Finite Coverings by Cones

Tijs, S.H.; Reijnierse, J.H.

Publication date:
2001

Citation for published version (APA):
Tijs, S. H., & Reijnierse, J. H. (2001). *Finite Coverings by Cones*. (CentER Discussion Paper; Vol. 2001-60). Operations research.

No. 2001-60

FINITE COVERINGS BY CONES

By Stef Tijs and Hans Reijnierse

September 2001

ISSN 0924-7815

Discussion paper

Finite Coverings by Cones

and an Application in Multi-Objective Programming[1]

Stef Tijs[2]
Hans Reijnierse[3]

August 6, 2001

Abstract: This paper considers analogues of statements concerning compactness and finite coverings, in which the roles of spheres are replaced by cones. Furthermore, one of the finite covering results provides an application in Multi-Objective Programming; infinite sets of alternatives are reduced to finite sets.

Key Words: cones, ε-domination, finite covering, multi-objective programming.

JEL-code: C00, C79

Introduction

An elementary observation gives that each bounded set in a finite-dimensional real space has a finite subset such that each element is close to one of the elements of the subset, irrespective of the quantifying of the notion close.
If the boundedness is relaxed to upper boundedness, the statement can be repaired by replacing 'close to' by 'almost dominated by'. This result is called the *ε-domination Theorem* and has been firstly proved by *Tijs* (1977). It has been used to derive ε-equilibrium point theorems for two person noncooperative games and resulted in a literature on approximate equilibria and other approximate solutions (see *Tijs* (1981), *Lucchetti, Patrone* and *Tijs* (1986), *Patrone* and *Tijs* (1987), *Jurg* and *Tijs* (1993), *Norde* and *Potters* (1997) and *Norde, Patrone* and *Tijs* (2000)).
Section 2 provides a new proof of the ε-domination Theorem. Furthermore, it shows that the characterizing property of compactness that every open covering of a compact set contains a finite subcovering, cannot be converted in a similar way.

[1]We would like to thank H.Norde, F. Patrone and J.-P. Penot for their helpful contributions.
[2]Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (S.H.Tijs@kub.nl).
[3]Department of Econometrics and CentER, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands (J.H.Reijnierse@kub.nl).

After a presentation of some of these results at the conference *'Variational Methods and Optimization, in memory of Francesco Ferro (1946-1999)'*(Genova, 2000), J.-P. Penot phrased the question whether the ε-domination Theorem, which deals with sets of the form $a + \mathbb{R}^m_-$, can be generalized to finite coverings by sets of the form $a + K$, in which K is a cone. First of all, the original set need not be upper bounded, but *K-bounded,* i.e. covered by some positive variety $a + K$.
To our surprise, the answer turns out to be yes and no. Section 3 provides a generalization concerning polyhedral cones. However, section 4 shows that in $\mathbb{R}^m$, $m \geqslant 3$, cones K and K-bounded sets V can be found such that V does not have the finite K-covering property. In the three-dimensional real space *each* non-polyhedral cone K contains a subset without the finite K-covering property. It remains an open question whether for every *closed* non-polyhedral cone in $\mathbb{R}^m$ $(m \geqslant 4)$ counterexamples can be constructed.

Section 5 provides an application of the generalization to polyhedral cones. In multicriteria optimization (see *Patrone* and *Tijs* (1987) and *Voorneveld, Grahn* and *Dufwenberg* (2000)) it is custom that there is one 'decision maker' who has to deal with several 'factors' (issues). We consider a situation in which a board (of a firm) has to make a decision, that exerts influence on several factors. Each member of the board weights the importance of factors differently. This is reflected by personal weight vectors on the factors. These vectors induce a polyhedral cone K in the factor-space. We show that if the set of alternatives is K-bounded, a finite set of alternatives ε-dominates the whole set.

