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Norde, H.W.; Reijnierse, J.H.

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By Henk Norde and Hans Reijnierse

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Henk Norde [1] Hans Reijnierse [1]

Abstract: A vector of balanced weights infers an inequality that games with a nonempty core obey. This paper gives a generalization of the notion 'vector of balanced weights'. Herewith it provides necessary and sufficient conditions to determine whether a TU-game has a population monotonic allocation scheme or not.
Furthermore it shows that every 4-person integer valued game with a population monotonic allocation scheme has an integer valued population monotonic allocation scheme and it gives an example of a 7-person integer valued game that has only non-integer valued population monotonic allocation schemes.

1. Introduction

In *Sprumont* (1990) the concept of a *population monotonic allocation scheme* (*pmas* for short) has been defined as a kind of extension of a core allocation (cf. *Moulin* (1989)). A pmas gives a core allocation for every subgame of a TU-game such that every player gets a weakly higher payoff in larger coalitions.
Games with a pmas have obviously a nonempty core. *Bondareva* (1963) and *Shapley* (1967) independently proved that a game (N, v) has a nonempty core if and only if it is *balanced*, that is, for each vector of balanced weights $\{\lambda_S\}_{S \subseteq N}$, the game obeys the corresponding inequality: $\lambda_N v(N) \geq \sum_{S \subsetneq N} \lambda_S v(S)$.
Here, a vector of balanced weights consists of nonnegative numbers with the property that $\lambda_N e_N = \sum_{S \subsetneq N} \lambda_S e_S$, in which e_S denotes the indicator vector of S. The class of balanced games is a finitely generated cone in the space of TU-games. The class of games with a pmas is a subcone of it, also finitely generated. Hence, there exists a collection of inequalities that describes this subcone. This collection is larger than the collection of inequalities describing the cone of balanced games and will be described in this paper by introducing the notion 'vector of subbalanced weights'. This description enables us to answer an open question postulated in *Reijnierse* (1995):
"Do there exist integer valued games with only non-integer valued pmasses?". We will show that the answer is negative for 4-person games. However, we give an example of such a game with 7 players.
This paper is organized as follows. After some preliminaries in Section 2 the concept of 'vectors of subbalanced weights' is introduced in Section 3. In Section 4 we provide

[1] Dept. of Econometrics, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands

an algorithm for checking whether some vector is a vector of subbalanced weights or not. In Section 5 we prove the main result of this paper, namely that a game has a pmas if and only if it satisfies all inequalities corresponding to vectors of subbalanced weights. Section 6 provides a complete description of the vectors of subbalanced weights for 4-person games, together with the result that every integer valued 4-person game with a pmas has an integer valued pmas. In Section 7 an integer valued 7-person game is given with only non-integer valued pmasses.

2. Preliminaries

Let N be a finite set and let $\mathcal{G}^N$ be the space of TU-games with player set N. Let $M := \{S \subseteq N \mid S \neq \phi\}$.

Definition 1: *A population monotonic allocation scheme or pmas of the game* (N, v) *is a table* $\boldsymbol{x} = \{x_{S,i}\}_{S \in M, i \in S}$ *with the properties:*

(i) $\sum_{i \in S} x_{S,i} = v(S)$ *for all* $S \in M$,

(ii) $x_{S,i} \leq x_{T,i}$ *for all* $S, T \in M,\ i \in S \subset T$.

The class of games with player set N that have a pmas is called PM^N, or PM if no confusion can occur. *Sprumont* (1990) was the first who proved the following result:

Theorem 2: *(Sprumont (1990)) The class* PM^N *is a cone and it is generated by the collection of all simple monotonic i-veto games in* $\mathcal{G}^N$, *united with the games* $-u_i$ $(i \in N)$.

This collection will be called $g(PM)$. A game is called *simple* if all its coalitional values are either 0 or 1, it is called *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T$ and it is called *i-veto* if $v(S) \neq 0$ implies $i \in S$. The simple game u_i is defined by $u_i(S) := 1$ if and only if $i \in S$.

Reijnierse (1995) submits a complete section to pmasses. Other results concerning pmasses can be found in *Derks* (1991).

3. Vectors of subbalanced weights

In *Sprumont* (1990) a brief dual characterization of the class PM^N is provided by introducing *vectors of subbalanced weights*. It is shown that a TU-game has a pmas if and only if it satisfies all inequalities corresponding to vectors of subbalanced weights. Although this characterization is theoretically correct, it is of very limited practical use. Vectors of subbalanced weights are defined by the principle of duality and it is a very tough task to check whether some vector is a vector of subbalanced weights or not. A systematic description of all vectors of subbalanced weights along these lines seems therefore impossible. Another remark concerning the dual characterization in *Sprumont* (1990) is the fact that only a 'dense' collection of elements of the dual cone of PM^N is provided and that many extreme directions of this dual cone are not generated by

