcal{E}$: $\Psi(0, A) = 0$, (ii) for any essential event A, $\Psi(y, A \cup B) - \Psi(x, A \cup B) > \Psi(y, B) - \Psi(x, B)$, for any set B in $\mathcal{E}$, and any $y > x$; (iii) for all $x > 0$, $\Psi(x, \mathcal{S}) > 0$ and $\Psi(x, .)/\Psi(x, \mathcal{S})$ is a normalized capacity.[5] And for all $\mathcal{P} = (A^1, \ldots, A^n)$, and for all acts of the form $x^1_{A^1}x^2_{A^2}\ldots x^{n-1}_{A^{n-1}}x^n$ for which $x^1 \geq \ldots \geq x^n$, the certainty equivalent function, $m(f)$ is implicitly defined by:

$$\sum_{i=1}^{n} \left(\Psi\left(x^i, C^i_{\mathcal{P}}\right) - \Psi\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right) = \Psi(m(f), \mathcal{S}). \tag{3}$$

The Choquet Expected Utility model (CEU) is the special case of 'multiplicative separability' in which $\Psi(x, A) = u(x)\nu(A)$, where u is an increasing function and ν is a normalized capacity. Furthermore, we shall only consider preferences that admit a *Gateau-differentiable* CU representation, which entails that, for any outcome $x \in \mathcal{X}$ and event $A \in \mathcal{E}$, the following derivatives are always well defined:

$$\begin{aligned}
\frac{\partial^+}{\partial x}\Psi(x, A) &\equiv \lim_{\varepsilon\to 0}\left(\Psi(x+\varepsilon, A) - \Psi(x, A)\right)/\varepsilon \\
\frac{\partial^-}{\partial x}\Psi(x, A) &\equiv \lim_{\varepsilon\to 0}\left(\Psi(x, A) - \Psi(x-\varepsilon, A)\right)/\varepsilon
\end{aligned}$$

To provide some insight into the nature of the CU model, consider the following diagram in which the value $\varphi(x, A)$ may be viewed as the measure of the 'rectangle' $[0, x] \times A$.[6]

---

[5] A (normalized) capacity is a function $\nu : \mathcal{E} \to [0, 1]$, satisfying:

1. $\nu(\emptyset) = 0$ and $\nu(\mathcal{S}) = 1$
2. $A \subset B \Rightarrow \nu(A) \leq \nu(B)$.

[6] The approach taken here is an adaptation of the measure representation of rank dependent models of decision making under risk, first proposed and developed by Segal (1989).

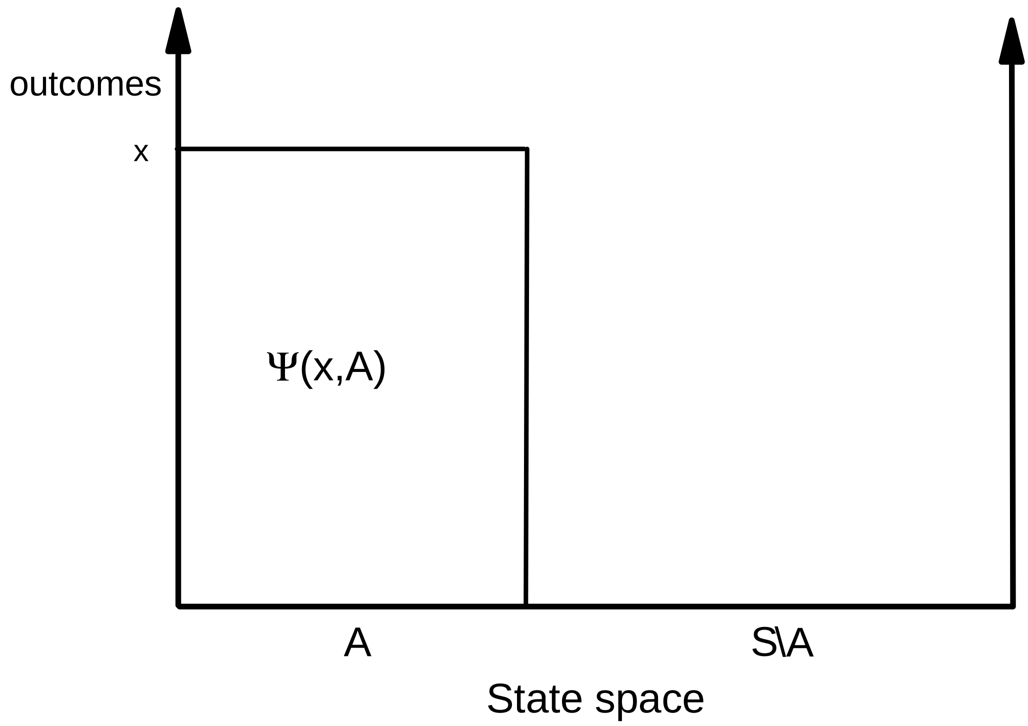

Figure 1: Measure representation of $\Psi(.,.)$

The implicit definition for the certainty equivalent of an act in (3) may be viewed as determining a 'generalized *quasi-linear* mean' (Chew 1983). The mean value $m(f)$, is that for which the 'area' between the 'graph' of the act f that lies above the graph of the constant act $m(f)$ is equal to the area between the graph of f that lies below the graph of the constant act $m(f)$. For example, fix the partition $\mathcal{P} = (A^1, \ldots, A^5)$ and suppose that for the act $f = x^1_{A^1} x^2_{A^2} x^3_{A^3} x^4_{A^4} x^5$ for which $x^1 > \ldots > x^5$, $m(f)$ the unique solution to (3) satisfies $x_2 > m(f) > x_3$. Figure two illustrates the measure-theoretic representation of $m(f)$, where the 'rectangle' marked $i \in \{1, 2, 3, 4, 5\}$ has a (signed) 'measure' of value

$$\left(\Psi\left(x^i, C^i_{\mathcal{P}}\right) - \Psi\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right) - \left(\Psi\left(m(f), C^i_{\mathcal{P}}\right) - \Psi\left(m(f), C^{i-1}_{\mathcal{P}}\right)\right)$$

attached to it. From the properties of $\Psi(.,.)$ we know the signed measures of rectangles 1 and 2 are positive while the rest are negative.