1 Preliminaries

Let us summarize the notations that are used throughout the paper. The considered objects live in some fixed finite dimensional real space, coordinate-wise extended with $-\infty$, and provided with the standard partial ordering. The character m is used to denote its dimension and the set of coordinates is denoted by $\underline{m}$:

$\underline{m} := \{1, \ldots, m\}$ and

$\underline{\mathbb{R}}^m := \{(x_1, \ldots, x_m) \mid -\infty \leqslant x_i < \infty \text{ for all } i \in \underline{m}\}$.

The sets of positive, non-negative, negative and non-positive vectors in $\mathbb{R}^m$ are denoted by $\mathbb{R}^m_{++}$, $\mathbb{R}^m_+$, $\mathbb{R}^m_{--}$ and $\mathbb{R}^m_-$ respectively. The vector with value $-\infty$ at each coordinate is denoted by $-\underline{\infty}$:

$-\underline{\infty} := (-\infty, \ldots, -\infty)$.

We recall that $K \subset \mathbb{R}^m$ is called a cone if for all elements a, b of K and all non-negative real numbers α and β also $\alpha a + \beta b$ is an element of K.
For a set B, the cone generated by B is denoted by $K(B)$; it consists of all finite non-negative linear combinations of elements in B. A cone generated by a finite set is called a *polyhedral* cone. If it is generated by a linear independent set, then $K(B)$ is called a

simplicial cone.
The *linear hull* of B, denoted by $L(B)$, consists of all finite linear combinations of elements of B. The relative interior of B is the set of elements b of B for which there exists an open neighborhood U_b such that $b \in (U_b \cap L(B)) \subset B$. The closure and relative interior of B are denoted by $\bar{V}$ and V° respectively.
For any subset K of $\mathbb{R}^m$ (usually we think of cones), V is called *K-bounded* if there exists a vector $a \in \mathbb{R}^m$ such that $V \subset a + K$. In the case that K equals $\mathbb{R}^m_-$, V is called *upper bounded.*

2 The finite covering property

Consider the following theorems:

Theorem 1: *Let V be a bounded subset of $\mathbb{R}^m$ and let $\varepsilon > 0$. Then, there is a finite subset W of V such that:*

$$V \subset \bigcup_{w \in W} B(w, \varepsilon).$$

Here, $B(w, \varepsilon)$ is the closed sphere with center w and radius ε.

Theorem 2: *Let V be a bounded and closed subset of $\mathbb{R}^m$ and let for $v \in V$, B_v be an open neighborhood of v. Then there is a finite subset W of V such that:*

$$V \subset \bigcup_{w \in W} B_w.$$

What happens if we relax the boundedness to upper boundedness in the previous theorems? Of course, the theorems will not hold anymore, but Theorem 1 can be modified by giving the roles of the spheres to 'negative orthants', i.e. sets of the form $a + \mathbb{R}^m_-$. We show that Theorem 2 cannot be generalized in a similar way. Before stating these results formally, let us introduce some terminology.
For $a, b \in \underline{\mathbb{R}}$, $[\![a, b]\!]$ denotes the set of extended real numbers between a and b, i.e.:

$$[\![a, b]\!] := \{x \in \underline{\mathbb{R}} \mid a \leqslant x \leqslant b\} \quad \text{if } a \leqslant b \quad \text{and}$$
$$\{x \in \underline{\mathbb{R}} \mid b \leqslant x \leqslant a\} \quad \text{otherwise.}$$

For $p, q \in \underline{\mathbb{R}}^m$, let $[\![p, q]\!]$ be the Cartesian product $\Pi_{i=1}^m [\![p_i, q_i]\!]$. The set $[\![p, q]\!]$ is called a *hypercube* if for all $i \in \underline{m}$ p_i unequals q_i. For a hypercube $[\![p, q]\!]$, the *center* $c(p, q)$ or $c([\![p, q]\!])$ is the vector in $\underline{\mathbb{R}}^m$ defined by:

$$\begin{aligned} c_i(p, q) &:= \tfrac{1}{2}(p_i + q_i) && \text{if } p_i, q_i \in \mathbb{R}, \\ &:= p_i - 1 && \text{if } q_i = -\infty, \\ &:= q_i - 1 && \text{if } p_i = -\infty. \end{aligned}$$

Note that $c(p, q) \in \mathbb{R}^m \cap [\![p, q]\!]$ and for all $i \in \underline{m}$ either $p_i < c_i(p, q) < q_i$ holds or $q_i < c(p, q) < p_i$.

