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Discussion paper

A Note on NTU Convexity and Population Monotonic Allocation Schemes

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Abstract

For cooperative games with transferable utility, convexity has turned out to be an important and widely applicable concept, mainly because it implies some very nice and handy properties. One of these is that every extended marginal vector constitutes a population monotonic allocation scheme. In this note, this well-known result is generalised to games with nontransferable utility.

1 Introduction

The notion of convexity for cooperative games with transferable utility (TU games) was introduced by Shapley (1971) and is one of the most analysed properties in cooperative game theory. One of the main reasons is that convexity implies some very nice properties. In this paper we focus on one of these: the existence of a population monotonic allocation scheme.

The two convexity notions for games with nontransferable utility (NTU games) that are used in this paper, individual merge and marginal convexity, have been introduced in Hendrickx et al. (2000). Both these properties generalise TU convexity, but the former is a stronger generalisation than the latter.

Sprumont (1990) shows that for TU games, every extended marginal vector is a population monotonic allocation scheme and poses the question, whether this result can be extended to NTU games. In this paper, we show that for individual merge convex games, this can indeed be established, while marginal convexity is not sufficient.

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2 Notation and basic definitions

The set of real numbers is denoted by $\mathbb{R}$ and the set of nonnegative reals by $\mathbb{R}_+$. By $\mathbb{R}^N$ we denote the set of all real-valued functions on N. An element of $\mathbb{R}^N$ is denoted by a vector $x = (x_i)_{i \in N}$. For $S \subset N, S \neq \emptyset$, we denote the restriction of x on S by $x_S = (x_i)_{i \in S}$. For $x, y \in \mathbb{R}^N$, $y \geq x$ denotes $y_i \geq x_i$ for all $i \in N$ and $y > x$ denotes $y_i > x_i$ for all $i \in N$.

A *cooperative game with nontransferable utility*, or *NTU game*, is described by a pair (N, V), where $N = \{1, \ldots, n\}$ is the set of players and V is the payoff map assigning to each coalition $S \subset N, S \neq \emptyset$ a subset $V(S)$ of $\mathbb{R}^S$ such that, for all $i \in N$,

$$V(\{i\}) = (-\infty, 0]$$

and for all $S \subset N, S \neq \emptyset$ we have

$V(S)$ is nonempty and closed,

$V(S)$ is comprehensive, ie, $x \in V(S)$ and $y \leq x$ imply $y \in V(S)$,

$V(S) \cap \mathbb{R}^S_+$ is bounded.

Moreover, we assume that (N, V) is *monotonic*: for all $S \subset T \subset N, S \neq \emptyset$ and for all $x \in V(S)$ there exists a $y \in V(T)$ such that $y_S \geq x$. Note that we do not define $V(\emptyset)$. For ease of notation, we sometimes use V rather than (N, V) to denote an NTU game. The set of all NTU games with player set N is denoted by NTU^N.

The set of *weak Pareto efficient allocations* for coalition $S \subset N, S \neq \emptyset$, denoted by $WPar(S)$, is defined by

$$WPar(S) = \{x \in V(S) \,|\, \nexists_{y \in V(S)} : y > x\}$$

and the set of *individually rational allocations* is defined by

$$IR(S) = \{x \in V(S) \,|\, \forall_{i \in S} : x_i \geq 0\}.$$

The *core* of an NTU game (N, V) consists of those elements of $V(N)$ for which it holds that no coalition $S \subset N, S \neq \emptyset$ can improve its payoff on its own:

$$C(V) = \{x \in V(N) \,|\, \forall_{S \subset N, S \neq \emptyset} \nexists_{y \in V(S)} : y > x_S\}.$$

An NTU game (N, V) is called *individually superadditive* if for all $i \in N$ and for all $S \subset N \backslash \{i\}, S \neq \emptyset$ we have

$$V(S) \times V(\{i\}) \subset V(S \cup \{i\}).$$

Note that individual superadditivity is stronger than monotonicity.

An *ordering* of the players in N is a bijection $\sigma : \{1, \ldots, n\} \to N$, where $\sigma(i)$ denotes which player in N is at position i. The set of all $n!$ orderings of N is denoted by $\Pi(N)$. We define the *marginal vector* $M^\sigma(V)$ corresponding to the ordering $\sigma \in \Pi(N)$ by

$$\begin{aligned} M^\sigma_{\sigma(k)}(V) = \quad & \max\{x_{\sigma(k)} \mid \; x \in V(\{\sigma(1), \ldots, \sigma(k)\}), \\ & \qquad\qquad\qquad \forall_{i \in \{1,\ldots,k-1\}} : x_{\sigma(i)} = M^\sigma_{\sigma(i)}(V)\} \end{aligned}$$

for all $k \in \{1, \ldots, n\}$. Note that the maximums are well defined as a result of the assumptions of monotonicity, closedness and boundedness. By construction, $M^\sigma(V) \in WPar(N)$. If a game is individually superadditive, then all marginal vectors belong to $IR(N)$.