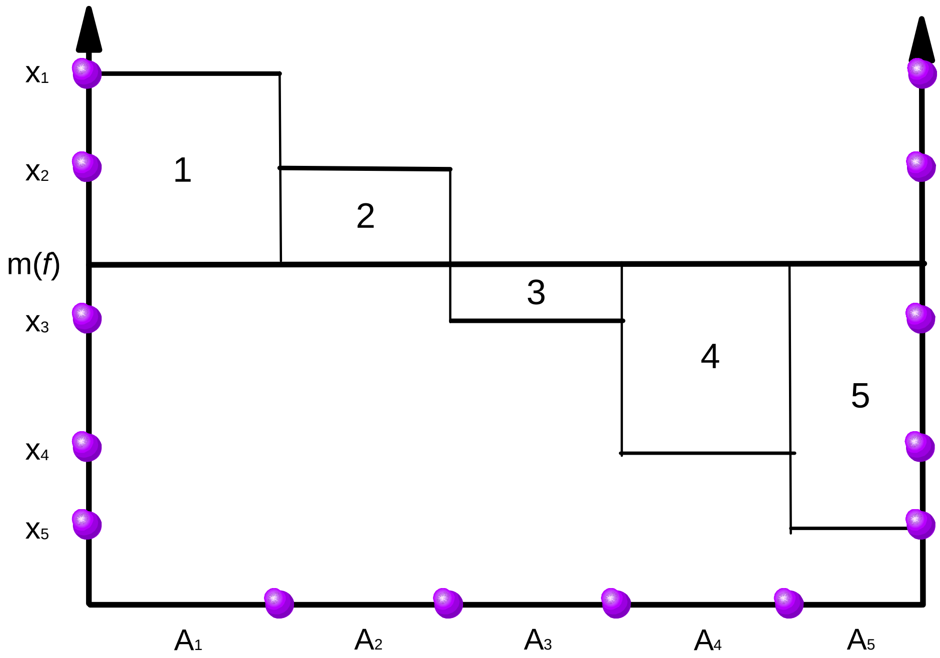

Figure 2: A graphical illustration of the certainty equivalent of an act – $m(f)$ is the certainty equivalent outcome of f since the sum of the signed measures attached to the 'rectangles' labelled $1, 2, 3, 4$ and 5 equals zero.

The next result gives the set of necessary and sufficient conditions for one CU preference relation to be more risk averse than another CU preference relation.

Proposition 8 *Suppose $\succsim$ and $\widehat{\succsim}$ both admit CU certainty equivalent representations $m(.)$ and $\widehat{m}(.)$, with associated outcome dependent capacities, Ψ and $\widehat{\Psi}$, respectively. Furthermore suppose both Ψ and $\widehat{\Psi}$ are Gateau-differentiable. Then $\succsim$ is at least as uncertainty averse as $\widehat{\succsim}$ if and only if the following condition holds for all finite ordered partitions $\mathcal{P} = (A^1, \ldots, A^n)$ and outcomes $x^1 \geq \ldots \geq x^n$,*

$$\inf_{\langle j \in \{2,\ldots,n\}\rangle} \frac{\sum_{i=1}^{j} \frac{\partial^+}{\partial x}\left[\widehat{\Psi}\left(x^i, C_{\mathcal{P}}^i\right) - \widehat{\Psi}\left(x^i, C_{\mathcal{P}}^{i-1}\right)\right] / \sum_{i=j+1}^{n} \frac{\partial^-}{\partial x}\left[\widehat{\Psi}\left(x^i, C_{\mathcal{P}}^i\right) - \widehat{\Psi}\left(x^i, C_{\mathcal{P}}^{i-1}\right)\right]}{\sum_{i=1}^{j} \frac{\partial^+}{\partial x}\left[\Psi\left(x^i, C_{\mathcal{P}}^i\right) - \Psi\left(x^i, C_{\mathcal{P}}^{i-1}\right)\right] / \sum_{i=j+1}^{n} \frac{\partial^-}{\partial x}\left[\Psi\left(x^i, C_{\mathcal{P}}^i\right) - \Psi\left(x^i, C_{\mathcal{P}}^{i-1}\right)\right]} \geq 1 \tag{4}$$

The essential idea behind (4) is illustrated in Figure 3 which depicts the changes in the graph of the act f that featured in Figure 2, resulting from the addition of the 'small' elementary gamble $(\alpha\varepsilon)_{A_1 \cup A_2 \cup A_3}(-\beta\varepsilon)$.

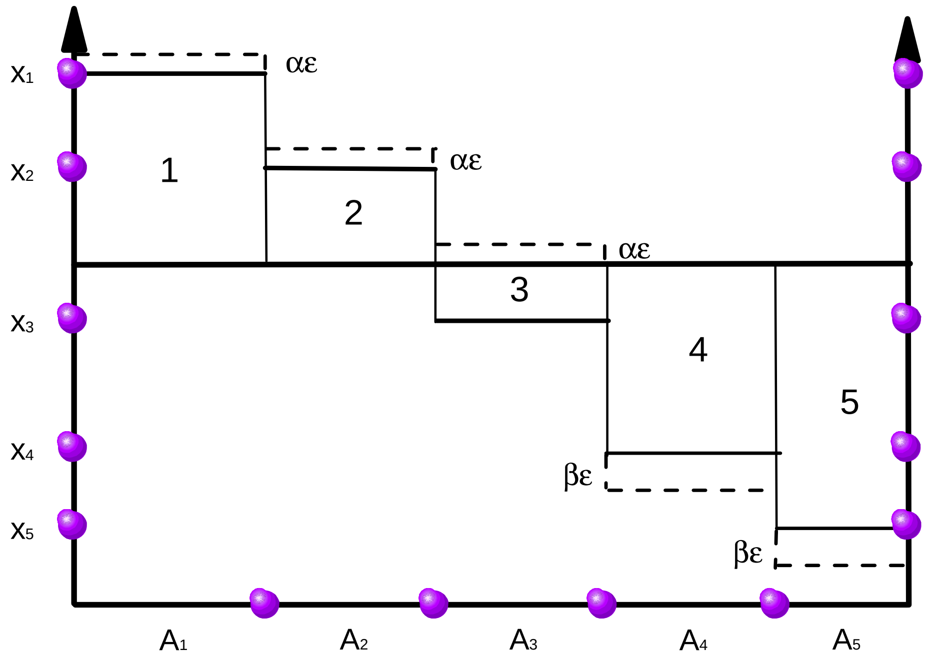

Figure 3: A graphical illustration of the increase in the uncertainty of f resulting from the addition of the elementary bet $(\alpha\varepsilon)_{A_1 \cup A_2 \cup A_3} (-\beta\varepsilon)$

Sufficiency requires that if the addition of this gamble is favorably viewed by %, then it should be favorably viewed by $\widehat{\%}$. Reading off the changes in the 'area' of the graph of the act f from Figure 3, sufficiency requires, if

$$\sum_{i=1}^{3} \left(\Psi\left(x^i + \alpha\varepsilon, C^i_{\mathcal{P}}\right) - \Psi\left(x^i + \alpha\varepsilon, C^{i-1}_{\mathcal{P}}\right)\right) - \left(\Psi\left(x^i, C^i_{\mathcal{P}}\right) - \Psi\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right)$$
$$> \sum_{i=4}^{5} \left(\Psi\left(x^i, C^i_{\mathcal{P}}\right) - \Psi\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right) - \left(\Psi\left(x^i - \beta\varepsilon, C^i_{\mathcal{P}}\right) - \Psi\left(x^i - \beta\varepsilon, C^{i-1}_{\mathcal{P}}\right)\right)$$

then

$$\sum_{i=1}^{3} \left(\widehat{\Psi}\left(x^i + \alpha\varepsilon, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i + \alpha\varepsilon, C^{i-1}_{\mathcal{P}}\right)\right) - \left(\widehat{\Psi}\left(x^i, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right)$$
$$> \sum_{i=4}^{5} \left(\widehat{\Psi}\left(x^i, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i, C^{i-1}_{\mathcal{P}}\right)\right) - \left(\widehat{\Psi}\left(x^i - \beta\varepsilon, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i - \beta\varepsilon, C^{i-1}_{\mathcal{P}}\right)\right).$$

By letting $\varepsilon \longrightarrow 0$, we see that (4) is sufficient for % to be at least as uncertainty averse as $\widehat{\%}$, for any 'elementary increase of uncertainty in the small'. It just remains to show, as we do in the formal proof, that such a comparative 'at least as uncertainty averse in the small' holding everywhere, 'integrates' to globally at least as uncertainty averse (that is, 'in the large'). Conversely, if (4) fails, then it is possible to construct an act f and a sufficiently small elementary gamble b, that is acceptable to add to f for %, but is unacceptable to add to f for $\widehat{\%}$. That is, we have $f + b \; \% \; f$ but $f \; \widehat{\succ} \; f + b$.