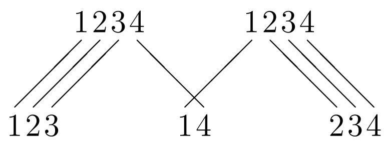
vectors of subbalanced weights. This automatically implies that, in order to see whether a game has a pmas or not, an infinite number of inequalities has to be checked.
The aim of this section is to provide a constructive definition of all elements of the dual cone of PM^N and to provide an interpretation of every such element. Since these elements can be defined by generalizing the notion of 'vectors of balanced weights' (*Bondareva* (1963) and *Shapley* (1967)), i.e. dual elements of the cone of balanced TU-games, in an appropiate way we will first focus on these vectors.
So, let $\{\lambda_S\}_{S \in M}$ be a vector of balanced weights, i.e. the vector consists of nonnegative numbers with the property that $\lambda_N e_N = \sum_{S \in M \setminus \{N\}} \lambda_S e_S$. Here e_S denotes the indicator vector of S. The inequality for some game (N, v), corresponding to this vector of balanced weights, is the following:

$$\lambda_N v(N) \geq \sum_{S \in M \setminus \{N\}} \lambda_S v(S).$$

The following interpretation can be given to this inequality: if every member of S works λ_S hours in coalition S, which generates a profit of $v(S)$ dollars per hour, and if every player in N works the same number of hours (λ_N) in total then it is more profitable for the whole society to work together all of the time in the grand coalition.
If $\{\lambda_S\}_{S \in M}$ corresponds to an extreme direction of the dual cone of the cone of balanced games we can assume that all weights λ_S are integer. An example of such a vector of balanced weights for $N = \{1, 2, 3, 4\}$ is the vector, defined by $\lambda_{1234} = 2$, $\lambda_{123} = \lambda_{14} = \lambda_{234} = 1$, and $\lambda_S = 0$ for all other S. This vector corresponds to the inequality

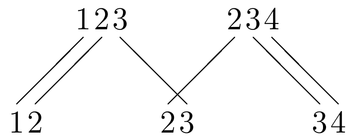
$$2v(1234) \geq v(123) + v(14) + v(234).$$

Geometrically this vector can be represented by the following matching between players in the grand coalition and identical players in proper subcoalitions:



In fact, this is the 'recipe' to construct all extreme elements of the dual cone of the cone of balanced games: start with an integer number of duplicates of the grand coalition, join these duplicates and regroup the various copies of players in proper subcoalitions, thereby of course taking care of the fact that no two copies of the same player are grouped together.
In order to get elements of the dual cone of PM^N this 'recipe' should be generalized in the sense that one can start with (duplicates of) other coalitions than the grand

coalition. Again, these coalitions are joined and regrouped, thereby taking care of the fact that every copy of some player is regrouped in some coalition which is a subset of the coalition to which he originally belonged (this condition is automatically satisfied if one has started with an integer number of duplicates of the grand coalition). An example of such a matching is



which corresponds to the inequality

$$v(123) + v(234) \geq v(12) + v(23) + v(34).$$

The following interpretation can be given to this inequality: if the society works according to the schedule 12, 23, 34, i.e. 1 and 2 work one hour together, generating a profit of $v(12)$ dollars per hour, 2 and 3 work one hour together, generating a profit of $v(23)$ dollars per hour, and 3 and 4 work one hour together, generating a profit of $v(34)$ dollars per hour, then it is more profitable for the whole society $\{1, 2, 3, 4\}$ to reschedule their efforts according to the schedule 123, 234, in which 1, 2 and 3 work one hour together and 2, 3, and 4 work one hour together. Note that according to the new schedule every player works the same amount of time as before, but in larger coalitions. The difference with vectors of balanced weights lies in the fact that the total amount of time that every player works need not be the same for every player.
Arbitrary elements of the dual cone of PM^N are obtained by considering inequalities which are nonnegative combinations of the inequalities as constructed above. This leads to the following formal definition.

Definition 3: *A vector of subbalanced weights, or vsw for short, is a tuple* $\langle \{\delta_S\}_{S \in \Delta}, \{\lambda_T\}_{T \in \Lambda} \rangle$ *with the following properties:*

(i) Δ *and* Λ *are disjoint subsets of M,*

(ii) $\delta_S > 0$ *and* $\lambda_T > 0$ *for all* $S \in \Delta$, $T \in \Lambda$,

(iii) *it is possible to assign to each triple* $(i, S, T) \in N \times \Delta \times \Lambda$ *with* $i \in T \subset S$, *a nonnegative number* $\mu^i_{S,T}$ *in such a way that:*

$$\sum_{T \in \Lambda : i \in T \subset S} \mu^i_{S,T} = \delta_S \text{ for each } S \in \Delta \text{ and } i \in S \quad \text{and}$$

$$\sum_{S \in \Delta : i \in T \subset S} \mu^i_{S,T} = \lambda_T \text{ for each } T \in \Lambda \text{ and } i \in T.$$

It is easy to infer that a vector of balanced weights is a *vsw*. Namely, if $\{\lambda_S\}_{S \in M}$ is a vector of balanced weights, take $\Delta := \{N\}$, $\Lambda := \{S \subsetneq N \mid \lambda_S > 0\}$ and $\delta_N := \lambda_N$.