An m-dimensional hypercube $[\![p,q]\!]$ has 2^m vertices. Each vertex r is of the form $(r_1,\ldots,r_m)\in\underline{\mathbb{R}}^m$ with $r_i\in\{p_i,q_i\}$ for each coordinate $i\in\underline{m}$. The *center sub-division* of $[\![p,q]\!]$ is the division of $[\![p,q]\!]$ into 2^m sub-hypercubes of the form $[\![c(p,q),r]\!]$, in which r is one of the 2^m vertices. Let us illustrate this sub-division by means of an example.

Example 3: Consider the hypercube (rectangle) H in $\underline{\mathbb{R}}^2$:

$$H := [\![(-\infty,-1),(-1,-3)]\!] = \{x\in\underline{\mathbb{R}}^2 \mid -\infty\leqslant x_1\leqslant -1,\ -3\leqslant x_2\leqslant -1\}.$$

The center $c(H)$ of H equals $(-2,-2)$. Figure 1 depicts the center sub-division generating four sub-rectangles, i.e:

$$H_{t\ell} := [\![(-\infty,-1),(-2,-2)]\!],\qquad H_{tr} := [\![(-1,-1),(-2,-2)]\!],$$
$$H_{b\ell} := [\![(-\infty,-3),(-2,-2)]\!],\qquad H_{br} := [\![(-1,-3),(-2,-2)]\!].$$

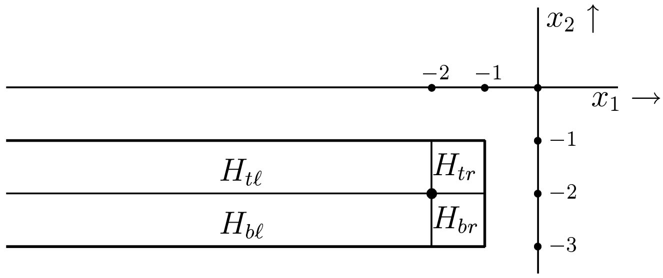

Figure 1: A center sub-division in the plane

Center sub-divisons of the four sub-rectangles lead to figure 2.

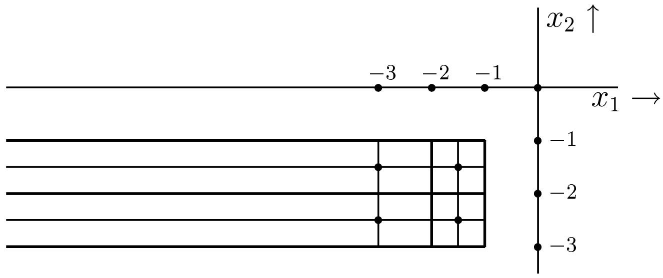

Figure 2: Four center sub-divisions

Definition 4: Let $\varepsilon\in\mathbb{R}^m_{++}$ be a strictly positive vector. A set V in $\mathbb{R}^m$ has the *ε-domination property* $D_\varepsilon(V)$ if there exists a finite subset W of V such that:

$$V\subset\bigcup_{w\in W}[\![-\underline{\infty},w+\varepsilon]\!].$$

The negation of $D_\varepsilon(V)$ is denoted by $ND_\varepsilon(V)$.

A set V in $\mathbb{R}^m$ has the *finite covering property* if it obeys the ε-domination property $D_\varepsilon(V)$ for each $\varepsilon\in\mathbb{R}^m_{++}$.

Now we have the equipment to give and prove the main theorem of this section. In comparison with the original proof, we have the opinion that the current one is less technical and more elegant.