Given the non-separability between the 'evaluation' of an outcome and the 'rank-dependent decision-weight' attached to an event that is one of the distinctive features of the general CU functional form, it is perhaps not surprising that (4) involves 'partial sums' over differences between cumulative events. More succinct and more readily interpretable conditions can be obtained if we consider the case of Choquet Expected Utility. That is, let us now restrict $\Psi(x, A)$ and $\widehat{\Psi}(x, A)$ to take the multiplicative forms $u(x)\,\nu(A)$ and $\widehat{u}(x)\,\widehat{\nu}(A)$, respectively, where u and $\widehat{u}$ are increasing functions and ν and $\widehat{\nu}$ are normalized capacities. Moreover, let us for the moment also assume that u and $\widehat{u}$ are both concave. Then for any partition $\mathcal{P} = (A^1, \ldots, A^n)$ and outcomes $x^1 \geq \ldots \geq x^n$, and any given $j \in \{2, \ldots, n-1\}$, we have for the partial sums that feature in (4), the following inequalities:

$$
\begin{aligned}
\sum_{i=1}^{j} \left[\widehat{u}'\left(x^i\right)\widehat{\nu}\left(C^i_{\mathcal{P}}\right) - \widehat{u}'\left(x^i\right)\widehat{\nu}\left(C^{i-1}_{\mathcal{P}}\right)\right] &\geq \widehat{u}'\left(x^1\right)\widehat{\nu}\left(C^j_{\mathcal{P}}\right) \\
\sum_{i=j+1}^{n} \left[\widehat{u}'\left(x^i\right)\widehat{\nu}\left(C^i_{\mathcal{P}}\right) - \widehat{u}'\left(x^i\right)\widehat{\nu}\left(C^{i-1}_{\mathcal{P}}\right)\right] &\leq \widehat{u}'\left(x^n\right)\left[1 - \widehat{\nu}\left(C^j_{\mathcal{P}}\right)\right] \\
\sum_{i=1}^{j} \left[u'\left(x^i\right)\nu\left(C^i_{\mathcal{P}}\right) - u'\left(x^i\right)\nu\left(C^{i-1}_{\mathcal{P}}\right)\right] &\leq u'\left(x^j\right)\nu\left(C^j_{\mathcal{P}}\right) \\
\sum_{i=j+1}^{n} \left[u'\left(x^i\right)\nu\left(C^i_{\mathcal{P}}\right) - u'\left(x^i\right)\nu\left(C^{i-1}_{\mathcal{P}}\right)\right] &\geq u'\left(x^{j+1}\right)\left[1 - \nu\left(C^j_{\mathcal{P}}\right)\right].
\end{aligned}
$$

Hence, applying (4), it follows that a *sufficient* condition for one CEU maximizer, (u, ν), with concave u, to be at least as uncertainty averse as another CEU maximizer, $(\widehat{u}, \widehat{\nu})$, also with concave $\widehat{u}$, is that for any event $E \in \mathcal{E}$ and any four outcomes $x^1 \geq x^2 \geq x^3 \geq x^4$

$$
\frac{\widehat{u}'\left(x^1\right)\widehat{\nu}(E)}{\widehat{u}'\left(x^4\right)\left(1 - \widehat{\nu}(E)\right)} \geq \frac{u'\left(x^2\right)\nu(E)}{u'\left(x^3\right)\left(1 - \nu(E)\right)} \tag{5}
$$

or equivalently,

$$
\lim_{\hat{y}\to\infty}\left(\frac{\widehat{u}'(\widehat{y})}{\widehat{u}'(0)}\right) \geq \sup_{\langle E\in\mathcal{E}\rangle}\left(\frac{\nu(E)\,/\,(1-\nu(E))}{\widehat{\nu}(E)\,/\,(1-\widehat{\nu}(E))}\right). \tag{6}
$$

This is similar to the condition that Chateauneuf, Cohen and Meilijson (1997) derive in the context of decision making under risk for a RDEU expected utility maximizer to be averse to every monotone increase in risk. Adapting their terminology, the left-hand side expression in (6) may be viewed as a measure of the *greediness* of the utility function, $\widehat{u}$. The ratio $\nu(E)/(1-\nu(E))$ may be interpreted as a measure of the *optimism* of the capacity ν about the event E obtaining. Hence the right-hand side expression measures the *relative* optimism of the capacity ν over the capacity $\widehat{\nu}$ about the event E. Thus (6) states that a sufficient condition for (u,ν) to be at least as uncertainty averse as $(\widehat{u},\widehat{\nu})$ is that the latter's greediness is never less than the former's relative optimism over any event.

Furthermore with both utility indexes concave, we can also derive the following sufficiency result.

Corollary 9 *Suppose* % *and* $\widehat{\%}$ *both admit CEU certainty equivalent representations* $\widehat{m}(.)$ *and* $m(.)$*, with associated* concave *utility indexes and capacities,* (u,ν) *and* $(\widehat{u},\widehat{\nu})$*, respectively. Furthermore suppose both* $\widehat{u}$ *and* u *are differentiable. If* u *is a concave transformation of* $\widehat{u}$ *and for all* E*,* $\nu(E) \leq \widehat{\nu}(E)$*, then* % *is at least as uncertainty averse as* $\widehat{\%}$.

Now consider expanding the class of CEU maximizers in such a way that neither u nor $\widehat{u}$ need be concave. By similar reasoning as was used to derive (5) above, we obtain the following sufficient condition:

$$\min_{\left\langle \substack{\widehat{y} > \widehat{x},\, y > x \\ \min(\widehat{y},y) \geq \max(\widehat{x},x)} \right\rangle} \left(\frac{\widehat{u}'(y)/\widehat{u}'(x)}{u'(y)/u'(x)} \right) \geq \sup_{\langle E \in \mathcal{E} \rangle} \left(\frac{\nu(E)/(1-\nu(E))}{\widehat{\nu}(E)/(1-\widehat{\nu}(E))} \right). \qquad (7)$$

Furthermore, if either u or $\widehat{u}$ is linear (that is, one of the individuals is a "Yaari-CEU maximizer") then it readily follows from Proposition (8) that (7) is both *necessary* and sufficient. One implication of this result is that a CEU maximizer with a non-concave utility index can be more uncertainty averse than a Yaari-CEU maximizer (or even CEU maximizer with a strictly concave utility index) provided the degree of 'pessimism' embodied in his capacity, as measured by the ratio $(1-\nu(E))/\nu(E)$, is sufficiently strong enough to outweigh any region of non-diminishing marginal utility. Again, this accords with similar results derived in the context of decision making under risk for RDEU maximizers.

