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Abstract

Preferences of a set of n individuals over a set of alternatives can be represented by a preference profile being an n-tuple of preference relations over these alternatives. A *social choice correspondence* assigns to every preference profile a subset of alternatives that can be viewed as the 'most prefered' alternatives by the society consisting of all individuals.

Two new social choice correspondences are introduced and analyzed. Both are Pareto optimal and are refinements of the well known Top cycle correspondence in case the corresponding simple majority win digraph is a tournament. One of them even is such a refinement for arbitrary preference profiles.

JEL classification: D71
Keywords: Social choice, Condorcet social choice correspondence, Top cycle, Pareto optimality, β-social choice correspondence, λ-social choice correspondence.

1 Introduction

Preferences of an individual i over a set of alternatives A can be represented by a *preference relation* p_i on A. We denote xp_iy if individual i prefers alternative x to alternative y. For a society consisting of a finite number of n agents a *preference profile* p is an n-tuple of such preference relations, each one representing the preferences of one individual. We refer to a triple (N, A, p) as above as a *social choice situation*. Although it is straightforward to find the most prefered alternative(s) in an individual preference relation that is complete and transitive, this is not the case for a preference profile which consists of n such individual preference relations. In the literature various *social choice correspondences* are defined which assign to every social choice situation a subset of alternatives which can be viewed as the 'most prefered' alternatives by the society consisting of all individuals. Examples of social choice correspondences can be found in, e.g., Schwartz (1972, 1990), Slater (1961), Fishburn (1977), Miller (1980), Banks (1985), Moulin (1986), Dutta (1988) and Laffond, Laslier and LeBreton (1993, 1995).

For a social choice situation (N, A, p) the corresponding *simple majority win digraph* is the binary relation $D_p \subset A \times A$, where the *arc* (x, y) belongs to D_p if alternative x 'defeats' alternative y by simple majority vote. So, alternative x defeats alternative y if and only if the number of individuals that (strictly) prefer x to y in their individual preference relation exceeds the number of individuals that (strictly) prefer y to x. A social choice correspondence is called *majoritarian* if it only depends on the corresponding simple majority win digraphs. In this paper we introduce two new majoritarian social choice correspondences which are defined using two *relational power measures*. Applied to social choice situations, a relational power measure is a function that assigns to every alternative in a simple majority win digraph a real number. These numbers induce an absolute ranking over the alternatives. Given a relational power measure one can define a social choice correspondence as the correspondence that assigns to a social choice situation the set of alternatives with highest power measure in the corresponding simple majority win digraph.

We use the relational power measures β and λ as introduced, respectively, in van den Brink and Borm (1995) and Borm, van den Brink and Slikker (2000), to derive two new social choice correspondences. These two new social choice correspondences, the β- and λ-social choice correspondence, turn out to perform well. Both are Pareto optimal and, in case the simple majority win digraph is a tournament, are refinements of Schwartz's Top cycle correspondence. The λ-social choice correspondence even is such a refinement of for arbitrary social choice situations. On the other hand, the β-social choice correspondence is monotone.

The paper is organized as follows. In Section 2 we discuss some preliminaries on social choice correspondences. In Section 3 we define the β-measure and give an axiomatic characterization. Section 4 discusses the corresponding β-social choice correspondence on the basis of elementary properties. In Section 5 we define the λ-measure and characterize it for strongly connected digraphs by means of determinants of special matrices. Section 6 discusses the corresponding λ-social choice corespondence. Finally, Section 7 gives some examples.

2 Preliminaries

In this paper we assume that individual preferences over a non-empty finite set of alternatives A can be represented by *weak orders*, i.e., preference relations p_i that are complete[1] and transitive[2]. A *preference profile* of a finite set of N of individuals N over a set of alternatives A is a tuple $p = (p_i)_{i \in N}$ of individual preference relations on A. A *linear order* is a weak order that is asymmetric[3]. If all orders are linear we call the profile a linear preference profile. A triple (N, A, p) is a *social choice situation*. It is called linear if p is linear. The class of all social choice situations is denoted by $\mathcal{S}$.

A *social choice correspondence* C on $S \subset \mathcal{S}$ assigns to each $(N, A, p) \in S$ a non-empty subset $C(N, A, p)$ of A. The alternatives in the subset $C(N, A, p)$ can be seen as the most prefered ones by the society consisting of all individuals. Given social choice situation (N, A, p) the *simple majority win digraph* D_p with $D_p \subset A \times A$ is defined as follows:

$$(x, y) \in D_p \Leftrightarrow n_p(x, y) > n_p(y, x),$$

where $n_p(x, y) = |\{i \in N \mid xp_iy \text{ and } \neg yp_ix\}|$ is the number of individuals that (strictly) prefer x to y in the profile p. Clearly, D_p is asymmetric[4]. If, e.g., the number of individuals is odd and all individual preferences are linear orders then D_p is a tournament[5]. Given a social choice correspondence C on S and a social choice situation $(N, A, p) \in S$ we call $C(N, A, p)$ the corresponding *social choice set*. A standard requirement for a social choice correspondence is that it satisfies the Condorcet principle. A *Condorcet winner* in (N, A, p) is an alternative $x \in A$ such that $(x, y) \in D_p$ for all $y \in A \setminus \{x\}$. The *Condorcet principle* states that a social choice correspondence on $S \subset \mathcal{S}$ should

[1] A preference relation p_i on A is complete if for every pair of distinct alternatives $x, y \in A$ at least one of the following is true: xp_iy or yp_ix.

[2] A preference relation p_i on A is transitive if for every triple of alternatives $x, y, z \in A$, xp_iy and yp_iz implies that xp_iz.

[3] A preference relation p_i on A is asymmetric if for every pair of alternatives $x, y \in A$, xp_iy implies that $\neg yp_ix$.

[4] Note that D_p also defines a preference relation. To make clear the distinction between individual preference relations and simple majority win digraphs we use different terminology for these. Definitions of completeness, transitivity and asymmetry given before for preference relations can be stated for digraphs in a straightforward way.

[5] A digraph (A, D) is a tournament if it is complete and asymmetric.

choose the Condorcet winner in any social choice situation in S which has a Condorcet winner. A social choice correspondence that satisfies the Condorcet principle is called a *Condorcet social choice correspondence*. A social choice correspondence on S is called *majoritarian* if the social choice set assigned to each social choice situation $(N, A, p) \in S$ only depends on the simple majority win digraph D_p.

The simple majority win digraph D_p does not represent the preferences of an individual but summarizes the preferences of all individuals into one preference relation that can be seen as reflecting the preferences of society. Note that even if the individual preference relations are linear, the relation D_p need not be a weak order. Consider, for example, the linear social choice situation (N, A, p) with $N = \{1, 2, 3\}$, $A = \{x, y, z\}$ and linear preference profile $p_1 : (x, y, z)$, $p_2 : (y, z, x)$ and $p_3 : (z, x, y)$.