Theorem 5: (Finite covering property for upper bounded sets) [cf. *Tijs* (1977)] *Each upper bounded set in* $\mathbb{R}^m$ *has the finite covering property.*

Proof: The proof will be by contradiction. Suppose there is an upper bounded set V in $\mathbb{R}^m$ and a positive vector $\varepsilon \in \mathbb{R}^m_{++}$ such that $ND_\varepsilon(V)$.
Take an upper bound a_1 in $\mathbb{R}^m$ of V. Define H_1 to be $[\![-\underline{\infty}, a_1]\!]$, so $V \subset H_1$. Because $ND_\varepsilon(V)$, there are a sequence of vectors $c_1, c_2, \ldots$ and a sequence of hypercubes $H_1, H_2, \ldots$ such that for all n in $\mathbb{N}$:

(1) c_n is the center of H_n,

(2) H_{n+1} is one of the sub-hypercubes of the center sub-division of H_n and

(3) $V_n := V \cap H_n$ has the property $ND_\varepsilon(V_n)$.

Let $\Pi_i : \mathbb{R}^m \longrightarrow \mathbb{R}$ be the projection of $\mathbb{R}^m$ on its i^{th} axis:

$\Pi_i(x) := x_i$ for all $x \in \mathbb{R}^m$.

We partition $\underline{m}$ into two sets; F and I. The set F will contain all coordinates i such that the i^{th} projection of H_n is bounded for sufficiently large numbers n:

$F := \{i \in \underline{m} \mid \exists n \in \mathbb{N}\ [\Pi_i(H_n) \text{ is bounded}]\},$ and
$I := \underline{m} \backslash F.$

Choose N in $\mathbb{N}$ such that $\Pi_i(H_n)$ is bounded for all $i \in F,\ n \geqslant N$. Note that:

(1.1) $\operatorname{diam}(\Pi_i(H_{n+1})) = \frac{1}{2}\operatorname{diam}(\Pi_i(H_n))$ if $i \in F$ and $n \geqslant N$

(1.2) $\sup(\Pi_i(H_{n+1})) = \sup(\Pi_i(H_n)) - 1.$ if $i \in I$.

To obtain a contradiction with $ND_\varepsilon(V)$, we pinpoint an element a of V and a number n^* in $\mathbb{N}$ such that:

$V_{n^*} \subset [\![-\underline{\infty}, a + \varepsilon]\!]$.

If F is the empty set, choose a arbitrarily. If F is non-empty, there exists by assertion (1.1) a number n_1 such that for all $i \in F$ and $v, w \in V_{n_1}$:

$|\Pi_i(v) - \Pi_i(w)| < \varepsilon_i$.

Choose a in V_{n_1}. Then:

(1.3) $\Pi_i(V_{n_1+s}) \subset \Pi_i(V_{n_1}) \subset (-\infty, a_i + \varepsilon_i]$ for all $i \in F$ and all $s \in \mathbb{N}$.

Take n_2 in $\mathbb{N}$ such that for all $i \in I$:

$\Pi_i(V_{n_1}) \subset (-\infty, a_i + \varepsilon_i + n_2)$.

Then, by (1.2):

(1.4) $\Pi_i(V_{n_1+n_2}) \subset (-\infty, a_i + \varepsilon_i)$.

Now, (1.3) and (1.4) imply that for $n^* := n_1 + n_2$:

$V_{n^*} \subset [\![-\underline{\infty}, a + \varepsilon]\!]$. □

Theorem 2 cannot be generalized in the same way, as the following counterexample shows.

Example 6: Let V be the closed and upper bounded subset of $\mathbb{R}^2$ defined by:

$$V := \{x \in \mathbb{R}^2_- \mid x_1 x_2 \geqslant 1\}.$$

Define the neighborhoods B_v by:

$$B_v := \{x \in \mathbb{R}^2_- \mid x_1 < \tfrac{1}{2}v_1,\ x_2 < \tfrac{1}{2}v_2\}. \qquad (v \in V)$$

For any finite subset W of V, the element x of V is not covered by $\bigcup_{w \in W} B_w$, in which:

$$x_1 := \frac{1}{4}\max_{w \in W} w_1 \text{ and } x_2 := \frac{1}{x_1}.$$

3 Finite coverings by polyhedral cones

This section gives a generalization of Theorem 1. Here, upper boundedness is replaced by K-boundedness in which K is a polyhedral cone. Let us generalize the finite covering property.