Then the tuple $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ satisfies the properties (i) and (ii) of the previous definition.
Define for each $T \in \Lambda$ and every $i \in T$: $\mu^i_{N,T} := \lambda_T$. Then, for all $i \in N$:

$$\sum_{T\in\Lambda:T\ni i} \mu^i_{N,T} = \sum_{T\subsetneq N:T\ni i} \lambda_T = \lambda_N = \delta_N.$$

Moreover, for each $T \in \Lambda$ and every $i \in T$:

$$\sum_{S\in\Delta:T\subset S} \mu^i_{S,T} = \mu^i_{N,T} = \lambda_T.$$

Hence, the third property of Definition 3 has been satisfied as well.

Example 4: Let $\Delta = \{(123), (234)\}$ and $\Lambda = \{(12), (23), (34)\}$. Let $\delta_S = \lambda_T = 1$ for all $S \in \Delta$, $T \in \Lambda$. Taking

$$\mu^1_{(123),(12)} = \mu^2_{(123),(12)} = \mu^2_{(234),(23)} = \mu^3_{(123),(23)} = \mu^3_{(234),(34)} = \mu^4_{(234),(34)} = 1$$
$$\mu^2_{(123),(23)} = \mu^3_{(234),(23)} = 0,$$

one verifies that the vector $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ is a *vsw*. The *vsw* corresponds to the inequality:

$$v(123) + v(234) \geq v(12) + v(23) + v(34),$$

which is the inequality mentioned just before Definition 3. Note that the 'matches' in the geometrical representation correspond to elements (i, S, T) with $\mu^i_{S,T} = 1$.

4. Verifying whether a tuple is a *vsw*

Let $\langle\{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a tuple with properties (i) and (ii) of Definition 3. How can we find numbers $\mu^i_{S,T}$ such that property (iii) is satisfied or show that such numbers do not exist?
Let $i \in N$. Define $\Delta^i := \{S \in \Delta \mid i \in S\}$ and $\Lambda^i := \{T \in \Lambda \mid i \in T\}$. Because

$$\sum_{S\in\Delta^i} \delta_S = \sum_{S\in\Delta^i}\left(\sum_{T\in\Lambda^i:T\subset S} \mu^i_{S,T}\right) = \sum_{T\in\Lambda^i}\left(\sum_{S\in\Delta^i:S\supset T} \mu^i_{S,T}\right) = \sum_{T\in\Lambda^i} \lambda_T,$$

the first test the tuple has to pass to be a *vsw*, is that $\sum_{S\in\Delta^i} \delta_S = \sum_{T\in\Lambda^i} \lambda_T$. If so, a flow network $\Gamma_i = \langle V, E\rangle$ is constructed as follows. The node set V consists of a *source*, a *sink* and a node for each coalition T in $\Delta^i \cup \Lambda^i$. The nodes will be called So, Si and $node(T)$ ($T \in \Delta^i \cup \Lambda^i$). The arc set E consists of directed arcs. For all $S \in \Delta^i$ there is an arc from the source to $node(S)$, called $arc(S)$. The capacity of this arc is δ_S. For all $T \in \Lambda^i$ there is an arc called $arc(T)$ from $node(T)$ to the sink with capacity λ_T. If $S \in \Delta^i$, $T \in \Lambda^i$ and $S \supset T$, there is an arc called $arc(S,T)$ from $node(S)$ to $node(T)$ with a large capacity, i.e. strictly larger than $\sum_{T\in\Lambda^i} \lambda_T$.
Find a maximal source to sink flow with the maximal flow algorithm of *Ford* and *Fulkerson* (1956). If its value f equals $\sum_{T\in\Lambda^i} \lambda_T$, take $\mu^i_{S,T}$ equal to the flow in $arc(S,T)$.
On the other hand, if there exist appropriate numbers $\mu^i_{S,T}$ (*for this particular player i*), f will equal $\sum_{T\in\Lambda^i} \lambda_T$. Namely, take the flow which uses the arcs from the source

and the arcs to the sink with full capacity and which uses the other arcs $arc(S,T)$ with capacity $\mu^i_{S,T}$.

These observations lead to the following Proposition:

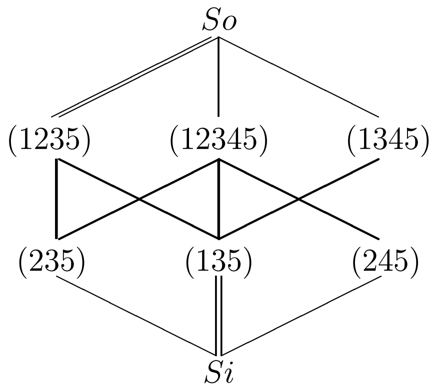
Proposition 5: *Let $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ be a tuple with properties (i) and (ii) of Definition 3. Then $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ is a vsw if and only if for every player $i \in N$:* $\sum_{S\in\Delta^i} \delta_S = \sum_{T\in\Lambda^i} \lambda_T$, *and the network Γ_i has value* $\sum_{T\in\Lambda^i} \lambda_T$.

Let us give an example of such a network.

