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On Three Shapley-like Solutions for Cooperative Games with Random Payoffs

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Abstract

Three solution concepts for cooperative games with random payoffs are introduced. These are the marginal value, the dividend value and the selector value. Inspiration for their definitions comes from several equivalent formulations of the Shapley value for cooperative TU games. An example shows that these solutions can all be different for cooperative games with random payoffs. Properties are studied and two characterizations on subclasses of games are provided.

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1 Introduction

In this paper we introduce and study three solution concepts for cooperative games with random payoffs. An example of a cooperative situation with uncertain payoffs is the following. Two firms will be temporarily working together in an R&D project. Although the profit of this project is yet uncertain, the firms sign a contract beforehand in which their profit shares are written down.

Cooperative games with random payoffs are introduced in Timmer, Borm and Tijs (2000). In these games the payoff to a coalition is not known with certainty and is modelled as a random variable. Further, the preferences of the players and the possible allocations of the payoffs are of a specific type. Another model of games where the payoffs to the coalitions are random variables is the model of stochastic cooperative games as discussed in Suijs (2000). The difference between these games and cooperative games with random payoffs lies in the assumptions on the preferences and the structure of the set of possible allocations of the payoffs (see Timmer et al. (2000)).

The Shapley value (Shapley (1953)) is a solution concept for cooperative TU games for which several equivalent formulations exist. One of these formulations is that the Shapley value equals the average of the marginal vectors. Suijs (2000) considered this formulation of the Shapley value but was not able to extend it to his model of stochastic cooperative games because, among others, a marginal vector of a stochastic cooperative game need not be uniquely defined. Nevertheless, the nucleolus, a solution concept for TU games that we do not discuss here, has been successfully extended to stochastic cooperative games (cf. Suijs (1996, 2000)).

Inspired by the equivalent formulations of the Shapley value for TU games we define three solution concepts for cooperative games with random payoffs. These are the marginal value, the dividend value and the selector value. We study properties of these solution concepts and give two characterizations on subclasses of games. The first one is on the class of games where all players have identical preferences of a specific 'linear' type. On this class of games with random payoffs the three solution concepts coincide. The second one is a characterization on the class of one-person and two-person games, where again the three solutions coincide. These two characterizations are based on characterizations of the Shapley value for cooperative TU games by Young (1985), and by Myerson (1980), Hart and Mas-Colell (1989) and Ortmann (1998), respectively. Further, an example shows that the solutions may all be different for three-person games.

This paper is organized in four sections. In section 2 we briefly recall the main basic features of cooperative games with random payoffs. The three solution concepts are introduced in section 3. In section 4 properties of the solution concepts are studied and the two characterizations are provided. Finally, an appendix contains the proofs that are omitted in the text.

2 Cooperative games with random payoffs

A cooperative game with random payoffs is a tuple $(N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$. N is the finite player set. A coalition is a nonempty subset of N. The nonnegative random payoff to coalition S is denoted by $R(S)$ and $\mathcal{S}$ is the set of coalitions with a nonzero payoff. The set $\mathcal{A}$ contains all possible individual payoffs that a player may receive from the coalitional payoffs and α_i is a function that describes how player i compares two random payoffs. Below we explain these ingredients in more detail.

Let $N = \{1, \ldots, n\}$. Denote by $|S|$ the cardinality of coalition S. Let $\mathcal{L}_+$ be the set of all nonnegative random variables with finite expectation. The payoff zero for sure is denoted by 0. Notice that $0 \in \mathcal{L}_+$.

The payoff $R(S)$ to coalition S is assumed to be an element of $\mathcal{L}_+$. $\mathcal{S}$ is the set of coalitions with a nonzero payoff, $\mathcal{S} = \{S \subset N | R(S) \neq 0,\ S \neq \emptyset\}$. We assume the following about the payoffs. The reason for this assumption is explained in section 3.