Three famous majoritarian Condorcet social choice correspondences are the following. The *Copeland correspondence COP* is given by

$$COP(N, A, p) = \{x \in A \mid c_x(D_p) \geq c_y(D_p) \text{ for all } y \in A\},$$

where $c_x(D) = |\{y \in A \mid (x, y) \in D\}| \Leftrightarrow |\{y \in A \mid (y, x) \in D\}|$ is the *Copeland score* of alternative x in digraph (A, D). So, the Copeland score of an alternative in a simple majority win digraph is the difference between the number of alternatives that it defeats and the number of alternatives by which it is defeated. The social choice set according to the Copeland correspondence consists of the alternatives with the highest Copeland score in the corresponding simple majority win digraph[6].

Let $(A, \overline{D}_p)$ be the *transitive closure* of digraph D_p, i.e., $(x, y) \in \overline{D}_p$ if and only if there exist $x_1, \ldots, x_t \in A$ such that (i) $x_1 = x$, (ii) $(x_k, x_{k+1}) \in D_p$ for all $k \in \{1, \ldots, t \Leftrightarrow 1\}$, and (iii) $x_t = y$. A subset $T \subset A$ is a *Top cycle* in D_p if

(i) $x, y \in T, x \neq y \Rightarrow (x, y) \in \overline{D}_p$, and

(ii) $x \notin T, y \in T \Rightarrow (x, y) \notin \overline{D}_p$.

Schwartz's *Top cycle correspondence TOP* assigns to every social choice situation (N, A, p) the union of all Top cycles in D_p. Note that if the simple majority win digraph D_p is

[6] The Copeland score has been used in ranking alternatives in digraphs in, e.g., Rubinstein (1980) and Henriet (1985).

a tournament on A then it has exactly one Top cycle[7].

An alternative y is *covered* by x in a social choice situation (N, A, p) if

(i) $(y, z) \in D_p \Rightarrow (x, z) \in D_p$ for all $z \in A$, and

(ii) $(x, y) \in D_p$.

The *uncovered set correspondence UNC* assigns to every social choice situation (N, A, p) the set of alternatives that are not covered by some other alternative in (N, A, p).

In this paper we only consider domains $S \subset \mathcal{S}$ that are *comprehensive*, i.e., $(N, A, p) \in S$, $N' \subset N$ and $A' \subset A$ implies that $(N', A, p|_{N'}) \in S$ and $(N, A', p|_{A'}) \in S$, where $p|_{N'}$ and $p|_{A'}$ denote the restriction of p to N' and A', respectively. Next we recall some elementary properties of a social choice correspondence C on a comprehensive domain $S \subset \mathcal{S}$.

- **Homogeneity:** If $(N, A, p), (M, A, q) \in S$ are such that there is a $k \in \mathbb{N}$ with $|\{j \in M \mid q_j = p_i\}| = k$ for all $i \in N$, then $C(N, A, p) = C(M, a, q)$;

- **Monotonicity:** If $(N, A, p), (N, A, q) \in S$ and $x \in A$ are such that for every $i \in N$ it holds that

  (i) $y p_i z \Leftrightarrow y q_i z$ for all $y, z \in A \setminus \{x\}, y \neq z$,

  (ii) $x p_i y \Rightarrow x q_i y$ for all $y \in A \setminus \{x\}$, and

  (iii) $x \in C(N, A, p)$,

  then $x \in C(N, A, q)$;

- **Pareto optimality:** If $(N, A, p) \in S$ and $x, y \in A$ are such that

  (i) $y p_i x$ for all $i \in N$, and

  (ii) there is an $i \in N$ such that $\neg x p_i y$,

[7] Suppose there are two different Top cycles, T and T', in the tournament D_p. For every $x \in T$, $y \in T'$ it then holds that either $(x, y) \in D_p \subset \overline{D}_p$ in which case T' cannot be a Top cycle, or $(y, x) \in D_p \subset \overline{D}_p$ in which case T cannot be a Top cycle.

then $x \notin C(N, A, p)$.

- **Smith's Condorcet principle:** If for every $(N, A, p) \in S$ such that A can be partioned into nonempty subsets A_1 and A_2 with $(x, y) \in D_p$ for all $x \in A_1, y \in A_2$, it holds that $A_2 \cap C(N, A, p) = \emptyset$;
- **Condorcet transitivity:** If $(N, A, p) \in S$, $y \in C(N, A, p)$ and $(x, y) \in D_p$ then $x \in C(N, A, p)$;
- **Subset condition 1:** For every $(N, A, p) \in S$ with $|A| \geq 2$ and $x \in C(N, A, p)$, there exists an $y \in A \setminus \{x\}$ such that $x \in C(N, A \setminus \{y\}, p \mid_{A \setminus \{y\}})$;
- **Subset condition 2:** For every $(N, A, p) \in S$ with $|A| \geq 2$ and $x \in C(N, A, p)$, there exists an $y \in A \setminus \{x\}$ such that $x \in C(N, \{x, y\}, p \mid_{\{x,y\}})$.

For a discussion on these properties we refer to Fishburn (1977). A well known result that is used in this paper is that all majoritarian social choice correspondences satisfy homogeneity. Above we showed how every social choice situation (N, A, p) leads to a simple majority win digraph D_p. The next proposition shows that for every asymmetric digraph (A, D) one can construct a social choice situation such that (A, D) is the corresponding simple majority win digraph[8].

Proposition 2.1 *For every asymmetric digraph (A, D) there is a social choice situation (N, A, p) such that $D = D_p$.*

Proof

Let $A = \{x_1, \ldots, x_n\}$. For every $(x_i, x_j) \in D$ with $i < j$ consider the social choice situation (N, A, p) with $N = \{1, 2\}$, $A = \{x_1, \ldots, x_n\}$, and preference profile p given by

$$\begin{aligned} p_1 &: \quad x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n, x_i, x_j \\ p_2 &: \quad x_i, x_j, x_n, \ldots x_{j+1}, x_{j-1}, \ldots, x_{i+1}, x_{i-1}, \ldots, x_1 \end{aligned}$$

Naturally, if $i > j$ one can construct a similar profile. The simple majority win digraph of this profile is given by $D_p = \{(x_i, x_j)\}$. Combining these $|D|$ preference profiles

[8]In McGarvey (1953) this is shown for tournaments.

yields a preference profile with $2|D|$ preference relations which simple majority win digraph is (A, D).