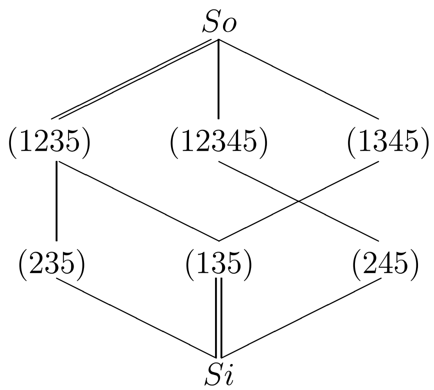
Example 6: Consider the tuple that corresponds to the inequality:

$$2v(1235) + v(12345) + v(1345) \geq v(12) + v(235) + 2v(135) + v(245) + v(134).$$

Let $i = 5$. The following figure illustrates the corresponding flow network (a node is represented by So, Si or its corresponding coalition):



Thick arcs have large capacity, single tiny arcs have capacity 1, double tiny arcs have capacity 2. The network has a unique maximal flow, depicted in the following figure:



Hence, the value f of the maximal flow equals 4, which is necessary for the tuple to be a *vsw*. The flow shows how the numbers $\mu^5_{S,T}$ can be chosen:

$\mu^5_{(1235),(235)} = \mu^5_{(1235),(135)} = \mu^5_{(12345),(245)} = \mu^5_{(1345),(135)} = 1$ and

$\mu^5_{(12345),(235)} = \mu^5_{(12345),(135)} = 0.$

For each player in N we can perform this test. Since all tests have a positive answer, the tuple is a *vsw*.

5. The duality result

We start this section with an example.

Example 7: If a game (N, v), with $N = \{1,2,3,4\}$, has a *pmas* $\boldsymbol{x}$, then it obeys the inequality

$$v(123) + v(234) \geq v(12) + v(23) + v(34),$$

which corresponds to the *vsw* in Example 4, since

$$\begin{gathered} v(123) + v(234) = \\ x_{123,1} + x_{123,2} + x_{123,3} + x_{234,2} + x_{234,3} + x_{234,4} \geq \\ x_{12,1} + x_{12,2} + x_{23,3} + x_{23,2} + x_{34,3} + x_{34,4} = \\ v(12) + v(23) + v(34). \end{gathered}$$

Each relation corresponding to a *vsw* is a necessary condition for having a pmas:

Theorem 8: *Let the game (N, v) have a population monotonic allocation scheme, say $\boldsymbol{x}$, and let $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda} \rangle$ be a vsw. Then v obeys the inequality:*

$$\sum_{S\in\Delta} \delta_S v(S) \geq \sum_{T\in\Lambda} \lambda_T v(T).$$

Proof: Let for $i \in N$, $S \in \Delta$ and $T \in \Lambda$ with $i \in T \subset S$ the numbers $\mu^i_{S,T}$ be as in Definition 3. We have:

$$\begin{aligned} &\sum_{S\in\Delta} \delta_S v(S) &&= \sum_{S\in\Delta} \delta_S \sum_{i\in S} x_{S,i} &&= \sum_{S\in\Delta} \sum_{i\in S} \delta_S x_{S,i} &&= \\ &\sum_{S\in\Delta} \sum_{i\in S} \sum_{T\in\Lambda: i\in T\subset S} \mu^i_{S,T} x_{S,i} &&= \sum_{(i,S,T): i\in T\subset S} \mu^i_{S,T} x_{S,i} &&\geq \sum_{(i,S,T): i\in T\subset S} \mu^i_{S,T} x_{T,i} &&= \\ &\sum_{T\in\Lambda} \sum_{i\in T} \sum_{S\in\Delta: i\in T\subset S} \mu^i_{S,T} x_{T,i} &&= \sum_{T\in\Lambda} \sum_{i\in T} \lambda_T x_{T,i} &&= \sum_{T\in\Lambda} \lambda_T v(T). \end{aligned}$$

$\triangleleft$

Corollary 9: *Let (N, v) have a population monotonic allocation scheme $\boldsymbol{x}$ and let $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda} \rangle$ be a vsw with associated numbers $\{\mu^i_{S,T} \mid i \in T \in \Lambda, T \subset S \in \Delta\}$. Suppose that $\sum_{S\in\Delta} \delta_S v(S) = \sum_{T\in\Lambda} \lambda_T v(T)$. Then $x_{S,i} = x_{T,i}$ for every triple (i, S, T) with $i \in T \subset S$ and $\mu^i_{S,T} > 0$.*

The converse of Theorem 8 is also true:

Theorem 10: *Let the game (N, v) obey all inequalities that arise from vsw's. Then v has a population monotonic allocation scheme.*

Proof: The dual of the cone *PM*, called PM^*, is defined by: $\{w \in \mathcal{G}^N \mid \langle w, v \rangle \geq 0$ for all $v \in PM\}$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathcal{G}^N$, i.e. $\langle w, v \rangle := \sum_{S\in M} w(S)v(S)$.

Because the cone PM is generated by the finite collection $g(PM)$ (Theorem 2), we have $PM = PM^{**} = \{v \in \mathcal{G}^N \mid \langle w, v \rangle \geq 0$ for all $w \in PM^*\}$.