A particularly interesting application of (4), is in the context of Epstein and Zhang's (2001) model of a CEU maximizer, (u,ν), for whom, just from the behavioral implications of the preference relation, an outside analyst is able to classify each event as being either 'ambiguous' or 'unambiguous' for that preference relation.[7] Let $\mathcal{E}_{\nu}^{UA} \subset \mathcal{E}$, denote the set

[7] Among the many important contributions made by Epstein and Zhang (2001) is their preference-based and model-free definition of an unambiguous event. An event T is unambiguous if (a) for all disjoint subevents $A, B \subset \mathcal{S}\backslash T$, acts h, and outcomes x^*, x, z, z', $x^*_{A} x_{B} z_{T} h$ % $x_{A} x^*_{B} z_{T} h$ implies $x^*_{A} x_{B} z'_{T} h$ % $x_{A} x^*_{B} z'_{T} h$ and (b) the condition obtained if T is everywhere replaced by $\mathcal{S}\backslash T$ in (a) is also satisfied. Otherwise, T is ambiguous.

of unambiguous events for (u, ν). The set of axioms that they impose on the preference relation guarantees that the set of 'unambiguous' events is rich enough so that 'beliefs' over these events can be represented by a countably additive, convex-valued probability measure $\pi : \mathcal{E}^{UA} \to [0,1]$. Moreover, for each $A \in \mathcal{E}^{UA}$, $\nu(A) = \phi(\pi(A))$, for some strictly increasing and onto map $\phi : [0,1] \to [0,1]$. Hence for any (measurable) finite partition, $\mathcal{P} = (A^1, \ldots, A^n)$, and for all acts of the form $f = x^1_{A^1} x^2_{A^2} \ldots x^{n-1}_{A^{n-1}} x^n$ for which $x^1 \geq \ldots \geq x^n$, the certainty equivalent function, $m(f)$ for the CEU maximizer, (u, ν), is defined by:

$$m(f) = u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\nu\left(C^i_{\mathcal{P}}\right) - u\left(x^i\right)\nu\left(C^{i-1}_{\mathcal{P}}\right)\right)\right).$$

Furthermore, if for each $i = 1, \ldots, n$, $A^i \in \mathcal{E}^{UA}$, then f is an unambiguous act and

$$m(f) = u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\phi\left(\pi\left(C^i_{\mathcal{P}}\right)\right) - u\left(x^i\right)\phi\left(\pi\left(C^{i-1}_{\mathcal{P}}\right)\right)\right)\right).$$

If we take another such CEU maximizer $(u, \widehat{\nu})$, for whom $\mathcal{E}^{UA}_{\widehat{\nu}} = \mathcal{E}$ (that is, every event is unambiguous for this individual) and $\widehat{\nu}(A) = \nu(A)$ for every $A \in \mathcal{E}^{UA}_{\nu}$, then by construction the two CEU maximizers, (u, ν) and $(u, \widehat{\nu})$, agree over any pair of acts that are measurable with respect to $\mathcal{E}^{UA}_{\nu}$. Furthermore, since every event is unambiguous for $(u, \widehat{\nu})$, this CEU-maximizer is *probabilistically sophisticated* in the sense of Machina and Schmeidler (1992), and so corresponds to Epstein's (1999) notion of an *ambiguity neutral* preference relation. Thus there exists a countably additive, convex-valued probability measure, $\widehat{\pi}$ that extends π to $\mathcal{E}$. That is, for any $E \in \mathcal{E}$, $\widehat{\nu}(E) = \phi(\widehat{\pi}(E))$, and for any (measurable) finite partition, $\mathcal{P} = (A^1, \ldots, A^n)$, and for all acts of the form $f = x^1_{A^1} x^2_{A^2} \ldots x^{n-1}_{A^{n-1}} x^n$ for which $x^1 \geq \ldots \geq x^n$, the certainty equivalent function, $\widehat{m}(f)$ for the CEU maximizer, $(u, \widehat{\nu})$, is defined by:

$$\begin{aligned}\widehat{m}(f) &= u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\widehat{\nu}\left(C^i_{\mathcal{P}}\right) - u\left(x^i\right)\widehat{\nu}\left(C^{i-1}_{\mathcal{P}}\right)\right)\right) \\ &= u^{-1}\left(\sum_{i=1}^{n}\left(u\left(x^i\right)\phi\left(\widehat{\pi}\left(C^i_{\mathcal{P}}\right)\right) - u\left(x^i\right)\phi\left(\widehat{\pi}\left(C^{i-1}_{\mathcal{P}}\right)\right)\right)\right).\end{aligned}$$

According to Epstein's (1999) definition, (u, ν) is *ambiguity averse* if for any pair of acts f and h, such that h is measurable with respect to $\mathcal{E}^{UA}_{\nu}$,

$$\widehat{m}(h) \geq \widehat{m}(f) \text{ implies } m(h) \geq m(f).$$

Epstein and Zhang (2001) show that (u, ν) is *ambiguity averse* if and only if

$$\widehat{\pi}(E) \geq \phi^{-1}(\nu(E)) \text{ for all } E \in \mathcal{E}.$$

The following corollary to Proposition (8) establishes the connection of our definition of more uncertainty averse to Epstein's (1999) definition of ambiguity averse.

Corollary 10 *Let* $\succsim$ *and* $\widehat{\succsim}$ *be preference relations corresponding to the CEU-maximizers* (u, ν) *and* $(u, \widehat{\nu})$ *defined above. Then* (u, ν) *is* ambiguity averse *in the sense of Epstein (1999), if and only if* $\succsim$ *is more uncertainty averse than* $\widehat{\succsim}$.

From this corollary we can conclude that a CEU-maximizer is ambiguity averse in the sense of Epstein (1999) if and only if there is an probabilistically sophisticated CEU-maximizer, such that: (a) the two preference relations agree over the set of unambiguous acts; and (b) the former is more uncertainty averse than the latter.

To explore what risk aversion may entail in this setting, consider a third CEU maximizer $(\widehat{u}, \widehat{p})$, where $\widehat{u}(x) = x$ for all $x \in \mathbb{R}_+$. This individual is actually a subjective-expected-*value* maximizer whose beliefs about the likelihood of events agrees with the CEU-maximizer (u, v) for all events in $\mathcal{E}_\nu^{UA}$. Using the probability measure p defined over the events in $\mathcal{E}_\nu^{UA}$, we can form the partial ordering of second order stochastic dominance over the set of acts measurable with respect to $\mathcal{E}_\nu^{UA}$, as follows:

Definition 4 *Fix a probability measure* π *defined over the events in* $\mathcal{E}_\nu^{UA}$. *For any finite partition* $\mathcal{P} = (A^1, \ldots, A^n)$, *such that* $A^i \in \mathcal{E}_\nu^{UA}$, *for all* $i = 1, \ldots, n$, *and any pair of acts* $f = x^1_{A^1} \ldots x^{n-1}_{A^{x-1}} x^n$ *and* $g = y^1_{A^1} \ldots y^{n-1}_{A^{m-1}} y^n$ *where* $x^1 \geq \ldots \geq x^n$ *and* $y^1 \geq \ldots \geq y^n$, *we say* f *second order stochastically dominates* g *(with respect to* π*) if*

$$\sum_{j=1}^{i} \left(x^j - y^j\right) \pi\left(A^j\right) \geq 0, \text{ for all } i = 1, \ldots, n.$$

As is well-known, the CEU-maximizer (u, v) agrees with the partial ordering of second order stochastic dominance over the set of acts that are measurable with respect to $\mathcal{E}_\nu^{UA}$, if and only if u is concave and ϕ is convex (see for example, Chew, Karni and Safra, 1987). There are, however, a number of weaker notions of risk aversion. We shall consider two.[8] The first is the weakest notion of risk aversion that requires that a (constant) act that gives the 'π–mean outcome' of another act f in every state, is weakly preferred to f.