□

3 The β-measure

We denote the set of all digraphs on A by $\mathcal{D}^A$. A *relational power measure* on a set A of alternatives is a function $f\colon \mathcal{D}^A \to \mathbb{R}^A$ that assigns an $|A|$-dimensional real vector $f(D) \in \mathbb{R}^A$ to every digraph D on A. Applying a relational power measure to simple majority win digraphs corresponding to a social choice situations in some domain S within $\mathcal{S}$, we can define the corresponding social choice correspondence as the correspondence that assigns to every social choice situation in S the set of alternatives with highest measure. An example of a relational power measure is the Copeland score, which has the Copeland correspondence as corresponding social choice correspondence. In this section and Section 5 we discuss two other relational power measures. Sections 4 and 6 discuss the corresponding social choice correspondences.

For a digraph D and an alternative $x \in A$, the alternatives in $S_D(x) = \{y \in A \mid (x, y) \in D\}$ are called the *successors* of x in D, and the alternatives in $P_D(x) = \{y \in A \mid (y, x) \in D\}$ are called the *predecessors* of x in D. Assume that each alternative has an 'initial' weight equal to one. The *β-measure* redistributes these weights according to

$$\beta_x(D) = \sum_{y \in S_D(x) \cup \{x\}} \frac{1}{|P_D(y)| + 1} \text{ for all } x \in A. \tag{1}$$

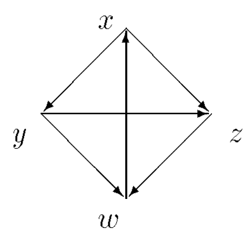
Thus, the β-measure distributes the initial weight of each alternative in a digraph equally over itself and all its predecessors[9]. Defining for every $D \in \mathcal{D}^A$ the *transition matrix* $\Pi(D)$ as the $|A| \times |A|$-matrix with entries

$$\pi_{xy} = \begin{cases} \frac{1}{|P_D(y)|+1} & \text{if } (x, y) \in D \text{ or } x = y \\ 0 & \text{otherwise,} \end{cases}$$

[9]We remark that this is not the 'original' β-measure considered in van den Brink and Gilles (1992), but the modified version considered in van den Brink and Borm (1994) who also provide a game theoretic analysis of the β-measure.

it readily follows that $\beta(D) = \Pi(D)\mathbb{1}_A$, where $\mathbb{1}_A$ is the $|A|$-dimensional vector which elements are all equal to one.

Example 3.1 Consider the (tournament) digraph D on $A = \{x, y, z, w\}$ given by $D = \{(x,y),(x,z),(y,z),(y,w),(z,w),(w,x)\}$.



The Copeland score of this digraph is given by $c(D) = (1, 1, -1, -1)$. According to the Copeland score alternatives x and y are ranked equally and are ranked higher than alternatives z and w (which are both ranked equal). However, the β-measure of this digraph is given by $\beta(D) = (\frac{4}{3}, \frac{7}{6}, \frac{2}{3}, \frac{5}{6})$. According to this β-measure alternative x is ranked higher than alternative y. □

Next we characterize the β-measure as a relational power measure for digraphs. Alternatives $x, y \in A$ are *connected* in digraph D if there is a sequence of alternatives $(x_1, \ldots, x_m)$ such that (i) $x_1 = x$, (ii) $\{(x_k, x_{k+1}), (x_{k+1}, x_k)\} \cap D \neq \emptyset$ for all $k \in \{1, \ldots, m-1\}$, and (iii) $x_m = y$. A subset of alternatives $T \subset A$ is a *maximally connected subset* in a digraph D if (i) every pair of alternatives $x, y \in T$ is connected in D, and (ii) no $x \in T$ and $y \in A \setminus T$ are connected in D.

A relational power measure is *component efficient* if the sum of the power measures assigned to all alternatives in a maximally connected subset of alternatives is equal to the number of alternatives in that maximally connected subset.

Axiom 3.2 (Component efficiency) *For every $D \in \mathcal{D}^A$ and every maximally connected subset T of alternatives in D it holds that $\sum_{x \in T} f_x(D) = |T|$.*

Consider a digraph $D \in \mathcal{D}^A$ and two alternatives $x, z \in A$ that both are predecessors of an alternative $y \in A$. The second axiom states that deleting the arc between x and

y has the same effect on the power measure of x as deleting the arc between z and y has on the power measure of z.

Axiom 3.3 (Equal loss property) *For every $D \in \mathcal{D}^A$, $y \in A$ and $x, z \in P_D(y)$, $f_x(D) \Leftrightarrow f_x(D \setminus \{(x,y)\}) = f_z(D) \Leftrightarrow f_z(D \setminus \{(z,y)\})$.*

The third axiom states that the power measures of the alternatives are 'locally' determined in the sense that the measure of alternative x does not change if the relation only changes 'far away' from x as described in the following axiom.

Axiom 3.4 (Local determinateness) *For every $D \in \mathcal{D}^A$, $x, y \in A$ satisfying $y \notin (S_D(x) \cup \{x\})$, and $z \in P_D(y)$, $f_x(D) = f_x(D \setminus \{(z,y)\})$.*

We refer to the difference $f_x(D) \Leftrightarrow f_x(D \setminus \{(z,y)\})$ as the *marginal difference* of x with respect to (z,y) in D. The fourth axiom states that for every alternative y that is defeated in D the sum of the marginal differences of all predecessors of y with respect to their arc with y minus the sum of marginal differences of y with respect to its arcs with all its predecessors equals one.

Axiom 3.5 (Marginal difference property) *For every $D \in \mathcal{D}^A$ and $y \in A$ with $P_D(y) \neq \emptyset$,*

$$\sum_{x \in P_D(y)} [(f_x(D) \Leftrightarrow f_x(D \setminus \{(x,y)\})) \Leftrightarrow (f_y(D) \Leftrightarrow f_y(D \setminus \{(x,y)\}))] = 1\,.$$

Thus the 'shift in power' resulting from one by one deleting the arcs on which one particular alternative is defeated equals the total power over that alternative. The four axioms stated above uniquely determine the β-measure as a relational power measure.