Assumption 2.1 *If $R(T) = 0$ for some coalition T then $R(S) = 0$ for all coalitions S such that $S \subset T$.*

An allocation of the payoff $R(S)$ to the members of S is a multiple $pR(S)$ with $p \in \mathbb{R}^S$ and where player $i \in S$ receives $p_i R(S)$. Such an allocation is efficient if $\sum_{i \in S} p_i = 1$. For ease of notation define $\Delta^*(S) = \{p \in \mathbb{R}^S | \sum_{i \in S} p_i = 1\}$. The set $\mathcal{A} = \{pR(S) | S \in \mathcal{S},\ p \in \mathbb{R}\}$ contains all the payoffs that a player may receive from an allocation of the coalitional payoffs with respect to $\mathcal{S}$. All nonzero payoffs in $\mathcal{A}$ are denoted by $\mathcal{A}_{-0} = \{pR(S) \in \mathcal{A} | p \neq 0\}$.

The preference relation of player i is denoted by $\succsim_i$ and it has the following interpretation. If $X \succsim_i Y$ then agent i weakly prefers X to Y. If he is indifferent between them, $X \sim_i Y$, then $Y \succsim_i X$ and $X \succsim_i Y$, and if he strictly prefers X to Y, $X \succ_i Y$ then $X \succsim_i Y$ and not $X \sim_i Y$. We assume the following about this preference relation.

Assumption 2.2 *For all $i \in N$ there exists a surjective, coordinatewise strictly increasing and continuous function $f^i : \mathbb{R} \to \mathbb{R}^{\mathcal{S}}$ such that*

1. *$f^i_S(t)R(S) \succsim_i f^i_T(t')R(T)$ if and only if $t \geq t'$, for all $S, T \in \mathcal{S}$; $t, t' \in \mathbb{R}$.*

2. *$f^i_S(0) = 0$ for all $S \in \mathcal{S}$.*

Some examples of preference relations that satisfy this assumption are the following. Let $\mathrm{E}(X)$ denote the expectation of X. If $X \succsim_i Y$ if and only if $\mathrm{E}(X) \geq \mathrm{E}(Y)$, $X, Y \in \mathcal{A}$, then $f^i_S(t) = t/\mathrm{E}(R(S))$ for all $S \in \mathcal{S}$, $i \in N$, $t \in \mathbb{R}$, represents this preference relation. This type of preferences is called 'expectation preferences'.

A second example involves quantiles of random variables. The β_i-quantile of the random variable X is $u^X_{\beta_i} = \sup\{t \in \mathbb{R} | \Pr\{X \leq t\} \leq \beta_i\}$ with $0 < \beta_i < 1$ such that $u^{R(S)}_{\beta_i} > 0$ for all $S \in \mathcal{S}$. Define the (utility) function $U_i : \mathcal{A} \to \mathbb{R}$ by $U_i(X) = u^X_{\beta_i}$ if $X \geq 0$ and $U_i(X) = u^X_{1-\beta_i}$ otherwise. If $X \succsim_i Y$ if and only if $U_i(X) \geq U_i(Y)$, $X, Y \in \mathcal{A}$, then the functions $f^i_S(t) = t/u^{R(S)}_{\beta_i}$ describe these so-called 'quantile preferences'. Notice that both expectation and quantile preferences have linear functions f^i_S for all $S \in \mathcal{S}$, that is, $f^i_S(t) = t f^i_S(1)$ for all $t \in \mathbb{R}$.

Define the function $\alpha_i : \mathcal{A} \times \mathcal{A}_{-0} \to \mathbb{R}$ by $\alpha_i(pR(S), qR(T)) = f_T^i((f_S^i)^{-1}(p))/q$. It is the unique number $\alpha_i \in \mathbb{R}$ such that $pR(S) \sim_i \alpha_i qR(T)$, for all $i \in N$, $pR(S) \in \mathcal{A}$ and $qR(T) \in \mathcal{A}_{-0}$. Further, define $\alpha_i(0,0) = 1$. We do not define $\alpha_i(pR(S), 0), pR(S) \in \mathcal{A}_{-0}$, because it can be derived from assumption 2.2 that we have $p_iR(S) \succ_i 0$ if $p_i > 0$ and $0 \succ_i p_iR(S)$ if $p_i < 0$. Hence, there exists no $\alpha_i \in \mathbb{R}$ such that $pR(S) \sim_i \alpha_i \cdot 0 = 0$.

Some interesting and often used properties of the functions α_i, $i \in N$, are given in the following lemma.