Definition 5 *Fix a probability measure* π *defined over the events in* $\mathcal{E}_\nu^{UA}$. *The preference relation* $\succsim$ *defined over the set of acts that are measurable with respect to* $\mathcal{E}_\nu^{UA}$, *is* weakly risk averse *(with respect to the probability measure* π*) if for any finite partition* $\mathcal{P} = (A^1, \ldots, A^n)$,

[8] The interested reader is referred to the excellent overview by Chateauneuf, Cohen and Meilijson (2001) that provides a taxonomy of five distinct characterizations of risk aversion for models of decision making under risk.

such that $A^i \in \mathcal{E}_\nu^{UA}$*, for all* $i = 1, \ldots, n$*, and any act* $f = x^1_{A^1} \ldots x^{n-1}_{A^{x-1}} x^n$*:* $\left(\sum_{i=1}^n \pi\left(A^i\right) x^i\right)_\mathcal{S}$ $\succsim f$*.*

There is no known characterization of weak risk aversion for a probabilistically sophisticated CEU-maximizer. Chateauneuf and Cohen (1994), do provide, however, sufficient conditions that do not imply the concavity of u.[9] The second alternative, aversion to monotone mean-preserving spreads, in terms of its strength, lies between the weak definition and the one based on the second-order stochastic dominance relation.

Definition 6 *Fix a probability measure* π *defined over the events in* $\mathcal{E}_\nu^{UA}$*. For any finite partitions* $\left(A^1, \ldots, A^n\right)$ *and* $\left(B^1, \ldots, B^n\right)$ *such that* $A^i, B^i \in \mathcal{E}_\nu^{UA}$*, and* $\pi\left(A^i\right) = \pi\left(B^i\right)$*, for all* $i = 1, \ldots, n$*, and any pair of acts* $f = x^1_{A^1} \ldots x^{n-1}_{A^{x-1}} x^n$ *and* $g = y^1_{B^1} \ldots y^{n-1}_{B^{n-1}} y^n$ *where* $x^1 \geq \ldots \geq x^n$ *and* $y^1 \geq \ldots \geq y^n$*, we say* g *is a* monotone mean-preserving increase in risk *(with respect to* π*) of* f *if*

$$\begin{aligned} \left[\left(x^i - y^i\right) - \left(x^j - y^j\right)\right](i-j) &\geq 0 \text{ for all } i, j \in \{1, \ldots, n\} \\ \text{and} \sum_{i=1}^n \left(x^i - y^i\right) \pi\left(A^i\right) &= 0. \end{aligned}$$

A preference relation $\succsim$ *defined over acts that are measurable with respect to* $\mathcal{E}_\nu^{UA}$ *is said to be* averse to monotone mean-preserving *increases in risk, if for any such pair of (unambiguous) acts* g *and* f *as defined above,* $f \succsim g$*.*

In the context of decision making under risk, where preferences are defined over lotteries, Chateauneuf, Cohen and Meilijson (1997) provide a complete characterization for a Rank-Dependent Expected Utility maximizer to be averse to all monotone mean preserving increases in risk. Adapting their result to the subjectively uncertain act-framework here, we have that the CEU-maximizer (u, ν) is (weakly) averse to any monotone mean-preserving increase in risk (with respect to π) if and only if

$$\inf_{E \in \mathcal{E}_\nu^{UA}} \left[\frac{\left(1 - \phi\left(\pi\left(E\right)\right)\right) / \phi\left(\pi\left(E\right)\right)}{\left(1 - \pi\left(E\right)\right) / \pi\left(E\right)}\right] \geq \sup_{x \geq y} u'\left(x\right) / u'\left(y\right). \tag{8}$$

They refer to the left-hand expression of (8) as the index of *pessimism* of ϕ. They dub the right-hand expression as the index of *greediness* and note that if it equals one, then u is concave. From (8), it follows that a necessary requirement for ϕ to satisfy is that

$$\inf_{E \in \mathcal{E}_\nu^{UA}} \left[\frac{\left(1 - \phi\left(\pi\left(E\right)\right)\right) / \phi\left(\pi\left(E\right)\right)}{\left(1 - \pi\left(E\right)\right) / \pi\left(E\right)}\right] \geq 1$$

[9] One implication of Chateauneuf and Cohen's (1994) result is that the claim by Epstein and Zhang (2001, p287) in Corollary 7.4 (a) that the concavity of u is necessary for a CEU-maximizer to be risk-averse (in the weak sense) over acts that are measurable with respect to the set of unambiguous acts is incorrect.

that is, $\phi(q) \leq q$, for all $q \in [0,1]$. However, as they emphasize, the most significant feature of (8) is that u need not be concave, for the CEU-maximizer (u, ν) to be monotone risk averse.

A hint at the connection between this notion of risk aversion and our model-free definition of increases in uncertainty, is suggested by the fact that if g is a monotone mean preserving increase in risk (with respect to π) of f and $A^i = B^i$, for all $i = 1, \ldots, n$, then $g\ \overline{U}\ f$. And indeed, the following corollary of Proposition 8 shows that the relationship is tight.

Corollary 11 *Let $\succsim$ and $\widehat{\succsim}$ be preference relations defined over acts that are measurable with respect to $\mathcal{E}_\nu^{UA}$, and that agree* on this restricted domain*, with the CEU-maximizers (u, ν) and $(\widehat{u}, \widehat{p})$, respectively. Then $\succsim$ is at least as uncertainty averse as $\widehat{\succsim}$ if and only if* (8) *is satisfied.*

4 Economic Implications

Choice sets in which all elements are ordered by $\overline{U}$ arise naturally in a range of economic problems. In the one safe asset, one risky asset portfolio problem, analyzed by Pratt (1964) and many others, $g\overline{U}f$ whenever g represents a portfolio with more of the risky asset. More generally, standard capital market theory implies that idiosyncratic risk can be completely traded away. Hence, net of such trades, all financial assets may be assumed to be comonotonic with the market portfolio. If the two fund separation property applies, and the difference between the state-contingent returns to the two funds are increasing and display a single-crossing property, then any portfolio may be represented as a mixture of two acts f and g with $g\overline{U}f$, and where the (normalized) prices of both acts are 1. We may also allow for fixed background risk h, representing, say, (normalized) returns to human capital, with the assumption that $(f+h)\overline{U}f$. This will be true if and only if h is comonotonic with f and takes both positive and negative values. Hence, we also have $(g+h)\overline{U}g$. Thus, the problem of an individual deciding how to allocate her (normalized) initial wealth of 1, is to find the optimal contingent income f^*, where

$$\begin{aligned} & f^* \in \mathcal{B} \equiv \left\{ \widetilde{f} \in \mathcal{F} : \widetilde{f} = h + \lambda f + (1-\lambda) g \text{ for some } \lambda \in [0,1] \right\} \\ \text{s.t } & f^* \succsim \widetilde{f}, \text{ for all } \widetilde{f} \in \mathcal{B}. \end{aligned}$$

The one safe asset, one risky asset portfolio problem arises when $h(s) \equiv 0$ and f is a constant act.