Proposition 3.6 *The β-measure is the unique relational power measure $f\colon \mathcal{D}^A \to \mathbb{R}^A$ that satisfies component efficiency, the equal loss property, local determinateness and the marginal difference property.*

Proof

It is easy to verify that β satisfies component efficiency and local determinateness.
Let $D \in \mathcal{D}^A$, $y \in A$ and $x, z \in P_D(y)$. Then

$$\beta_x(D) \Leftrightarrow \beta_x(D \setminus \{(x,y)\}) = \beta_z(D) \Leftrightarrow \beta_z(D \setminus \{(z,y)\}) = \frac{1}{|P_D(y)|+1},$$

showing that β satisfies the equal loss property.
Now, let $D \in \mathcal{D}^A$ and $y \in A$ be such that $P_D(y) \neq \emptyset$. Then

$$\sum_{x \in P_D(y)} [(\beta_x(D) \Leftrightarrow \beta_x(D \setminus \{(x,y)\})) \Leftrightarrow (\beta_y(D) \Leftrightarrow \beta_y(D \setminus \{(x,y)\}))] =$$

$$\sum_{x \in P_D(y)} \left(\frac{1}{|P_D(y)|+1} \Leftrightarrow \left(\frac{1}{|P_D(y)|+1} \Leftrightarrow \frac{1}{|P_D(y)|} \right) \right) = \sum_{x \in P_D(y)} \frac{1}{|P_D(y)|} = 1,$$

showing that β satisfies the marginal difference property.

In order to prove the theorem it remains to show that $f(D)$ is uniquely determined if $f \colon \mathcal{D}^A \to \mathbb{R}^A$ satisfies the four axioms. Therefore, suppose that $f \colon \mathcal{D}^A \to \mathbb{R}^A$ satisfies the four axioms. Let $D \in \mathcal{D}^A$. We proceed by induction on the number $|D|$ of elements in D.
If $|D| = 0$ then $D = \emptyset$, and component efficiency implies that $f_x(D) = 1$ for all $x \in A$.
Proceeding by induction, assume that $f(D')$ is uniquely determined for all digraphs $D' \in \mathcal{D}^A$ with $|D'| \leq k \Leftrightarrow 1$, and let $D \in \mathcal{D}^A$ with $|D| = k \geq 1$. Further, let $y \in A$ with $P_D(y) \neq \emptyset$, and take $z \in P_D(y)$.
For every $x \in A \setminus (P_D(y) \cup \{y\})$ local determinateness and the induction hypothesis imply that the values

$$f_x(D) = f_x(D \setminus \{(z,y)\}) \tag{2}$$

are uniquely determined. For every $x \in P_D(y) \setminus \{z\}$ the equal loss property implies that

$$f_x(D) \Leftrightarrow f_x(D \setminus \{(x,y)\}) = f_z(D) \Leftrightarrow f_z(D \setminus \{(z,y)\}).$$

With the induction hypothesis it follows that the values

$$f_x(D) \Leftrightarrow f_z(D) = f_x(D \setminus \{(x,y)\}) \Leftrightarrow f_z(D \setminus \{(z,y)\}) \tag{3}$$

are uniquely determined for all $x \in P_D(y) \setminus \{z\}$. This yields $|P_D(y)| \Leftrightarrow 1$ linear equations. The marginal difference property and the induction hypothesis imply that

$$\sum_{x \in P_D(y)} (f_x(D) \Leftrightarrow f_y(D)) = 1 + \sum_{x \in P_D(y)} (f_x(D \setminus \{(x,y)\}) \Leftrightarrow f_y(D \setminus \{(x,y)\})). \tag{4}$$

Finally, component efficiency[10] and equation (2) yield

$$\sum_{x \in P_D(y) \cup \{y\}} f_x(D) = |A| \Leftrightarrow \sum_{x \in A \setminus (P_D(y) \cup \{y\})} f_x(D \setminus \{(z,y)\}). \tag{5}$$

Deriving from equations (3), (4) and (5) the corresponding $(|P_D(y)|+1) \times (|P_D(y)|+1)$ matrix given by [11]

$$\begin{pmatrix} 1 & 0 & \dots & 0 & \Leftrightarrow 1 & 0 \\ 0 & 1 & \dots & 0 & \Leftrightarrow 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \Leftrightarrow 1 & 0 \\ 1 & 1 & \dots & 1 & 1 & \Leftrightarrow |P_D(y)| \\ 1 & 1 & \dots & 1 & 1 & 1 \end{pmatrix},$$

we can determine that the determinant of this matrix is not equal to zero. So, the equations (3), (4) and (5) yield $|P_D(y)| + 1$ linearly independent equations in the $|P_D(y)| + 1$ unknown variables $f_x(D)$, $x \in P_D(y) \cup \{y\}$, and thus these variables are uniquely determined. Uniqueness of $f(D)$ then follows with (2).

□

[10]Note that component efficiency implies *efficiency* stating that $\sum_{x \in A} f_x(D) = |A|$ for all digraphs $D \in \mathcal{D}^A$.

[11]The first $|P_D(y)| - 1$ rows follow from (3), the $|P_D(y)| - th$ row from (4), and the last row from (5). The first $|P_D(y)| - 1$ columns correspond to the alternatives $x \in P_D(y) \setminus \{z\}$, the $|P_D(y)| - th$ column to z, and the last column to y.

We remark that the independence of the axioms stated in Theorem 3.6 can be illustrated by presenting four alternative relational power measures that each satisfy three but not all four of the axioms[12].

We end this section by remarking that two alternative characterizations of β are obtained by replacing component efficiency and the equal loss property in Theorem 3.6 by efficiency and *symmetry*[13], or by replacing component efficiency in Theorem 3.6 by efficiency and symmetry for $D_\emptyset$. Note that the second alternative characterization uses weaker axioms than the ones in Theorem 3.6 in the sense that component efficiency implies efficiency and symmetry for $D_\emptyset$, but not the other way around.

4 The β-social choice correspondence

In this section we apply the β-measure to define the majoritarian social choice correspondence C_β given by

$$C_\beta(N, A, p) = \{x \in A \mid \beta_x(D_p) \geq \beta_y(D_p) \text{ for all } y \in A\}.$$

We verify some elementary properties.

Theorem 4.1 *The social choice correspondence C_β is a Condorcet social choice correspondence which satisfies homogeneity, monotonicity, Pareto optimality, Smith's Condorcet principle and Subset condition 2.*

PROOF

If x is the Condorcet winner in social choice situation (N, A, p) then $(S_{D_p}(y) \cup \{y\}) \subset (A \setminus \{x\})$ for all $y \in A \setminus \{x\}$. Asymmetry of D_p then implies that for every $y \in A \setminus \{x\}$,

$$\beta_x(D_p) = \sum_{z \in A} \frac{1}{|P_{D_p}(z)| + 1} > \sum_{z \in S_{D_p}(y) \cup \{y\}} \frac{1}{|P_{D_p}(z)| + 1} = \beta_y(D_p).$$

Thus, $C_\beta(N, A, p) = \{x\}$, showing that C_β is a Condorcet social choice correspondence.

[12] These alternative measures can be obtained from the authors on request.