Hence, to prove the Theorem, it is sufficient to prove that $v \in PM^{**}$, i.e. that $\langle w, v \rangle \geq 0$ for every $w \in PM^*$. In order to show this last statement, it suffices to prove that every

$w \in PM^*$ induces an inequality that corresponds to some *vsw*. Therefore, let $w \in PM^*$ and define $\langle \{\delta_S\}_{S\in\Delta}, \{\lambda_T\}_{T\in\Lambda}\rangle$ as follows:

$\Delta := \{S \in M \mid w(S) > 0\}$, $\Lambda := \{T \in M \mid w(T) < 0\}$,

$\delta_S := w(S)$ for every $S \in \Delta$ and $\lambda_T := -w(T)$ for every $T \in \Lambda$.

For each $i \in N$ we must show that the tuple satisfies property (iii) in Definition 3. Let $i \in N$. Define $N^{-i} := N\backslash\{i\}$ and define $w^{-i} \in \mathcal{G}^{N^{-i}}$ by $w^{-i}(S) := w(S \cup \{i\})$ for every $S \subseteq N^{-i}$. Define moreover $\Delta^{-i} := \{S \subseteq N^{-i} \mid w^{-i}(S) > 0\}$ and $\Lambda^{-i} := \{S \subseteq N^{-i} \mid w^{-i}(S) < 0\}$.

Let (N^{-i}, u) be a monotonic game in $\mathcal{G}^{N^{-i}}$. The game (N, z), defined by $z(S) := u(S\backslash\{i\})$ if $i \in S$ and $z(S) := 0$ otherwise, is a monotonic i-veto game in $\mathcal{G}^N$. So $z \in PM$ and hence $[w^{-i}, u] = \langle w, z\rangle \geq 0$. Here $[\cdot,\cdot]$ denotes the usual inner product of $\mathcal{G}^{N^{-i}}$.

So w^{-i} is an element of the dual cone of the cone of monotonic games in $\mathcal{G}^{N^{-i}}$. Since this cone is completely described by the finite set of inequalities $u(S) - u(T) \geq 0$, $T \subset S \subseteq N^{-i}$, the vector w^{-i} is a nonnegative linear combination of the vectors $e_S - e_T$, $T \subset S \subseteq N^{-i}$. One easily verifies that w^{-i} is even a nonnegative linear combination of the vectors $e_S - e_T$, with $T \subset S \subseteq N^{-i}$, $w^{-i}(S) > 0$, $w^{-i}(T) < 0$. So we may write

$$w^{-i} = \sum_{(S,T)\in\Delta^{-i}\times\Lambda^{-i}:T\subset S} \gamma_{S,T}(e_S - e_T)$$

for some nonnegative numbers $\gamma_{S,T}$, $(S,T) \in \Delta^{-i} \times \Lambda^{-i}, T \subset S$. Now define, for every $(S,T) \in \Delta \times \Lambda$ with $i \in T \subset S$, $\mu^i_{S,T} := \gamma_{S\backslash\{i\},T\backslash\{i\}}$. For every $S \in \Delta$ with $i \in S$ we have

$$\sum_{T\in\Lambda:i\in T\subset S} \mu^i_{S,T} = \sum_{T\in\Lambda:i\in T\subset S} \gamma_{S\backslash\{i\},T\backslash\{i\}} = \sum_{T\in\Lambda^{-i}:T\subset S\backslash\{i\}} \gamma_{S\backslash\{i\},T} = w^{-i}(S\backslash\{i\}) = w(S) = \delta_S$$

and for every $T \in \Lambda$ with $i \in T$ we have

$$\sum_{S\in\Delta:i\in T\subset S} \mu^i_{S,T} = \sum_{S\in\Delta:i\in T\subset S} \gamma_{S\backslash\{i\},T\backslash\{i\}} = \sum_{S\in\Delta^{-i}:T\backslash\{i\}\subset S} \gamma_{S,T\backslash\{i\}} = -w^{-i}(T\backslash\{i\}) = -w(T) = \lambda_T.$$

This finishes the proof. $\triangleleft$

6. Four person games

Consider for a (characteristic function of a) 4-person game v the following inequalities, which correspond to vsw's:

(A)	$v(ij) \geq v(i) + v(j)$	(6 inequalities)
(B)	$v(ijk) \geq v(i) + v(jk)$	(12 inequalities)
(C)	$v(1234) \geq v(i) + v(jkl)$	(4 inequalities)
(D)	$v(ijk) + v(jkl) \geq v(ij) + v(jk) + v(kl)$	(12 inequalities)
(E)	$2v(ijk) \geq v(ij) + v(ik) + v(jk)$	(4 inequalities)
(F)	$v(ijk) + v(1234) \geq v(ij) + v(jk) + v(ikl)$	(12 inequalities)
(G)	$v(1234) \geq v(ij) + v(kl)$	(3 inequalities)
(H)	$2v(1234) \geq v(ij) + v(jkl) + v(ikl)$	(6 inequalities)
(I)	$3v(1234) \geq v(123) + v(124) + v(134) + v(234)$	(1 inequality)