Definition 7: Let K be a set in $\mathbb{R}^m$ with a non-empty relative interior. A subset V of $\mathbb{R}^m$ has the *finite K-covering property* if for each $\varepsilon \in -K^\circ$ there is a finite subset W of V such that:

$$V \subset (\varepsilon + W + K) := \bigcup_{w \in W} \bigcup_{x \in K} \{\varepsilon + w + x\}.$$

Note that if K is a (closed) sphere, the finite K-covering property is equivalent with boundedness (Theorem 1). If K equals $\mathbb{R}^m_-$, we get back the original definition 4. For simplicial cones, the finite K-covering property is implied by K-boundedness:

Proposition 8: *Let K be a simplicial cone in $\mathbb{R}^m$. Then K-bounded subsets of $\mathbb{R}^m$ have the finite K-covering property.*

Proof: Let V be K-bounded, say $V \subset a + K$. Let $\varepsilon \in -K^\circ$. Let $f_1, \ldots, f_k$ form a maximal linear independent set of extreme vectors of K. So, $f_1, \ldots, f_k$ are uniquely determined up to their lengths and they form a base of the linear hull $L(K)$.
Define the linear map $A\colon L(K) \longrightarrow \mathbb{R}^k$ by $A(f_i) := -e_i$ for $i \in \{1, \ldots, k\}$. Note that $\varepsilon' := A\varepsilon \in \mathbb{R}^k_{++}$ and that $A(V - a) \subset A(K) = \mathbb{R}^k_-$. So $A(V-a)$ is upper bounded.
By Theorem 5, it follows that there is a finite subset W' of $A(V-a)$ such that:

$$A(V-a) \subset (\varepsilon' + W' + \mathbb{R}^k_-).$$

Put $W := a + A^{-1}(W')$. Then W is a finite subset of V and:

$$V - a \subset A^{-1}\left(\varepsilon' + W' + \mathbb{R}^k_-\right) = A^{-1}(\varepsilon') + A^{-1}(W') + K = \bigcup_{w \in W} (\varepsilon + (w-a) + K).$$

So, V is a subset of $\varepsilon + W + K$. □

The result is even valid for any polyhedral cone:

Theorem 9: *Let K in $\mathbb{R}^m$ be a polyhedral cone and let B be a finite set generating it. Then every K-bounded set has the finite K-covering property.*

Proof: Let $I(B)$ be the family of linear independent subsystems of B, i.e.:

$I(B) := \{J \subset B \mid J \text{ is a linear independent set}\}$.

Then, $K = \bigcup_{J\in I(B)} K(J)$. Let ε be any vector in $-K^\circ$. Choose $\lambda \in \mathbb{R}^B_{++}$ such that $\varepsilon = \sum_{b\in B} \lambda_b(-b)$. Such a positive combination can be found as follows. Since ε is situated in the relative interior of $-K$, it is a convex combination of some vector $d \in -K$ and minus the sum $c := -\sum_{b\in B} b$ of the elements of B:

$\varepsilon = \alpha d + (1-\alpha)c$ for some $\alpha \in (0,1)$.

So, ε is the convex combination of a non-positive combination and a strictly negative combination of the elements of B.

For $J \in I(B)$, define $\varepsilon_J := \sum_{j\in J} \lambda_j(-j)$. We have:

$\varepsilon_J \in -K(J)^\circ$ and $\varepsilon_J - \varepsilon \in K$.