[13] A relational power measure $f: \mathcal{D}^A \to \mathbb{R}^A$ satisfies symmetry if for every $D \in \mathcal{D}^A$ and $x, y \in D$ with $S_D(x) = S_D(y)$ and $P_D(x) = P_D(y)$, $f_x(D) = f_y(D)$. It satisfies symmetry for $D_\emptyset$ if this is only required for $D_\emptyset = \emptyset$.

Homogeneity follows by C_β being a majoritarian social choice correspondence.
To see that C_β is monotonic we consider social choice situations $(N, A, p), (N, A, q)$ such that there exists an $x \in A$ with, for every $i \in N$, (i) $yp_iz \Leftrightarrow yq_iz$ for all $y, z \in A \setminus \{x\}, y \neq z$, and (ii) $xp_iy \Rightarrow xq_iy$ for all $y \in A \setminus \{x\}$. We establish the following facts:

(a) $P_{D_q}(y) \supset P_{D_p}(y)$ for all $y \in A \setminus \{x\}$,

(b) $S_{D_q}(y) \subset S_{D_p}(y)$ for all $y \in A \setminus \{x\}$,

(c) $P_{D_q}(x) \subset P_{D_p}(x)$, and

(d) $S_{D_q}(x) \supset S_{D_p}(x)$.

Let $y \in A \setminus \{x\}$. Then these facts imply that

$$\beta_y(D_q) \Leftrightarrow \beta_y(D_p) =$$

$$= \sum_{z \in S_{D_q}(y)} \left(\frac{1}{|P_{D_q}(z)| + 1} \right) + \frac{1}{|P_{D_q}(y)| + 1} \Leftrightarrow \sum_{z \in S_{D_p}(y)} \left(\frac{1}{|P_{D_p}(z)| + 1} \right) \Leftrightarrow \frac{1}{|P_{D_p}(y)| + 1}$$

$$\leq \sum_{z \in S_{D_q}(y)} \left(\frac{1}{|P_{D_q}(z)| + 1} \right) \Leftrightarrow \sum_{z \in S_{D_p}(y)} \left(\frac{1}{|P_{D_p}(z)| + 1} \right)$$

$$\leq \sum_{z \in S_{D_q}(y) \setminus \{x\}} \left(\frac{1}{|P_{D_q}(z)| + 1} \right) \Leftrightarrow \sum_{z \in S_{D_p}(y) \setminus \{x\}} \left(\frac{1}{|P_{D_p}(z)| + 1} \right)$$

$$\leq \sum_{z \in S_{D_q}(y) \setminus \{x\}} \left(\frac{1}{|P_{D_q}(z)| + 1} \right) + \frac{1}{|P_{D_q}(x)| + 1} \Leftrightarrow \sum_{z \in S_{D_p}(y) \setminus \{x\}} \left(\frac{1}{|P_{D_p}(z)| + 1} \right) \Leftrightarrow \frac{1}{|P_{D_p}(x)| + 1}$$

$$\leq \frac{1}{|P_{D_q}(x)| + 1} \Leftrightarrow \frac{1}{|P_{D_p}(x)| + 1}$$

$$\leq \sum_{z \in S_{D_q}(x) \setminus S_{D_p}(x)} \left(\frac{1}{|P_{D_q}(z)| + 1} \right) + \frac{1}{|P_{D_q}(x)| + 1} \Leftrightarrow \frac{1}{|P_{D_p}(x)| + 1}$$

$$= \sum_{z \in S_{D_q}(x)} \left(\frac{1}{|P_{D_q}(z)|+1} \right) + \frac{1}{|P_{D_q}(x)|+1} \Leftrightarrow \sum_{z \in S_{D_p}(x)} \left(\frac{1}{|P_{D_p}(z)|+1} \right) \Leftrightarrow \frac{1}{|P_{D_p}(x)|+1}$$

$$= \beta_x(D_q) \Leftrightarrow \beta_x(D_p).$$

The first inequality follows from fact (a), the second inequality follows from facts (b) and (c), the third inequality follows from fact (c), the fourth inequality follows again from fact (a), and the first equality after the inequalities follows from fact (d).

Therefore x cannot profit less than any other alternative from replacing p with q, implying that C_β is monotonic.

To show Pareto optimality of C_β we take alternatives $y, x \in A$ such that yp_ix for all $i \in N$ and there exists an $i \in N$ such that $\neg xp_iy$. Then $(y,x) \in D_p$. Because the individual preferences are transitive we know that $(z,y) \in D_p \Rightarrow (z,x) \in D_p$, and $(x,z) \in D_p \Rightarrow (y,z) \in D_p$ for all $z \in A \setminus \{x,y\}$. Therefore $\beta_y(D_p) > \beta_x(D_p)$, and thus $x \notin C_\beta(N,A,p)$, proving Pareto optimality of C_β.

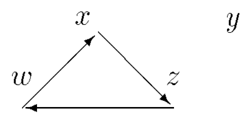
To show Smith's Condorcet principle, suppose that A can be partioned into nonempty subsets A_1 and A_2 with $(x,y) \in D_p$ for all $x \in A_1, y \in A_2$. Since $\beta_x(D_p) \geq \sum_{z \in A_2 \cup \{x\}} \frac{1}{|P_D(z)|+1} > \sum_{z \in A_2} \frac{1}{|P_D(z)|+1}$ for all $x \in A_1$, and $\beta_y(D_p) = \sum_{z \in S_D(y)} \frac{1}{|P_D(z)|+1} \leq \sum_{z \in A_2} \frac{1}{|P_D(z)|+1}$ for all $y \in A_2$, $A_2 \cap C_\beta(N,A,p) = \emptyset$, showing that C_β satisfies Smith's Condorcet principle.

To show Subset condition 2, it is clear that $\beta_x(D_p) \geq 1$ if $x \in C_\beta(N,A,p)$ since a best alternative must have at least the average β-measure. If there is no alternative $y \in A \setminus \{x\}$ with $(x,y) \in D_p$ then also there is no $y \in A \setminus \{x\}$ with $(y,x) \in D_p$, and thus $x \in \{x,y\} = C_\beta(N,\{x,y\},p|_{\{x,y\}})$ for all $y \in A \setminus \{x\}$. Otherwise, if there exists at least one alternative $y \in A \setminus \{x\}$ with $(x,y) \in D_p$ then $x \in C_\beta(N,\{x,y\},p|_{\{x,y\}})$.

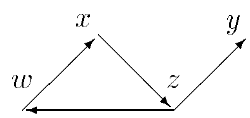
□

The social choice correspondence C_β does not satisfy all properties discussed in Section 2.

Example 4.2 Consider a social choice situation with simple majority win digraph D_p on $A = \{x,y,z,w\}$ given by $D_p = \{(x,z),(z,w),(w,x)\}$. (Existence of such a social choice situation follows from Proposition 2.1.)