Different characters are used to denote different players. Note that all these inequalities can be obtained following the 'recipe' described in Section 3. If a 4-person game has a pmas then this game satisfies all conditions (A)-(I). In this section we prove that these conditions are sufficient conditions in order to guarantee that a game has a pmas. As a byproduct we get that an integer-valued 4-person game with a pmas has an integer-valued pmas. Note that the conditions (A), (B), (C) and (G) imply superadditivity, the conditions (A), (C), (G), (H) and (I) imply balancedness and the conditions (A), (B), (C), (E), (G), (H) and (I) imply totally balancedness. Note moreover that for every condition in (A)-(I) the following statement is true: if v is monotonic and $v(S) = 0$ for some coalition S occuring in the right-hand side of this condition then v satisfies this condition.

In the sequel of this section we need the following definition.

Definition 11: Let $i \in N$ and let $S_1, S_2, \ldots, S_t$ be coalitions such that $i \in S_l$ for every $l \in \{1, 2, \ldots, t\}$. The game $u_{S_1,\ldots,S_t}$ is the simple monotonic i-veto game which has $S_1, \ldots, S_t$ as minimal winning coalitions, i.e. $u_{S_1,\ldots,S_t}(S) = 1$ if and only if $S \supseteq S_l$ for some $l \in \{1, 2, \ldots, t\}$.

If a 4-person game satisfies the conditions (A)-(I) then the corresponding 0-normalized game also satisfies these conditions. This statement is an immediate consequence of the fact that linear games satisfy all conditions (A)-(I) with equality. Moreover, due to conditions (A)-(C), this 0-normalized game is monotonic. Let v be a 0-normalized monotonic game. A 0-normalized monotonic simple veto game u is *subtractable* from v if $v - \varepsilon u$ is monotonic for some $\varepsilon > 0$. Note that $v(N) > v(N \backslash i)$ is a necessary and sufficient condition for the existence of a monotonic simple i-veto game which is subtractable from v. Moreover, if u_1 and u_2 are both monotonic simple i-veto games which are subtractable from v then also $u := \max\{u_1, u_2\}$ is subtractable from v. This enables us (in case $v(N) > v(N \backslash i)$) to define u_i^v as the maximal monotonic simple i-veto game which is subtractable from v. Moreover, the positive number $a_i^v := \min\{v(S) - v(T) : S \supset T, u_i^v(S) = 1, u_i^v(T) = 0\}$ indicates how many times u_i^v can be subtracted from v at most such that the remainder is still monotonic. If $v(N) = v(N\backslash i)$ then $u_i^v := 0$.

Lemma 12: *If a 0-normalized 4-person game v satisfies conditions (A)-(I) and $v(S) > 0$ for some $S \subseteq N$ then there is an $i \in S$ such that $u_i^v(S) = 1$.*

Proof: Without loss of generality we may assume that $v(S) > v(S\backslash j)$ for every $j \in S$ (if there is a $j \in S$ with $v(S) = v(S\backslash j) > 0$ it is sufficient to prove the statement for $S\backslash j$). We distinguish between three cases: i) $|S| = 4$; ii) $|S| = 3$; iii) $|S| = 2$.
Case i): $|S| = 4$. Then $S = (1234)$. Since $v(S) > v(S\backslash j)$ for every $j \in S$ the game u_{1234} is subtractable from v. Hence $u_i^v(S) = 1$ for every $i \in S$.
Case ii): $|S| = 3$. Without loss of generality assume that $S = (123)$. Since v satisfies condition (I) there is at least one $j \in S$ with $v(N) > v(N\backslash j)$, say $j = 1$. Then at least one of the games u_{123}, $u_{123,124}$, $u_{123,134}$, $u_{123,124,134}$ or $u_{123,14}$ is subtractable from v. Hence $u_1^v(S) = 1$.
Case iii): $|S| = 2$. Without loss of generality assume that $S = (12)$. Either $v(T) > v(T\backslash 1)$ for every $T \supset S$ or $v(T) > v(T\backslash 2)$ for every $T \supset S$ (otherwise there is a $T_1 \supset S$ with $v(T_1) = v(T_1\backslash 1)$ and a $T_2 \supset S$ with $v(T_2) = v(T_2\backslash 2)$ which contradicts condition $v(T_1) + v(T_2) \geq v(12) + v(T_1\backslash 1) + v(T_2\backslash 2)$, which is one of the conditions in (D)-(F) and (H)). Assume $v(T) > v(T\backslash 1)$ for every $T \supset S$. If $v(134) = v(34)$ then $v(123) > v(13)$ (because $v(123) + v(134) \geq v(12) + v(13) + v(34)$) and $v(124) > v(14)$ (because $v(124) + v(134) \geq v(12) + v(14) + v(34)$) and hence u_{12} is subtractable. If $v(134) > v(34)$ then the monotonic simple 1-veto game u defined by $u(S) := 1$ iff $1 \in S$ and $v(S) > 0$ is subtractable from v. Anyhow, $u_1^v(S) = 1$. $\triangleleft$