We immediately obtain:

Corollary 12 *Let $\succsim$ be* everywhere *at least as* uncertainty averse *as* $\widehat{\succsim}$ *and let f^* (respectively, $\widehat{f}^*$) be the optimal contingent income for $\succsim$ (respectively, $\widehat{\succsim}$) and λ^* (respectively, $\widehat{\lambda}^*$) the associated allocation to the less uncertain asset. Then $\widehat{f}^* \; \overline{U} \; f^*$, that is, $\lambda^* \leq \widehat{\lambda}^*$.*

5 Conclusion

Most economic analysis of choice under uncertainty, and particularly of increases in uncertainty, has been based on the assumption that decision-makers have well-defined subjective probabilities. On the other hand, the fundamental result of the literature, the proof of existence of equilibrium in state-contingent markets derived by Arrow and Debreu (1954), does not require decision-makers to possess subjective probabilities or to satisfy the postulates of any model specific to problems involving uncertainty. In this paper, definitions of increases in uncertainty, and comparative degrees of aversion to such increases in uncertainty, independent of subjective probabilities and of any particular model of choice under uncertainty have been presented. Moreover, it has been shown that, for a number of widely-used models of choice under risk and uncertainty, this definition is consistent with the definitions already in use.

Appendix

Proof of Proposition 3.

We first establish the following lemmas.

Lemma 13 *If for any pair of simple acts g and f, any pair of positive real numbers, α and β, and any three element partition of $\mathcal{S}$, (E_{-1}, E_0, E_1), we have*

$$g(s) - f(s) = \begin{cases} \alpha & \text{if } s \in E_1 \\ 0 & \text{if } s \in E_0 \\ -\beta & \text{if } s \in E\_1 \end{cases}$$

then there exists a simple act h for which gUh and hUf.

Proof. $\alpha > \beta/2$ then define

$$h(s) = \begin{cases} f(s) + \alpha - \beta/2 & \text{if } s \in E_1 \\ f(s) - \beta/2 & \text{if } s \in E_0 \\ f(s) - \beta/2 & \text{if } s \in E_{-1}. \end{cases}$$

Notice that

$$g(s)-h(s)=\begin{cases}\beta/2 & \text{if } s\in E_1\cup E_0\\ -\beta/2 & \text{if } s\in E_{-1}\end{cases} \quad \text{and } h(s)-f(s)=\begin{cases}\alpha-\beta/2 & \text{if } s\in E_1\\ -\beta/2 & \text{if } s\in E_0\cup E_{-1}\end{cases}$$

as required. If $\alpha\le\beta/2$ then define

$$h(s)=\begin{cases} f(s)+\alpha/2 & \text{if } s\in E_1\\ f(s)+\alpha/2 & \text{if } s\in E_0\\ f(s)-\beta+\alpha/2 & \text{if } s\in E_{-1}.\end{cases}$$

Now we have

$$g(s)-h(s)=\begin{cases}\alpha/2 & \text{if } s\in E_1\\ -\alpha/2 & \text{if } s\in E_0\cup E_{-1}\end{cases} \quad \text{and } h(s)-f(s)=\begin{cases}\alpha/2 & \text{if } s\in E_1\cup E_0\\ -\beta+\alpha/2 & \text{if } s\in E_{-1}.\end{cases}$$

■

Lemma 14 *If for any pair of simple acts f and g, $g\overline{U}f$ then there exists a finite sequence of simple acts $\langle h_m\rangle_{m=1}^{M}$ such that $h_1=f$, $h_M=g$ and $h_{m+1}Uh_m$, $m=1,\dots M-1$.*

Proof. From the definition of $g\overline{U}f$ it follows that $g-f$ is pairwise co-monotonic with both g and f. Let $[E_{-J},\dots,E_1,E_0,E_1,\dots,E_I]$ be the coarsest partition of $\mathcal{S}$ for which $g-f$ is measurable and with the labelling monotonically ordered, that is for any $i>j$, and any $s\in E_i$ and $s'\in E_j$, $g(s)-f(s)>g(s')-f(s')$. Moreover, assume that for any $i<0$, and any $s\in E_i$, $g(s)-f(s)<0$;. for any $i>0$ and any $s\in E_i$, $g(s)-f(s)>0$; and for any $s\in E_0$, $g(s)=f(s)$ E_0 may be empty, but since $\inf_{s\in\mathcal{S}}g(s)<\inf_{s\in\mathcal{S}}f(s)$ and $\sup_{s\in\mathcal{S}}g(s)$ $>\sup_{s\in\mathcal{S}}f(s)$ it follows that $I\ge1$ and $J\ge1$. For each $i=-J,\dots,0,\dots,I$, and some $s_i\in E_i$, set $d_i:=g(s_i)-f(s_i)$. By construction, we have

$$d_{-J}<d_{-J+1}<\dots<d_{-1}<d_0=0<d_1<\dots<d_I.$$

Let $h_1:=f$. Define

$$h_3(s)=\begin{cases} f(s)+d_1 & \text{if } s\in E_1\cup E_2\cup\dots\cup\ E_I\\ f(s) & \text{if } s\in E_0\\ f(s)+d_{-1} & \text{if } s\in E_{-1}\cup E_{-2}\cup\dots\cup\ E_{-J}.\end{cases}$$

For $i = 2, \ldots, \min\{I, J\} - 1$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_i \cup \ldots \cup E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\ h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J}. \end{cases}$$

Proof. $I \geq J$, then for $i = J, \ldots, I$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + d_i - d_{i-1} & \text{if } s \in E_i \cup \ldots \cup E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-J+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{i-1} \\ h_{2i-1}(s) + (d_{-J} - d_{-J+1}) / (I - J + 1) & \text{if } s \in E_{-J}. \end{cases}$$

Notice, in this case $h_{2I+1} = g$.

If, however, $I < J$, then for $i = I, \ldots, J$, define

$$h_{2i+1}(s) = \begin{cases} h_{2i-1}(s) + (d_I - d_{I-1}) / (J - I + 1) & \text{if } s \in E_I \\ h_{2i-1}(s) & \text{if } s \in E_{-i+1} \cup \ldots \cup E_0 \cup \ldots \cup E_{I-1} \\ h_{2i-1}(s) + d_{-i} - d_{-i+1} & \text{if } s \in E_{-i} \cup \ldots \cup E_{-J} \end{cases}$$

and now $h_{2J+1} = g$.