Let V be any K-bounded set. There is a vector $a \in \mathbb{R}^m$ with $V \subset a + K$. Let for each $J \in I(B)$: $V_J := V \cap (a + K(J))$. Then V_J is a subset of $a + K(J)$, making V_J $K(J)$-bounded. By Proposition 8, a finite subset W_J of V_J can be found such that $V_J \subset \bigcup_{w\in W_J}(\varepsilon_J + w + K(J))$. Hence,

$$V = \bigcup_{J\in I(B)} V_J \subset \bigcup_{J\in I(B)} (\varepsilon_J + W_J + K(J)) \subset \varepsilon + W + K,$$

in which W is the finite set $\bigcup_{J\in I(B)} W_J$. The second inclusion follows from the fact that $\varepsilon_J - \varepsilon \in K$. We have proved that V has the finite K-property. □

The converse statement is valid for any full dimensional cone K. If K has a lower dimension than m, it is necessary that V is a subset of a variety of the linear hull of K.

Theorem 10: *Let K be a cone in $\mathbb{R}^m$. If V has the finite K-property and for some vector $c \in \mathbb{R}^m$, V is a subset of $c + L(K)$, then V is K-bounded.*

Proof: Let $\varepsilon \in -K^\circ$. By assumption, there is a finite subset W of V such that $V \subset \varepsilon + W + K$.

If $W = \{w\}$, there is nothing to prove; V is K-bounded by $\varepsilon + w + K$.

If W contains two elements, say $W = \{a, b\}$, take a subset $\{f_1, \ldots, f_k\}$ of K that forms a basis of $L(K)$. Because $a - b \in L(K)$, there are (unique) real numbers $\lambda_1, \ldots, \lambda_k$ such that: $a - b = \sum_{i=1}^k \lambda_i f_i$. Define t by:

$$t := a - \sum_{i:\lambda_i>0} \lambda_i f_i, \quad \text{or, equivalently:} \quad t := b + \sum_{i:\lambda_i<0} \lambda_i f_i.$$

So, $t \in c + L(K)$ and both a and b are elements of $t + K$, which gives $(a+K)\cup(b+K) \subset t + K$. Hence, V is a subset of $\varepsilon + t + K$.

If W contains more than two elements, take any pair in W and replace it (by the previous argument) by one vector in $c + L(K)$ (not necessarily in V). An induction argument finishes the proof. □

4 Cones with infinitely many extreme directions

Can we generalize any further? What happens if a cone is considered with infinitely many extreme directions? It turns out that in $\mathbb{R}^m$, $m \geqslant 3$, cones K and K-bounded sets V can be found such that V does not have the finite K-covering property. For $m = 3$ we can prove more: here *each* non-polyhedral cone K contains a subset without the finite K-covering property.

Theorem 11: *Let K be a convex cone in $\mathbb{R}^3$ with an infinite number of extreme directions. Then there exists a positive number $\varepsilon > 0$ and a subset V of K such that* no *finite subset W of V has the property:*

for all v in V, there exists a $w \in W$ with: $\quad d(v, w + K) < \varepsilon$,

in which $d(v, w + K) := \min_{k \in \bar{K}} |v - (w + k)|$.

Proof: Let E^3 be the unit-sphere of $\mathbb{R}^3$: $E^3 := \{x \in \mathbb{R}^3 \mid |x| = 1\}$. The collection of normalized extreme directions of K contains a convergent sequence $r_1, r_2, \ldots$ in E^3 with limit, say, r. Define for each j in $\mathbb{N}$ a supporting hyperplane and a corresponding normalized perpendicular vector h_j, pointed inward. So, $\langle h_j, k\rangle \geqslant 0 = \langle h_j, r_j\rangle$ for all $k \in K$ and $|h_j| = 1$.
Let $h := \lim_{j\to\infty} h_j$ (if this limit does not exist, use a subsequence such that it does). The hyperplane $H := \{x \in \mathbb{R}^3 \mid \langle h, x\rangle = 0\}$ is tangent to K at r, i.e. $\langle h, k\rangle \geqslant 0 = \langle h, r\rangle$ for all $k \in K$.
Because H is a two-dimensional subspace, it contains at most two extreme directions of K. If one of them, or both, are elements of the sequence $r_1, r_2, \ldots$, remove them from the sequence. Hence, without loss of generality we can assume that $\langle h, r_j\rangle > 0$ for all $j \in \mathbb{N}$. Define V as follows:

$$V := K \cap \{x \in \mathbb{R}^3 \mid \langle h, x\rangle = 1\}.$$

For all j in $\mathbb{N}$, the half-line $\{\lambda r_j \mid \lambda \geqslant 0\}$ intersects V, say at a_j. Choose ε in $(0, 1)$ arbitrarily and let W be any finite subset of V. The theorem now follows immediately from the assertion that for any $w \in W$:

$$\liminf_{j\to\infty} \; d(a_j, w + K) \geqslant 1 > \varepsilon.$$

Define $\delta_j := \min_{k\in\bar{K}} d(a_j - w, k)$ and define k_j to be the argument of this minimum, resulting in $\delta_j = |a_j - w - k_j|$. We have:

$$\liminf_{j\to\infty} \; \delta_j = \liminf_{j\to\infty} \; |h_j| \cdot |k_j + w - a_j| \underset{(1)}{\geqslant} \liminf_{j\to\infty} \; \langle h_j, k_j + w - a_j\rangle \underset{(2)}{=}$$

$$\liminf_{j\to\infty} \langle h_j, k_j + w\rangle \underset{(3)}{=} \liminf_{j\to\infty} \langle h_j, k_j\rangle + 1 \underset{(4)}{\geqslant} 1.$$

Inequality (1) is the famous inequality of Schwarz, equality (2) holds by $\langle h_j, a_j\rangle = |a_j|\langle h_j, r_j\rangle = 0$, equality (3) is valid because h_j converges to h and $\langle h, w\rangle = 1$, and finally inequality (4) applies since k_j is an element of $\bar{K}$ and h_j is pointed inward.
We conclude that: $$\liminf_{j\to\infty} d(a_j - w, K) \geqslant 1.$$ □

Theorem 11 does not apply in four or more dimensions, as Example 12 shows. In this example the cone K is not closed. It is an open question whether Theorem 11 holds in general for *closed* cones.

Example 12: Let K be the following cone in $\mathbb{R}^4$:

$$K := \{x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 \leqslant x_3^2,\ x_3 \leqslant 0,\ x_4 = 0\} \cup \{x \in \mathbb{R}^4 \mid x_4 < 0\}.$$

Then $(t, \sqrt{1-t^2}, -1, 0)$ is an extreme direction of K for every t in $[0, 1]$. Furthermore, $-K^\circ$ is the set of vectors with a positive fourth coordinate.
For every subset V of K and $\varepsilon \in -K^\circ$, $\varepsilon + w + K$ covers V in which w is an element of V such that:

$$\sup_{x\in V} x_4 < w_4 + \varepsilon_4.$$

5 A multi-objective problem

The purpose of this section is to describe a multi-objective situation, in which our finite covering Theorem 9 for polytopes can help to reduce an initially infinite action space to a finite one, that is approximately just as good as the original action space.
We consider the situation that a board of a firm has to choose an action a from the set of possible actions A. There are m important factors or issues for the firm (we think of gains, returns, working load, market shares, reputation, et cetera), that depend on the chosen action. The set of factors is denoted by F and is assumed to be $\{1, \ldots, m\}$. The functions $v_1, \ldots, v_m$ from A to $\mathbb{R}$ measure the 'benefits' or 'values' with respect to the different factors; $v_k(a)$ represents the benefit for factor k if a is chosen.
The board N of the firm consists of n members: 1 up to n. Each board member, with his own backgrounds and responsibilities, will weight the factors in his own manner. We suppose that board member $i \in N$, has non-negative weights $w_{1i}, \ldots, w_{mi}$ on the factors. So member i wants to maximize the utility-function $u_i : A \longrightarrow \mathbb{R}$, given by:

$$u_i(a) := \sum_{k=1}^{m} v_k(a) w_{ki}.$$

A classical multi-objective programming problem corresponds to the situation that m equals n and $w_{ki} = 1$ if $i = k$ and $w_{ki} = 0$ otherwise. Here, each board member only cares for his own factor. In case $w_{ki} = w_{k1}$ for all $k \in F$ and $i \in N$, we obtain an ordinary optimization problem.