Lemma 13: *Let v be a 0-normalized 4-person game that satisfies conditions (A)-(I) and let $N^v \subseteq N$ be the set of players i with $u_i^v \neq 0$. Let $i^* \in N^v$ be such that $u_{i^*}^v$ has a minimal number of veto players. Then $v - a_{i^*}^v u_{i^*}^v$ also satisfies conditions (A)-(I).*

Proof: Let $v' := v - a_{i^*}^v u_{i^*}^v$. We will distinguish between four cases.
Case i): $u_{i^*}^v$ has only one veto player, say $i^* = 1$. Then $u_1^v(123) = u_1^v(124) = u_1^v(134) = 1$. In order to show that v' satisfies all conditions (A)-(I), consider an arbitrary condition in (A)-(I), to be referred to as condition $(*)$. If u_1^v satisfies condition $(*)$ with equality then clearly v' satisfies condition $(*)$. If u_1^v satisfies condition $(*)$ with strict inequality then in the right-hand side of this inequality occurs some coalition S with $1 \in S$ and $u_1^v(S) = 0$. So $|S| \leq 2$. If $|S| = 1$ then clearly $v(S) = 0$ and hence $v'(S) = 0$. If $|S| = 2$ then $u_1^v(S) = 0$ implies $v(S) = 0$ and we also get $v'(S) = 0$. Now v' satisfies condition $(*)$ because of monotonicity of v'.
Case ii): $u_{i^*}^v$ has two veto players, say 1 and 2. So, $u_{i^*}^v = u_{12}$ or $u_{i^*}^v = u_{123,124}$. If $u_1^v(13) = 1$ then u_1^v has (13) and (124) as winning coalitions and therefore only one veto player, giving a contradiction. Hence, $u_1^v(13) = 0$. If $u_3^v(13) = 1$ then $u_3^v = u_{13}$ is subtractable and hence $u_1^v(13) = 1$ leading, as before, to a contradiction. So, $u_1^v(13) = u_3^v(13) = 0$

and hence, according to Lemma 12, $v(13) = 0$. Analogously we get $v(14) = v(23) = v(24) = 0$. Hence $v'(13) = v'(14) = v'(23) = v'(24) = 0$ and v' satisfies all conditions (A)-(F) by monotonicity. Condition (G) with $(ij) = (13)$ or (14) is satisfied by v' due to monotonicity. Condition (G) with $(ij) = (12)$ is satisfied by v' due to monotonicity in case $v(12) = v'(12) = 0$ and due to the fact that $u^v_{i^*} = u_{12}$ satisfies this condition with equality in case $v(12) > 0$. Condition (H) with $(ij) \in \{(13), (14), (23), (24)\}$ is satisfied by monotonicity of v', condition (H) with $(ij) = (34)$ is satisfied because $u^v_{i^*}$ satisfies this condition with equality and condition (H) with $(ij) = (12)$ is satisfied because $v(134) = v(34)$ (and hence $v'(134) = v'(34)$) and $v'(1234) \geq v'(12) + v'(34)$. Condition (I) is satisfied by v' because $v'(134) = v'(34)$, v' satisfies condition (H) with $(ij) = (34)$ and monotonicity of v'.
Case iii): $u^v_{i^*}$ has three veto players, say 1, 2 and 3. Then $v(S) = 0$ if $S \neq (123)$ and $S \neq (1234)$ and the statement is trivial.
Case iv): $u^v_{i^*}$ has four veto players. Then $v(S) = 0$ if $S \neq (1234)$ and the statement is trivial. $\triangleleft$

Lemmas 12 and 13 provide the basis for an algorithm in order to determine whether a 0-normalized 4-person game v has a pmas or not: compute in each step the games u^v_i and subtract that game u^v_i which has a minimal number of veto players (a^v_i times). If the game v satisfies conditions (A)-(I) then Lemma 13 guarantees that after such a step we are left with a game $v' := v - a^v_i u^v_i$ which also satisfies conditions (A)-(I). Moreover, if $v' \neq 0$, Lemma 12 guarantees that there is at least one player i with $u^{v'}_i \neq 0$ and hence the algorithm does not stop. Eventually a game v, satisfying conditions (A)-(I), is written as a positive linear combination of monotonic simple veto games and therefore, according to Theorem 2, has a pmas. Note also that if v is integer valued, all a^v_i's are integer. We have proved the following theorem.

Theorem 14: *If v is a 4-person game satisfying conditions (A)-(I) then v has a pmas. If, moreover, v is integer valued then v has an integer valued pmas.*

Example 15: Let v be the 0-normalized game, given by $v(1234) = 7$, $v(123) = v(124) = v(134) = 4$, $v(234) = 6$, $v(12) = v(13) = v(14) = 2$, $v(23) = v(24) = 3$, and $v(34) = 4$. Computing the u^v_i's we get, e.g., $u^v_2 = u_{21,23,24}$ (one veto player) with $a^v_2 = 2$. Determination of $v' = v - a^v_i u^v_i$ yields $v'(1234) = 5$, $v'(123) = v'(124) = 2$, $v'(134) = v'(234) = 4$, $v'(12) = 0$, $v'(13) = v'(14) = 2$, $v(23) = v(24) = 1$, and $v'(34) = 4$. Proceeding in the same way we subtract $u_{31,32,34}$ in the second step, $u_{31,34}$ in the third step, $u_{41,42,43}$ in the fourth step, $u_{41,43}$ in the fifth step, and u_{1234} in the sixth step, after which the algorithm ends. So, $v = 2u_{21,23,24} + u_{31,32,34} + u_{31,34} + u_{41,42,43} + u_{41,43} + u_{1234}$.