Here $\beta(D_p) = (1,1,1,1)$, and thus $C_\beta(N,A,p) = A$. Even though $y \in C_\beta(N,A,p)$ it holds that $C_\beta((N, A\backslash\{x\}, p|_{A\backslash\{x\}}) = \{z\}$, $C_\beta(A \setminus \{w\}, p|_{A\backslash\{w\}}) = \{x\}$, and $C_\beta(N, A \setminus \{z\}, p|_{A\backslash\{z\}}) = \{w\}$. This shows that C_β does not satisfy Subset condition 1.



Consider a social choice situation with simple majority win digraph D_p on $A = \{x,y,z,w\}$ given by $D_p = \{(x,z),(z,w),(w,x),(z.y)\}$, we see that $\beta(D_p) = (1, \frac{1}{2}, 1, 1\frac{1}{2})$, and thus $C_\beta(N,A,p) = \{z\}$. But $(x,z) \in D_p$ showing that C_β does not satisfy Condorcet transitivity[14]. $\square$

If the simple majority win digraph corresponding to a social choice situation (N,A,p) is a tournament then the β-social choice set is a subset of the Top cycle of D_p. Since we showed that C_β is Pareto optimal, for social choice situations yielding simple majority win tournaments, C_β is a Pareto optimal refinement of *TOP*. For arbitrary social choice situations C_β is a Pareto optimal refinement of the uncovered set correspondence *UNC*.

Theorem 4.3 *For every social choice situation (N,A,p) it holds that $C_\beta(N,A,p) \subset UNC(N,A,p)$. If D_p is a tournament, then $C_\beta(N,A,p) \subset TOP(N,A,p)$.*

Proof

Let (N,A,p) be a social choice situation. If $x \notin UNC(N,A,p)$ then there is a $y \in A\backslash\{x\}$ with $(y,x) \in D_p$ and $S_{D_p}(x) \subset S_{D_p}(y)$. But then $\beta_x(D_p) = \sum_{z \in S_D(x)\cup\{x\}} \frac{1}{|P_D(z)|+1} < \sum_{z \in S_D(x)\cup\{x\}} \frac{1}{|P_D(z)|+1} + \frac{1}{|P_D(y)|+1} \leq \sum_{z \in S_D(y)\cup\{y\}} \frac{1}{P_D(y)|+1} = \beta_y(D_p)$, implying that

[14]The failure of Condorcet transitivity also follows from C_β satisfying Pareto optimality, since every Pareto optimal social choice correspondence does not satisfy Condorcet transitivity.

$x \notin C_\beta(N, A, p)$. So, $C_\beta(N, A, p) \subset UNC(N, A, p)$.
If D_p is a tournament then $C_\beta(N, A, p) \subset TOP(N, A, P)$ follows from the fact that D_p being a tournament implies that $UNC(N, A, p) \subset TOP(N, A, p)$, and the assertion shown above.

□

As shown below, C_β is not a refinement of Schwartz's Top cycle correspondence for every social choice situation.

Example 4.4 Consider a social choice situation (N, A, p) with $A = \{x, y, z, w, v\}$ which simple majority win digraph is given by $D_p = \{(x, y), (y, z), (y, w), (y, v)\}$. Then $TOP(N, A, p) = \{x\}$, while $C_\beta(N, A, p) = \{y\}$.

5 The λ-measure

In this section we discuss an alternative relational power measure that can be used in defining a social choice correspondence. In the β-measure discussed in the previous sections it is implicitly assumed that every alternative in a digraph has an initial weight equal to one, and measuring relational power is seen as fairly redistributing these weights according to the arcs in the digraph. Instead of taking initial weights equal to one, it seems natural to take weights that already reflect the preference of society over the alternatives. If the measure $\beta^1 = \beta$ determines the weights in the redistribution method discussed in Section 3, then one obtains the second order measure β^2. Of course, this second order measure can be used as new input weights, and so on, yielding higher order measures:

$$\beta^1(D) = \beta(D),$$

and

$$\beta^t_x(D) = \sum_{y \in S_D(x) \cup \{x\}} \frac{\beta^{t-1}_y(D)}{|P_D(y)| + 1} \text{ for all } x \in A \text{ and } t \in \{2, 3, \ldots\}.$$

Van den Brink, Borm, and Slikker (2000) show the existence of the limit of the iterative process which repeats the procedure described above by considering the $(k \Leftrightarrow 1)^{th}$-order β-measure as new input weights at the k^{th} step.

Definition 5.1 *The relational power measure* $\lambda\colon \mathcal{D}^A \to \mathbb{R}^A$ *is defined by*

$$\lambda(D) = \lim_{t\to\infty} \Pi^t(D)\mathbb{1}_A .$$

Example 5.2 The λ-measure of the digraph D given in Example 3.1 is $\lambda(D) = \frac{4}{23}(8, 6, 3, 6)$. So, the λ-measures of y and w are the same, while the β-measure of y exceeds that of w. □

A digraph D is *strongly connected* if A is its unique Top cycle. It turns out that the λ-measure of an alternative x in such a digraph is proportional to the cofactors of the main diagonal entries of the matrix $(I \Leftrightarrow \Pi(D))$, I being the $|A| \times |A|$-identity matrix. This follows from well-known results on Markov processes. For a survey see, e.g., Iosifescu (1980).

Proposition 5.3 *If* $D \in \mathcal{D}^A$ *is strongly connected then*

$$\lambda(D) = \frac{|A|}{\sum_{x\in A} \det(I \Leftrightarrow \Pi(D))_{-x})} \cdot (\det(I \Leftrightarrow \Pi(D))_{-x}))_{x\in N}.$$

6 The λ-social choice correspondence

Next we apply the λ-measure to define the majoritarian λ-social choice correspondence C_λ given by

$$C_\lambda(N, A, p) = \{x \in A \mid \lambda_x(D_p) \geq \lambda_y(D_p) \text{ for all } y \in A\}.$$

Compared to the properties satisfied by C_β, C_λ satisfies the properties stated in Proposition 4.1 except monotonicity.

Theorem 6.1 *The social choice correspondence* C_λ *is a Condorcet social choice correspondence which satisfies homogeneity, Pareto optimality, Smith's Condorcet principle and Subset condition 2.*

Proof

If x is the Condorcet winner in (N, A, p) then $\{x\}$ is the unique Top cycle in D_p. Borm, van den Brink and Slikker (2000) show that in that case $\lambda_x(D_p) > 0$ and $\lambda_y(D_p) = 0$ for all $y \in A \setminus \{x\}$. So, $C_\lambda(N, A, p) = \{x\}$, showing that C_λ is a Condorcet social choice correspondence.