Formalizing these ideas, we get the definition of a multi-objective programming problem. It is a tuple

$$P := \langle A, v_1, \ldots, v_m, N, W \rangle,$$

in which $W := [w_{ki}]_{k \in F, i \in N}$ is a non-negative matrix without zero-columns.

We denote the set of all possible evaluations $v(A)$ with respect to the factors by V:

$$V := v(A) = \{(v_1(a), \ldots, v_m(a)) \in \mathbb{R}^m \mid a \in A\}.$$

The cone of all directions in the factor-space $\mathbb{R}^m$ that are unfavorable for all board members is called K:

$$K := \{x \in \mathbb{R}^m \mid x^T W \leqslant 0\}.$$

Hence, K-boundedness of V has a natural interpretation; it means that the personal benefits for any board member are limited, even if he could choose action a alone.

Let $\delta \in \mathbb{R}^n_{++}$. The amount δ_i can be interpreted as an amount of utility that is *not* worthwhile negotiating for, according to board member i.

Now we come to the main result of this section:

Theorem 13: *Let $P = \langle A, v_1, \ldots, v_m, N, W \rangle$ be a multi-objective programming situation and let $\delta \in \mathbb{R}^n_{++}$. Let K be the polyhedral cone in $\mathbb{R}^m$ defined by*

$$K := \{x \in \mathbb{R}^m \mid x^T W \leqslant 0\}.$$

Let again $v(A)$ be denoted by V. If $v(A)$ is K-bounded, then there exists a finite subset B of A such that for all actions $a \in A$ there exists an alternative $b \in B$ with:

$$u_i(b) \geqslant u_i(a) - \delta_i \text{ for all } i \in N.$$

Proof: Take $\varepsilon \in \mathbb{R}^m_{++}$ such that $0 < \varepsilon^T W \leqslant \delta$. Then $\varepsilon \in -K^\circ$, the relative interior of $-K$. By Theorem 9, a finite subset V' of V can be selected such that:

$$V \subset \varepsilon + V' + K.$$

Take a minimal subset B of A such that $v(B) = V'$, so the cardinalities of B and V' coincide. Now, let $a \in A$. There exists an element $b \in B$ such that:

$$v(a) - \varepsilon - v(b) \in K,$$

so

$$(v(a) - \varepsilon - v(b))^T W \leqslant 0.$$

Therefore:

$$u(a) - u(b) = v(a)^T W - v(a)^T W \leqslant \varepsilon^T W \leqslant \delta.$$

Hence, $u(b) \geqslant u(a) - \delta$. □

The Theorem states that as soon as every member pronounces a number he considers to be insignificant, a finite selection of the actions can be filtered. The choice of an action within this finite set is another problem which we leave to the board of the firm.

References

P. Jurg, S. Tijs (1993) *On the determinateness of semi-infinite bimatrix games.* International Journal of Game Theory Vol. 21, 361-369.

R. Lucchetti, F. Patrone and S. Tijs (1986) *Determinateness of two-person games.* Italiana (6) 5-B, 907-924.

H. Norde, F. Patrone, S. Tijs (2000) *Characterizing properties of approximate solutions for optimization problems.* Mathematical Social Sciences Vol. 40, 297-311.

H. Norde, J. Potters (1997) *On the determinateness of bimatrix games.* Mathematics of Operations Research Vol. 22, 631-638.

F.Patrone, S. Tijs (1987) *A unified approach to approximate solutions in games and multiobjective programming.* Journal of Optimization Theory and Applications Vol. 52, No 2, 273-278.

S. Tijs (1977) *ε-Equilibrium point theorems for two-person games.* Methods of Operations Research Vol. 26, 755-766.

S. Tijs (1981) *Nash equilibrium for noncooperative n-person games in normal form.* SIAM Review Vol. 23 No 2, 225-237.

M. Voorneveld, S. Grahn, M. Dufwenberg (2000) *Ideal equilibria in noncooperative multicriteria games.* Mathematical Methods of Operations Research Vol. 52, 65-77.