Lemma 2.3 *For all players $i \in N$ it holds that $\alpha_i(hX, X) = h$ for any $h \in \mathbb{R}$, $X \in \mathcal{A}_{-0}$. If for player $i \in N$ the functions f_S^i, $S \in \mathcal{S}$, are linear then*

1. $\alpha_i(pR(S), qR(T)) = pf_T^i(1)/(qf_S^i(1))$ *for all* $pR(S) \in \mathcal{A}$ *and* $qR(T) \in \mathcal{A}_{-0}$*,*

2. $pR(S) \succsim_i qR(T)$ *if and only if* $p/f_S^i(1) \geq q/f_T^i(1)$ *for all* $pR(S), qR(T) \in \mathcal{A}$*.*

Proof. Let $i \in N$, $h \in \mathbb{R}$ and $X \in \mathcal{A}_{-0}$. By definition of α_i it holds that $hX \sim_i \alpha_i(hX, X)X$. From assumption 2.2 we derive that the preference relation $\succsim_i$ is monotone increasing and this implies that $h = \alpha_i(hX, X)$.

Secondly, let player i have linear functions f_S^i, $S \in \mathcal{S}$. For $pR(S) \in \mathcal{A}$ and $qR(T) \in \mathcal{A}_{-0}$ we get

$$\alpha_i(pR(S), qR(T)) = f_T^i((f_S^i)^{-1}(p))/q = f_T^i(p/f_S^i(1))/q = pf_T^i(1)/(qf_S^i(1))$$

where the first equality is by definition of α_i and the other equalities follow from the linearity of f_S^i, $S \in \mathcal{S}$. For $pR(S), qR(T) \in \mathcal{A}$ we obtain

$$pR(S) \succsim_i qR(T) \Leftrightarrow t \geq t' \text{ with } f_S^i(t) = p \text{ and } f_T^i(t') = q \Leftrightarrow p/f_S^i(1) \geq q/f_T^i(1)$$

where the first equivalence comes from assumption 2.2 and the second one from the linearity of the functions f_S^i, $S \in \mathcal{S}$. □

3 The marginal, dividend and selector values

The Shapley value for cooperative TU games is a solution for which several equivalent formulations exist. Based on these formulations, we define three solutions for cooperative games with random payoffs.

We start with some definitions. A cooperative TU game is a pair (N, v) where $N = \{1, \dots, n\}$ is the finite set of players, $v(\emptyset) = 0$ and $v(S) \in \mathbb{R}$ is the worth of coalition S. Let $\Pi(N)$ be the set of all bijections $\sigma : \{1, \dots, n\} \to N$ of N, $S_i^\sigma = \{\sigma(1), \dots, \sigma(i)\}$, $i = 1, \dots, n$, and $S_0^\sigma = \emptyset$. The marginal vector $m^\sigma(v)$ is a vector in $\mathbb{R}^N$ where player $\sigma(i)$ receives his marginal contribution to coalition S_{i-1}^σ,

$$m_{\sigma(i)}^\sigma(v) = v(S_i^\sigma) - v(S_{i-1}^\sigma),$$

for $i = 1, \ldots, n$. The Shapley value $\phi(v)$ is equal to the average of the marginal vectors:

$$\phi_i(v) = (n!)^{-1} \sum_{\sigma \in \Pi(N)} m_i^\sigma(v), \tag{3.1}$$

for all $i \in N$. For cooperative games with random payoffs we define marginal vectors as follows. Let $\sigma \in \Pi(N)$ and $G = (N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N})$. Define $Y^\sigma_{\sigma(1)} = R(\{\sigma(1)\})$, player $\sigma(1)$ receives his individual payoff. Further,

$$Y^\sigma_{\sigma(i)} = \left[1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(Y^\sigma_{\sigma(k)}, R(S_i^\sigma))\right] R(S_i^\sigma)$$

for $i = 2, \ldots, n$. $Y^\sigma_{\sigma(i)}$ is the marginal contribution of player $\sigma(i)$ to coalition S^σ_{i-1}. This contribution is the remainder of $R(S_i^\sigma)$ after the players in S^σ_{i-1} received parts that they find equivalent to their marginal contributions. Assumption 2.1 is necessary to avoid situations where $\alpha_{\sigma(k)}(Y^\sigma_{\sigma(k)}, R(S_i^\sigma))$ is not defined, that is, where $Y^\sigma_{\sigma(k)} \neq 0$ and $R(S_i^\sigma) = 0$. The marginal vector M^σ corresponding to permutation $\sigma \in \Pi(N)$ is that allocation of $R(N)$ where player i receives a multiple of $R(N)$ that is equivalent for him to Y_i^σ: $M_i^\sigma(G) = m_i^\sigma(G)R(N)$ with