An NTU game (N, V) is called *individual merge convex* (cf. Hendrickx et al. (2000)) if it is individually superadditive and the individual merge property, ie, for all $k \in N$ and all $S \subsetneq T \subset N \backslash \{k\}$ such that $S \neq \emptyset$, the following statement is true: for all $p \in WPar(S) \cap IR(S)$, all $q \in V(T)$ and all $r \in V(S \cup \{k\})$ such that $r_S \geq p$ there exists an $s \in V(T \cup \{k\})$ such that

$$\begin{cases} \forall_{i \in T} : s_i \geq q_i \\ s_k \geq r_k. \end{cases}$$

The idea behind the individual merge property is as follows: whatever allocations the coalitions S and T agree upon separately, given an allocation for coalition $S \cup \{i\}$ such that the members of S are willing to let player i join, player i can obtain a (weakly) better payoff by joining the larger coalition T.

An NTU game (N, V) is called *marginal convex* if for all $\sigma \in \Pi(N)$ we have

$$M^\sigma(V) \in C(V).$$

In Hendrickx et al. (2000), it is shown that individual merge convexity is a stronger property than marginal convexity.

One important aspect of both convexity properties is that within the class of NTU games that correspond to TU games, they are equivalent and coincide with TU convexity.

Another property of these concepts is the following: if an NTU game (N, V) satisfies some form of convexity, then all its subgames do, where the subgame of

(N,V) with respect to coalition $S \subset N, S \neq \emptyset$ is defined as the NTU game (S, V^S) with $V^S(T) = V(T)$ for all $T \subset S, T \neq \emptyset$.

3 Population monotonic allocation schemes

A *population monotonic allocation scheme* or *pmas* for an NTU game (N,V) is a collection of vectors $y^S \in \mathbb{R}^S$ for all $S \subset N, S \neq \emptyset$ such that

$$y^S \in WPar(S) \tag{3.1}$$

for all $S \subset N, S \neq \emptyset$ and

$$y_i^S \leq y_i^T \tag{3.2}$$

if $S, T \subset N$ and $i \in N$ are such that $S \subset T$ and $i \in S$. Note that this definition generalises the definition of pmas for TU games as given in Sprumont (1990).

A pmas induces a core element in every subgame, as stated in the following lemma.

Lemma 3.1 *Let $(N,V) \in NTU^N$ and let $\{y^S\}_{S \subset N, S \neq \emptyset}$ be a pmas for (N,V). Then $y^S \in C(V^S)$ for all $S \subset N, S \neq \emptyset$.*

The *extended marginal vector* of (N,V) with respect to ordering $\sigma \in \Pi(N)$ is the set of vectors $\{M^{\sigma_{|S}}(V^S)\}_{S \subset N, S \neq \emptyset}$, where $\sigma_{|S} \in \Pi(S)$ is such that $\sigma^{-1}(i) < \sigma^{-1}(j)$ implies $\sigma_{|S}^{-1}(i) < \sigma_{|S}^{-1}(j)$ for all $i, j \in S$.

Sprumont (1990) shows that for TU games, convexity implies that each of the extended marginal vectors consitutes a pmas. In his Conclusion, Sprumont poses the question whether this result can be generalised to NTU games. Moulin (1989) shows that in an ordinally convex NTU game, the extended marginal vector need not constitute a pmas.

The following example shows that marginal convexity is not sufficient either. Note that, contrary to Ichiishi (1993), ordinal convexity does not imply marginal convexity, as shown in Hendrickx et al. (2000).

Example 3.1 Consider the following NTU game with player set $N = \{1, 2, 3\}$:

$$V(\{i\}) = (-\infty, 0] \text{ for all } i \in N,$$

$$V(\{1,2\}) = \{x \in \mathbb{R}^{\{1,2\}} \mid 10x_1 + x_2 \leq 10\},$$

$$V(\{1,3\}) = \{x \in \mathbb{R}^{\{1,3\}} \mid x_1 + 10x_3 \leq 10\},$$

$$V(\{2,3\}) = \{x \in \mathbb{R}^{\{2,3\}} \mid x_2 + x_3 \leq 1\},$$

$$V(N) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq 11\}.$$

It is readily verified that this game is marginal convex. Take $\sigma = (3,1,2)$ and $S = \{1,2\}$. Then $M_2^\sigma(V) = 1$ and $M_2^{\sigma|S}(V^S) = 10$, so the extended marginal vector corresponding to σ is not a pmas. Note that each of the five other extended marginal vectors does constitute a pmas. $\triangleleft$

Individual merge convexity does turn out to be sufficient for the extended marginal vector to be a pmas, as shown in the following theorem.