Homogeneity follows from C_λ being majoritarian.

To show Pareto optimality of C_λ, take alternatives x, y, such that (i) yp_ix for all $i \in N$, and (ii) there is an $i \in N$ such that $\neg xp_iy$. In that case $(y, x) \in D_p$. Because the individual preferences are transitive we know that $(z, y) \in D_p$ implies $(z, x) \in D_p$, and $(x, z) \in D_p$ implies $(y, z) \in D_p$ for all $z \in A \setminus \{x, y\}$. Since (i) $\pi_{xz} \leq \pi_{yz}$ for all $z \in A \setminus \{y\}$, (ii) $\pi_{xy} = 0$, (iii) $\pi_{yy} > 0$, and (iv) λ is a *stationary* power measure[15], it follows that

$$\lambda_x(D_p) = \sum_{z \in S_D(x) \cup \{x\}} \frac{\lambda_z(D_p)}{|P_{D_p}(z)| + 1} = \sum_{z \in A} \pi_{xz} \lambda_z(D_p) < \sum_{z \in A} \pi_{yz} \lambda_z(D_p) = \lambda_y(D_p).$$

So, $x \notin C_\lambda(N, A, p)$, showing that C_λ satisfies Pareto optimality.

Suppose that A can be partioned into nonempty subsets A_1 and A_2 with $(x, y) \in D_p$ for all $x \in A_1$, $y \in A_2$. Then there is no alternative in A_2 that belongs to any Top cycle and consequently $\lambda_y(D_p) = 0$ for all $y \in A_2$, implying that $A_2 \cap C_\lambda(N, A, p) = \emptyset$, showing that C_λ satisfies Smith's Condorcet principle.

The validity of Subset condition 2 follows from a similar reasoning used for C_β in the proof of Theorem 4.1.

□

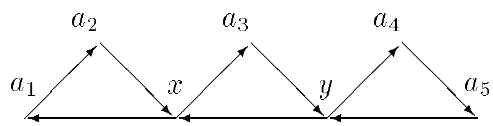
The first simple majority win digraphs given in Example 4.2 illustrates that C_λ does not satisfy Subset condition 1. Since C_λ satisfies Pareto optimality, it cannot satisfy Condorcet transitivity. One difference with the β-social choice correspondence is that C_λ does not satisfy monotonicity.

[15]Borm, van den Brink and Slikker (2000) show that for arbitrary digraphs $D \in \mathcal{D}^A$, $\lambda(D)$ is a stationary power measure meaning that $\lambda_x(D) = \sum_{y \in S_D(x) \cup \{x\}} \frac{\lambda_y(D)}{|P_D(y)|+1}$ for all $x \in A$.

Example 6.2 Consider a social choice situation (N, A, p) with $N = \{1, \ldots, 8\}$, $A = \{x, y, a_1, a_2, a_3, a_4, a_5\}$ and p given by

$$\begin{array}{llllllll} p_1: & a_1 & a_2 & x & a_3 & y & a_4 & a_5 \\ p_2: & a_5 & y & a_4 & x & a_3 & a_1 & a_2 \\ p_3: & a_2 & x & a_3 & y & a_4 & a_5 & a_1 \\ p_4: & a_1 & a_4 & a_5 & a_3 & y & a_2 & x \\ p_5: & x & a_1 & a_5 & y & a_2 & a_3 & a_4 \\ p_6: & a_4 & a_3 & a_2 & a_5 & y & x & a_1 \\ p_7: & x & y & a_1 & a_2 & a_3 & a_4 & a_5 \\ p_8: & a_5 & a_4 & a_3 & a_2 & a_1 & x & y \end{array} \tag{6}$$

This social choice situation has the simple majority win digraph illustrated by



which yields $C_\lambda(N, A, p) = \{x, y\}$ since $\lambda_x(D_p) = \lambda_y(D_p) = \frac{21}{16}$, and $\lambda_{a_i} = \frac{7}{8}$ for $i = 1, \ldots, 5$. If q is the preference profile obtained if the first individual changes its preferences to $(a_1, x, a_2, a_3, y, a_4, a_5)$, then arc (a_2, x) in the digraph disappears and we obtain $\lambda_y(A, D_q) = \frac{21}{11}$ while $\lambda_x(A, D_q) = \frac{14}{11}$. Consequently $x \notin C(N, A, q)$. □

Considering the properties discussed here, it seems that C_β is more attractive than C_λ for social choice situations having simple majority win tournament digraphs because of the monotonicity of C_β. However, C_λ is a (Pareto optimal) refinement of Schwartz's Top cycle correspondence for arbitrary social choice situations while C_β is such a refinement only if the corresponding simple majority win digraph is a tournament.

Theorem 6.3 *For every social choice situation (N, A, p) it holds that $C_\lambda(N, A, p) \subset UNC(N, A, p)$ and $C_\lambda(N, A, p) \subset TOP(N, A, p)$.*

PROOF

Let (N, A, p) be a social choice situation.
If $x \notin UNC(N, A, p)$ then there is a $y \in A \setminus \{x\}$ with $(y, x) \in D_p$ and $S_{D_p}(x) \subset S_{D_p}(y)$. But then But then $\lambda_x(D_p) = \sum_{z \in \{x\} \cup S_{D_p}(x)} \frac{\lambda_z(D_p)}{|P_{D_p}(z)|+1} \leq \sum_{z \in S_{D_p}(y)} \frac{\lambda_z(D_p)}{|P_{D_p}(z)|+1} < \sum_{z \in \{y\} \cup S_{D_p}(y)} \frac{\lambda_z(D_p)}{|P_{D_p}(z)|+1} = \lambda_y(D_p)$, and thus $x \notin C_\lambda(N, A, p)$.
So, $C_\lambda(N, A, p) \subset UNC(N, A, p)$.
Borm, van den Brink and Slikker (2000) show that well-known results on stochastic matrices as discussed in, e.g., Berger (1993) imply that $\lambda_x(D_p) = 0$ for all $x \in N \setminus TOP(N, A, p)$, and $\lambda_x(D_p) > 0$ for all $x \in TOP(N, A, p)$. But then $C_\lambda(N, A, p) \subset TOP(N, A, p)$.