For each $i = 1, \ldots, \max\{I, J\}$, it follows from Lemma (13) that there exists a simple act h_{2i} for which $h_{2i+1} U h_{2i}$ and $h_{2i} U h_{2i-1}$. Hence we have

$$g = h_{2\max\{I,J\}+1} U h_{2\max\{I,J\}} U \ldots U h_1 = f$$

as required. ■

We are now in a position to prove the proposition. Set $\underline{x} := \inf_{s \in \mathcal{S}} f(s)$, $\overline{x} := \sup_{s \in \mathcal{S}} f(s)$, $\underline{y} := \inf_{s \in \mathcal{S}} g(s)$ and $\overline{y} := \sup_{s \in \mathcal{S}} g(s)$. Set $\overline{x}_0 := \sup_{s \in \mathcal{S}} \{f(s) \mid f(s) \succsim g(s)\}$ and $\underline{x}_0 := \inf_{s \in \mathcal{S}} \{f(s) \mid g(s) \succsim f(s)\}$. Since $g \overline{U} f$ it follows that $\underline{y} < \underline{x} \leq \overline{x} < \overline{y}$ which combined with (2) implies that $\overline{x}_0$ and $\underline{x}_0$ are well defined. Set $E_1^1 := \{s \in \mathcal{S} \mid f(s) \succ \overline{x}_0\}$, $E_0^1 :=$

$\{s \in \mathcal{S} \mid f(s) = g(s)\}$ and $E_{-1}^{1} := \mathcal{S} \setminus (E_{1}^{1} \cup E_0)$. Define

$$f_1(s) = \begin{cases} \overline{x} & \text{if } s \in E_1^1 \\ \overline{x}_0 & \text{if } s \in E_0^1 \\ \underline{x} & \text{if } s \in E_{-1}^1 \end{cases}$$

$$g_1(s) = \begin{cases} \overline{y} & \text{if } s \in E_1^1 \\ \overline{x}_0 & \text{if } s \in E_0^1 \\ \underline{y} & \text{if } s \in E_{-1}^1. \end{cases}$$

From Lemma (13) it follows there exists a simple act h for which hUf_1 and g_1Uh, as required. For step n, divide each of the intervals $[\underline{x}, \underline{x}_0)$ $[\underline{x}_0, \overline{x}_0]$and $(\overline{x}_0, \overline{x}]$ into n equal length subintervals: that is,

$$[\underline{x}, \underline{x} + (\underline{x}_0 - \underline{x})/n), \ [\underline{x} + (\underline{x}_0 - \underline{x})/n, \underline{x} + 2(\underline{x}_0 - \underline{x})/n), \ 2026, \ [\underline{x} + (n-1)(\underline{x}_0 - \underline{x})/n, \underline{x}_0),$$

$$[\underline{x}_0, \underline{x}_0 + (\overline{x}_0 - \underline{x}_0)/n], \ [\underline{x}_0 + (\overline{x}_0 - \underline{x}_0)/n, \underline{x}_0 + 2(\overline{x}_0 - \underline{x}_0)/n], \ 2026, \ [\underline{x}_0 + (n-1)(\overline{x}_0 - \underline{x}_0)/n, \overline{x}_0]$$

and

$$[\overline{x}_0, \overline{x}_0 + (\overline{x} - \overline{x}_0)/n], \ (\overline{x}_0 + (\overline{x} - \overline{x}_0)/n, \overline{x}_0 + 2(\overline{x} - \overline{x}_0)/n], \ 2026, \ (\overline{x}_0 + (n-1)(\overline{x} - \overline{x}_0)/n, \overline{x}].$$

For each $m = 1, \ldots, n$, set

$$\begin{aligned} E_m^n &:= \{s \in \mathcal{S} \mid f(s) \in (\overline{x}_0 + (m-1)(\overline{x} - \overline{x}_0)/n, \overline{x}_0 + m(\overline{x} - \overline{x}_0)/n]\} \\ E_{0m}^n &:= \{s \in \mathcal{S} \mid f(s) \in (\underline{x}_0 + (m-1)(\overline{x}_0 - \underline{x}_0)/n, \underline{x}_0 + m(\overline{x}_0 - \underline{x}_0)/n]\} \\ E_{-m}^n &:= \{s \in \mathcal{S} \mid f(s) \in [\underline{x}_0 - m(\underline{x}_0 - \underline{x})/n, \underline{x}_0 - (m-1)(\underline{x}_0 - \underline{x})/n)\}. \end{aligned}$$

Define

$$f_n(s) = \begin{cases} x_0 + m(\overline{x} - x_0)/n & \text{if } s \in E^n_m \\ \underline{x}_0 + m(\overline{x}_0 - \underline{x}_0)/n & \text{if } s \in E^n_{0m} \\ x_0 - m(x_0 - \underline{x})/n & \text{if } s \in E^n_{-m} \end{cases}$$

$$g_n(s) = \begin{cases} \sup_{t \in E^n_m} g(t) & \text{if } s \in E^n_m \\ \underline{x}_0 + m(\overline{x}_0 - \underline{x}_0)/n & \text{if } s \in E^n_{0m} \\ \inf_{t \in D^n_m} g(t) & \text{if } s \in E^n_{-m}. \end{cases}$$

Notice by construction all acts f_n and g_n are simple with $f_n \to f$ and $g_n \to g$. Moreover: $f_n(s) - g_n(s) > 0$ if $s \in E^n_{-m}$, for some $m = 1, \ldots, n$; $f_n(s) = g_n(s)$, if $s \in E^n_{0m}$, for some $m = 1, \ldots, n$; and $g_n(s) - f_n(s) > 0$ if $s \in E^n_m$ for some $m = 1, \ldots, n$. Finally, it also follows from (2) that for any $s \in E^n_m$ and any $t \in E^n_{m'}$ with $m > m'$, $g_n(s) - f_n(s) > g_n(t) - f_n(t) > 0$. Similarly, for any $s \in E^n_{-m}$ and any $t \in E^n_{-m'}$ with $m > m'$, $f_n(s) - g_n(s) > f_n(t) - g_n(t)$. Hence $g_n \overline{U} f_n$ and applying Lemma (14) we can transform f_n into g_n by a finite sequence of elementary increases in uncertainty. ■

Proof of Proposition 7.

Sufficiency is obvious. For necessity of the equality of π and $\widehat{\pi}$, consider choices in a neighborhood of a constant act x. For any real-valued function $d : \mathcal{S} \to \mathsf{R}$ and sufficiently small $\varepsilon > 0$, the certainty equivalent of the act $x + \varepsilon d$ (in the neighborhood of x) under m is approximately

$$x + \varepsilon \int_s d(s)\pi(ds)$$

and, similarly for $\widehat{m}$, the certainty equivalent is

$$x + \varepsilon \int_s d(s)\widehat{\pi}(ds).$$

If $\pi(E) > \tilde{\pi}(E)$ for some $E \subset \mathcal{S}$ then

$$\frac{\pi(E)}{1 - \pi(E)} > \frac{\widehat{\pi}(E)}{1 - \widehat{\pi}(E)} - \frac{\delta}{(1 - \pi(E))(1 - \widehat{\pi}(E))}$$

for some $\delta > 0$. Thus if we take

$$d(s) = \begin{cases} 1 - \widehat{\pi}(E) - \delta & \text{if } s \in E \\ -\widehat{\pi}(E) - \delta & \text{if } s \notin E, \end{cases}$$

then for any $\varepsilon > 0$ we have

$$\varepsilon \int_s d(s)\pi(ds) > 0 > \varepsilon \int_s d(s)\widehat{\pi}(ds)$$

and $x + \varepsilon d \, \overline{U} \, x$. So for sufficiently small $\varepsilon > 0$, it follows from continuity and monotonicity of % and $\widehat{\%}$, that $x \mathrel{\widehat{\succ}} (x + \varepsilon d)$ but $(x + \varepsilon d) \succ x$.