Theorem 3.2 *Let $(N,V) \in NTU^N$ be an individual merge convex game and let $\sigma \in \Pi(N)$. Then $\{M^{\sigma|S}(V^S)\}_{S \subset N, S \neq \emptyset}$ is a pmas for (N,V).*

Proof: First note that property (3.1) is satisfied by construction. For (3.2), let $S \subset T \subset N, S \neq \emptyset$ and let $i \in S$. Define $\bar{S} = \{j \in S \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$ and $\bar{T} = \{j \in T \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$. We show that $M_i^{\sigma|S}(V^S) \leq M_i^{\sigma|T}(V^T)$ by distinguishing between three cases:

- If $\bar{S} = \bar{T}$, then $M_i^{\sigma|S}(V^S) = M_i^{\sigma|T}(V^T)$ by construction.
- Otherwise, if $\bar{S} = \emptyset$, then $M_i^{\sigma|S}(V^S) = 0$ and by individual superadditivity, $M_i^{\sigma|T}(V^T) \geq 0$.
- Otherwise, apply the individual merge property to i, $\bar{S}$ and $\bar{T}$. Since (N,V) is individual merge convex, the game V and all its subgames are also marginal convex. Since $M^{\sigma|S}(V^S) \in C(V^S)$, there exists a $p \in WPar(\bar{S}) \cap IR(\bar{S})$ such that $M_{\bar{S}}^{\sigma|S}(V^S) \geq p$. Taking $q = M_{\bar{T}}^{\sigma|T}(V^T) \in V(\bar{T})$ and $r = M_{\bar{S}\cup\{i\}}^{\sigma|S}(V^S) \in V(\bar{S} \cup \{i\})$, the individual merge property states that there exists an $s \in V(\bar{T} \cup \{i\})$ such that $s_i \geq M_i^{\sigma|S}(V^S)$ and $s_{\bar{T}} \geq M_{\bar{T}}^{\sigma|T}(V^T)$. By construction $M^{\sigma|T}(V^T) \geq s_i$ and hence, $M_i^{\sigma|S}(V^S) \leq M_i^{\sigma|T}(V^T)$.

From these three cases it follows that (3.2) is satisfied as well and hence, $\{M^{\sigma_{|S}}(V^S)\}_{S\subset N,S\neq\emptyset}$ is a pmas for (N,V). □

An immediate consequence of the TU equivalent of Theorem 3.2, as noted by Sprumont (1990), is that for convex TU games, the extended Shapley value is a pmas.

An NTU generalisation of the Shapley value is the *marginal based compromise value* or *MC value*, introduced in Otten et al. (1998). It is defined as

$$MC(V) = \alpha_V \sum_{\sigma\in\Pi(N)} M^\sigma(V),$$

where $\alpha_V = \max\{\alpha \in \mathbb{R}_+ \mid \alpha \sum_{\sigma\in\Pi(N)} M^\sigma(V) \in V(N)\}$. The *extended MC value* is the set of vectors $\{MC(V^S)\}_{S\subset N,S\neq\emptyset}$.

Using this generalisation of the Shapley value, the result in Sprumont (1990) can be extended to NTU games, as shown in the following proposition.

Proposition 3.3 *Let (N,V) be an individual merge convex game such that $C(V^S)$ is convex for all $S\subset N, S\neq\emptyset$. Then $\{MC(V^S)\}_{S\subset N,S\neq\emptyset}$ is a pmas for (N,V).*

Proof: Because the core of every subgame is a convex set, the MC value equals the average of the marginal vectors in each subgame. Using Theorem 3.2, the assertion readily follows. □

Note that in Proposition 3.3, we need the extra condition that $C(V^S)$ is convex for all S. Without this, the extended MC value need not be a pmas.

As was the case for the extended marginal vectors, marginal convexity is not sufficient for the extended MC value, even if the core of every subgame is a convex set.

Example 3.2 Consider the 3-person game of Example 3.1. The MC value of the whole game equals $(5\frac{1}{6}, 3\frac{2}{3}, 2\frac{1}{6})$, while in the subgame consisting of players 1 and 2, the MC value equals $(\frac{1}{2}, 5)$. Monotonicity condition (3.2) is violated for player 2. ◁

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