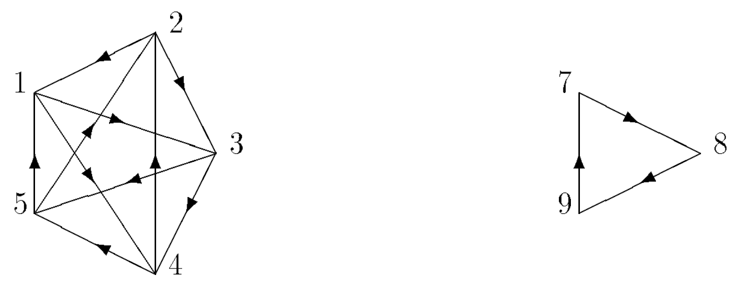
□

Without being precise, Fishburn (1977) also considers the 'discriminability' of social choice correspondences: *'The most discriminating social choice functions [read: correspondences] tend to produce choice sets that contain a single candidate; less discriminating functions are inclined to produce tied candidates in which case further means are needed to obtain unique winners'.* Indeed discriminability seems to be a desirable property. It is clear that a social choice correspondence C that assigns to every social choice situation a strict subset of the choice set assigned by social choice correspondence C' is more discriminating than C'. However, it is difficult to compare social choice correspondences which are no refinements of one another with respect to their discriminability. Loosely speaking it is clear that the β- and (more strongly) the λ-social choice correspondences mostly will assign 'few' alternatives to social choice situations with simple majority tournament digraphs, and also do well with respect to discriminability for arbitrary social choice situations.

7 Some examples

We conclude the paper by giving some examples. The first example shows that according to C_λ the alternative that 'defeats' the lowest number of other alternatives still can be the unique element of the choice set.

Example 7.1 Consider the 9-player (tournament) digraph D on $A = \{1, \ldots, 9\}$ that is represented by figures 1 and 2 together.



a: Subdigraph on $\{1, 2, 3, 4, 5\}$ b: Subdigraph on $\{7, 8, 9\}$

Figure 1: Two subdigraphs

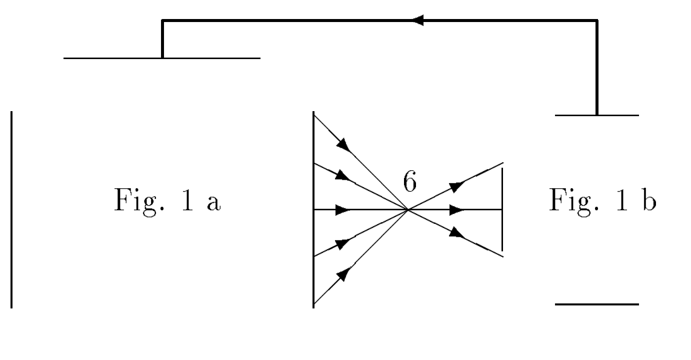


Figure 2: Digraph (A, D)

This digraph represents a tournament in which the alternatives can be divided in three groups. Group A consist of alternatives 1 through 5, group B of alternative 6 alone, and group C of alternatives 7 through 9. According to the subtournament on group A each alternative defeats 2 other alternaitives within this group and is defeated by 2 alteratives within this group. Similar results within group C, where each alternative defeats and is defeated by one alternative. It remains to describe the wins between alternatives of different groups. Here, a circular structure can be observed. Every alternative of group A defeats alternative 6, alternative 6 defeats all alternatives of group C, and every alternative of group C defeats all alternatives of group A.

The values attributed to the alternatives by λ can be computed and shown to be equal to $\frac{1}{31}(18, 18, 18, 18, 18, 54, 45, 45, 45)$. Though alternative 6 defeats only 3 alternatives, which is the least number of wins of the alternatives, it is the λ-winner[16].

We observe a surprising result if we reverse all arcs in D. The digraph that represents this situation will be denoted by D^*. Hence, alternative i defeats alternative j in D^* if and only if alternative j defeats alternative i in D. The values of the alternatives according to λ in D^* are $\frac{1}{101}(108, 108, 108, 108, 108, 180, 63, 63, 63)$. Note that alternative 6 is once more the λ-winner, i.e., reversing all results does not change the λ-winner in this example.

Example 7.2 A social choice correspondence C satisfies *Weak Condorcet consistency* on if for every pair of social choice situations $(N, A, p), (M, A, q)$ with $(x, y) \in D_p$ for all $x \in C(M, A, q)$ and $y \in A\backslash\{x\}$, it holds that $C(M, A, q) \cap C(N \cup M, A, (p, q)) \neq \emptyset$, where (p, q) is the union of the preference profiles p and q on A (see Fishburn (1977)). The two social choice correspondences discussed in this paper do not satisfy this property[17] as illustrated by the social choice situations (N, A, p) and (M, A, q) given by $N = \{1, 2\}$,

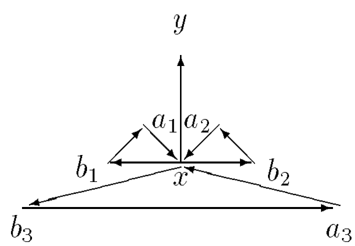
[16] This example also shows that C_λ and C_β do not satisfy *composition consistency* as defined in Laffond, Laine and Laslier (1996).

[17] This example also shows that C_β and C_λ do not satisfy the Exclusive and Inclusive Condorcet Principles as considered in Fishburn (1977).

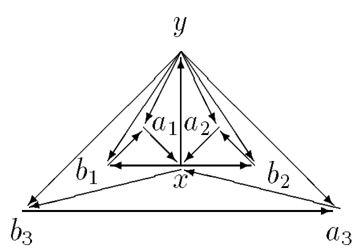
$M = \{3a, 3b, 4a, 4b, 5a, 5b, 6a, 6b, 7a, 7b, 8a, 8b\}$, $A = \{x, y, a_1, a_2, a_3, \ b_1, b_2, b_3\}$, and preference profiles p and q given by

$p_1:$	x	y	a_1	a_2	a_3	b_1	b_2	b_3
$p_2:$	x	y	b_3	b_2	b_1	a_3	a_2	a_1
$q_3:$	a_1	a_2	a_3	x	b_1	b_2	b_3	y
$q_4:$	a_3	a_2	a_1	x	b_3	b_2	b_1	y
$q_5:$	y	b_1	b_2	b_3	a_1	a_2	a_3	x
$q_6:$	y	x	b_3	b_2	b_1	a_3	a_2	a_1
$q_7:$	b_1	a_1	b_2	a_2	b_3	a_3	x	y
$q_8:$	x	y	b_3	a_3	b_2	a_2	b_1	a_1

where q_i is the preference relation of both q_{ia} and q_{ib}, $i \in \{3, \ldots, 8\}$. While (M, A, q) defines the simple majority win digraph that is illustrated by



with $C_\beta(M, A, q) = C_\lambda(M, A, q) = \{x\}$ and $(x, z) \in D_p$ for all $z \in A \setminus \{x\}$, the profile $(N \cup M, A, (p, q))$ defines the simple majority win digraph that is illustrated by



with $C_\beta(N \cup M, A, (p, q)) = C_\lambda(N \cup M, A, (p, q)) = \{y\}$. □

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