7. An integer game with only non-integer pmasses

In Section 6 we have presented an algorithm which produces for every 4-person integer valued game with a pmas an integer valued pmas. In division problems with indivisible goods the existence of such a pmas can be relevant. We will show, however, that such an algorithm does not exist for games with 7 (or more) players. We will show that there exists an integer game that has only non-integer pmasses.

Let $N = (1234567)$ and consider the inequality:

$$2v(12345) + 2v(12346) + 2v(12347) \geq$$
$$3v(1234) + v(125) + v(136) + v(147) + v(237) + v(246) + v(345).$$

It is easy to verify that it arises from a *vsw*, namely take:

$\Delta := \{(12345), (12346), (12347)\}$,
$\Lambda := \{(1234), (125), (136), (147), (237), (246), (345)\}$,
$\delta_S := 2$ for all $S \in \Delta$, $\lambda_{(1234)} := 3$, $\lambda_T := 1$ for all $T \in \Lambda\backslash\{(1234)\}$,
$\mu^i_{S,T} := 1$ for all $(i, S, T) \in N \times \Delta \times \Lambda$ with $i \in T \subset S$.

Suppose that we have a game v with a pmas $\boldsymbol{x}$ such that $v(S) = 2$ for $S \in \Delta \cup \{(1234)\}$ and $v(T) = 1$ for $T \in \Lambda\backslash\{(1234)\}$. Then the inequality is tight. By Corollary 9 we can infer that there exist numbers $\alpha_1, \ldots, \alpha_7$, such that for all $S \in \Delta \cup \Lambda$: $x_{S,i} = \alpha_i$ for all $i \in S$.

We have: $\alpha_5 = \sum_{i=1}^{5} \alpha_i - \sum_{i=1}^{4} \alpha_i = \sum_{i=1}^{5} x_{(12345),i} - \sum_{i=1}^{4} x_{(1234),i} = v(12345) - v(1234) = 0.$

Because of the symmetric roles of the players $5, 6$ and 7, $\alpha_6 = \alpha_7 = 0$ as well.

Let i and j be players in (1234). Then there is a 3-person coalition $T \in \Lambda$ which contains (ij) and one player of the coalition (567). Therefore $\alpha_i + \alpha_j = v(T) = 1$.

This makes $\alpha_i = \frac{1}{2}$ for every $i \in (1234)$.

Hence, in order to find an example we have to find a game v with the previous properties. This can be done by defining:

$v(S) = 0$	if there is no $T \in \Delta \cup \Lambda$ with $T \subseteq S$,
$v(S) = 1$	if there are $T \in \Lambda\backslash\{(1234)\}$, $U \in \Delta$ such that $T \subseteq S \subsetneq U$,
$v(S) = 2$	if $S \in \Delta \cup \{(1234)\}$ and
$v(S) = \vert S \cap (1234)\vert$	else.

Let $\boldsymbol{x}$ be defined as follows:

$x_{S,i} = 0$	if $v(S) = 0$ or $i \in (567)$,
$x_{S,i} = 0$	if $v(S) = 1$, $i \in S \cap (1234)$ and $S\backslash i \in \Lambda$,
$x_{S,i} = \frac{1}{2}$	if $v(S) = 1$, $i \in S \cap (1234)$ and $S\backslash i \notin \Lambda$,
$x_{S,i} = \frac{1}{2}$	if $S \in \Delta \cup \{(1234)\}$ and $i \in (1234)$ and
$x_{S,i} = 1$	else.

Then $\boldsymbol{x}$ is a pmas of v.

We have not been able to find examples with less than 7 players.

References:

Bondareva, O. (1963). *Certain applications of the methods of linear programming to the theory of cooperative games.* Problemy Kibernetiki 10, 119-139 (in Russian).

Derks, J. (1991). *On polyhedral cones of cooperative games.* Ph.D. Thesis, Maastricht.

Ford, L. and Fulkerson, D. (1956). *Maximal flow through a network.* Canadian Journal of Mathematics 8, 399-404.

Moulin, H. (1989). *Cores and large cores when population varies.* Mimeo, Department of Economics, Duke University.

Reijnierse, J. (1995). *Games, Graphs and Algorithms.* Ph.D. Thesis, University of Nijmegen, The Netherlands.

Shapley, L. (1967). *On balanced sets and cores.* Naval Research Logistics Quarterly 14, 453-460.

Sprumont, Y. (1990). *Population monotonic allocation schemes for cooperative games with transferable utility.* Games and Economic Behavior 2, 378-394.