To demonstrate the necessity of u being a concave transformation of $\widehat{u}$, suppose the contrary, that is, u is not a concave transform of $\tilde{u}$. Then there must exist utility levels v_1, v_2 and v_3 in the *range* of $\widehat{u}$, and λ in $(0,1)$, such that

$$\lambda v_1 + (1-\lambda) v_3 = v_2$$
$$\lambda u \circ \widehat{u}^{-1}(v_1) + (1-\lambda) u \circ \widehat{u}^{-1}(v_3) > u \circ \widehat{u}^{-1}(v_2).$$

Since π is non-atomic, there exists an event $E \subset \mathcal{S}$ for which $\pi(E) = \lambda$. So consider the act $f := \widehat{u}^{-1}(v_1)_E \, \widehat{u}^{-1}(v_3)$ and the constant act $x := \widehat{u}^{-1}(v_2)$. By construction we have $f \, \overline{U} \, x$, $x \mathrel{\widehat{\%}} f$ and $f \succ x$. ¥

Proof of Proposition 8.

Sufficiency: Consider any pair of comonotonic simple acts f and g, for which $g - f = \alpha_A(-\beta)$, for some A in $\mathcal{E}$, and for some $\alpha, \beta > 0$. Further suppose $g \succ f$. We need to show that $g \mathrel{\widehat{\succ}} f$. By the implicit function theorem and the fact the preference relation admits a Gateau-differentiable CU representation, it follows that

$$\begin{aligned}\frac{d}{d\varepsilon} m(f + (g-f)\overline{\varepsilon}) &= \frac{\alpha \sum_{i=1}^{j} \left(\frac{\partial^+}{\partial x} \left[\Psi\left(x^i + \alpha\overline{\varepsilon}, C^i_{\mathcal{P}}\right) - \Psi\left(x^i + \alpha\overline{\varepsilon}, C^{i-1}_{\mathcal{P}}\right) \right] \right)}{\frac{\partial^+}{\partial x} \Psi\left(m(f + (g-f)\overline{\varepsilon}), \mathcal{S}\right)} \\ &\quad - \frac{\beta \sum_{i=j+1}^{n} \left(\frac{\partial^-}{\partial x} \left[\Psi\left(x^i - \beta\overline{\varepsilon}, C^i_{\mathcal{P}}\right) - \Psi\left(x^i - \beta\overline{\varepsilon}, C^{i-1}_{\mathcal{P}}\right) \right] \right)}{\frac{\partial^+}{\partial x} \Psi\left(m(f + (g-f)\overline{\varepsilon}), \mathcal{S}\right)} \\ &> 0 \text{ for all } \overline{\varepsilon} \text{ in } [0,1].\end{aligned}$$

From condition (4) it follows that

$$\frac{\sum_{i=1}^{j} \frac{\partial^+}{\partial x} \widehat{\Psi}\left(x^i, A^i\right)}{\sum_{i=j+1}^{n} \frac{\partial^-}{\partial x} \widehat{\Psi}\left(x^i, A^i\right)} \geq \frac{\sum_{i=1}^{j} \frac{\partial^+}{\partial x} \Psi\left(x^i, A^i\right)}{\sum_{i=j+1}^{n} \frac{\partial^-}{\partial x} \Psi\left(x^i, A^i\right)} > \frac{\beta}{\alpha}.$$

And since

$$\begin{aligned}&\operatorname{sign}\left(\frac{d}{d\varepsilon} \widehat{m}(f + (g-f)\overline{\varepsilon}) \right) \\ = \; &\operatorname{sign}\left[\alpha \sum_{i=1}^{j} \left(\frac{\partial^+}{\partial x} \left[\widehat{\Psi}\left(x^i + \alpha\overline{\varepsilon}, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i + \alpha\overline{\varepsilon}, C^{i-1}_{\mathcal{P}}\right) \right] \right) \right. \\ &\left. -\beta \sum_{i=j+1}^{n} \left(\frac{\partial^-}{\partial x} \left[\widehat{\Psi}\left(x^i - \beta\overline{\varepsilon}, C^i_{\mathcal{P}}\right) - \widehat{\Psi}\left(x^i - \beta\overline{\varepsilon}, C^{i-1}_{\mathcal{P}}\right) \right] \right) \right]\end{aligned}$$

it further follows that

$$\frac{d}{d\varepsilon}\widehat{m}\left(f+(g-f)\,\overline{\varepsilon}\right)>0 \text{ for all } \overline{\varepsilon} \text{ in } [0,1]$$

and hence

$$\left(\widehat{m}\left(g\right)-\widehat{m}\left(f\right)\right)=\int_0^1\left[\frac{d}{d\varepsilon}\widehat{m}\left(f+(g-f)\,\overline{\varepsilon}\right)\right]d\overline{\varepsilon}>0$$

as required.

Necessity: Suppose for the ordered partitions $(A^1,\ldots,A^n)$ condition (4) does not hold in an open set around the set of outcomes $x^1\geq\ldots\geq x^n$. That is, for some $j\in\{1,\ldots,n\}$ and some $\gamma>0$, we have

$$\frac{\sum_{i=1}^{j}\frac{\partial^+}{\partial x}\left[\widehat{\Psi}\left(x^i,C^i_{\mathcal{P}}\right)-\widehat{\Psi}\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]}{\sum_{i=j+1}^{n}\frac{\partial^-}{\partial x}\left[\widehat{\Psi}\left(x^i,C^i_{\mathcal{P}}\right)-\widehat{\Psi}\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]}<\gamma<\frac{\sum_{i=1}^{j}\frac{\partial^+}{\partial x}\left[\Psi\left(x^i,C^i_{\mathcal{P}}\right)-\Psi\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]}{\sum_{i=j+1}^{n}\frac{\partial^-}{\partial x}\left[\Psi\left(x^i,C^i_{\mathcal{P}}\right)-\Psi\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]}.$$

So consider $f=x^1_{A^1}\ldots x^{n-1}_{A^{n-1}}x^n$ and the act $g(\varepsilon)=f+(\varepsilon)_{\cup_{i=1}^{j}A^i}(-\gamma\varepsilon)$. For sufficiently small ε,

$$\begin{aligned}\operatorname{sign}\left(m\left(g\left(\varepsilon\right)\right)-m\left(f\right)\right) &= \operatorname{sign}\left[\sum_{i=1}^{j}\frac{\partial^+}{\partial x}\left[\Psi\left(x^i,C^i_{\mathcal{P}}\right)-\Psi\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]\right.\\ &\qquad\left.-\gamma\sum_{i=j+1}^{n}\frac{\partial^-}{\partial x}\left[\Psi\left(x^i,C^i_{\mathcal{P}}\right)-\Psi\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]\right]\end{aligned}$$

and

$$\begin{aligned}\operatorname{sign}\left(\widehat{m}\left(g\left(\varepsilon\right)\right)-\widehat{m}\left(f\right)\right) &= \operatorname{sign}\left[\sum_{i=1}^{j}\frac{\partial^+}{\partial x}\left[\widehat{\Psi}\left(x^i,C^i_{\mathcal{P}}\right)-\widehat{\Psi}\left(x^i+\varepsilon,C^{i-1}_{\mathcal{P}}\right)\right]\right.\\ &\qquad\left.-\gamma\sum_{i=j+1}^{n}\frac{\partial^-}{\partial x}\left[\widehat{\Psi}\left(x^i,C^i_{\mathcal{P}}\right)-\widehat{\Psi}\left(x^i,C^{i-1}_{\mathcal{P}}\right)\right]\right]\end{aligned}$$

Hence for sufficiently small ε, we have $g(\varepsilon)\,Uf$, but $g(\varepsilon)\succ f$ and $f\,\widehat{\succ}\,g(\varepsilon)$. ¥

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