$$m_i^\sigma(G) = \alpha_i(Y_i^\sigma, R(N)),$$

for all $i \in N$. Let $\mathcal{G}^N$ be the class of games $(N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N})$ with random payoffs and with player set N. A solution for cooperative games with random payoffs is a function Ψ on $\mathcal{G}^N$ such that $\Psi(G)$ is an allocation $pR(N)$ for the game $G \in \mathcal{G}^N$.

In a straightforward way we define the *marginal value*[3] Φ^m for cooperative games with random payoffs as the average of the marginal vectors.

$$\Phi_i^m(G) = \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} m_i^\sigma(G)\right] R(N)$$

A second formulation of $\phi(v)$ uses the dividends per capita $d_S(v)$ of the coalitions S, as introduced by Harsanyi (1959). These numbers are calculated in a recursive way:

$$d_S(v) = \begin{cases} v(S), & |S| = 1, \\ |S|^{-1}\left(v(S) - \sum_{T \subsetneq S} |T| d_T(v)\right), & |S| > 1. \end{cases}$$

Now the Shapley value of (N, v) can be written as

$$\phi_i(v) = \sum_{S: i \in S} d_S(v) \tag{3.2}$$

for all $i \in N$. For a cooperative game with random payoffs G we define the dividend per capita $d_S(G)$ of coalition S as follows:

$$d_S(G) = \begin{cases} R(S), & |S| = 1, \\ |S|^{-1}\left[1 - \sum_{T \subsetneq S} \sum_{j \in T} \alpha_j(d_T(G), R(S))\right] R(S), & |S| > 1. \end{cases}$$

[3]In Timmer et al. (2000) this value is called the Shapley value. Here, we consider three values based on the Shapley value for cooperative TU games. To avoid confusion, we have renamed this value as the marginal value.

The dividend per capita of a one-person coalition is equal to its payoff. If S contains more than one player then we start with its payoff $R(S)$. Given a subset T of S, $T \neq S$, we give each player $j \in T$ the dividend per capita $d_T(G)$ expressed as a multiple of $R(S)$. After we have done so for all sets $T \subset S$, $T \neq S$, we divide the remainder of $R(S)$ by $|S|$ to obtain the dividend per capita. The *dividend value* Φ^d is an extension of (3.2) and is defined by

$$\Phi_i^d(G) = \left[\sum_{S:i\in S} \alpha_i(d_S(G), R(N))\right] R(N)$$

for all $i \in N$. Player i receives the dividends per capita, expressed in multiples of $R(N)$, of all the coalitions to which he belongs.

A third formulation is given by Derks, Haller and Peters (2000) who show that the Shapley value is the average of the so-called selector vectors. Define $2^N = \{S|S \subset N\}$ and $\Delta_S(v) = |S|d_S(v)$, the dividend of coalition S. The function $\gamma : 2^N \setminus \{\emptyset\} \to N$ with $\gamma(S) \in S$ for all coalitions S is called a selector function. The family of selector functions for games with player set N is denoted by $\Gamma(N)$ and $|\Gamma(N)| = \prod_{k=2}^n k^{\binom{n}{k}}$. The selector vector $m^\gamma(v)$ corresponding to $\gamma \in \Gamma(N)$ is defined by

$$m_i^\gamma(v) = \sum_{S:\gamma(S)=i} \Delta_S(v)$$

for all $i \in N$, player i receives the dividends of those coalitions S for which $\gamma(S) = i$, and we have for all $i \in N$

$$\phi_i(v) = |\Gamma(N)|^{-1} \sum_{\gamma\in\Gamma(N)} m_i^\gamma(v).$$

For a cooperative game with random payoffs G define the dividend of coalition S, $\Delta_S(G)$, by

$$\Delta_S(G) = \begin{cases} R(S), & |S| = 1, \\ \left[1 - \sum_{T\subsetneq S} \sum_{j\in T} \alpha_j(\Delta_T(G)/|T|, R(S))\right] R(S), & |S| > 1. \end{cases}$$

The dividend $\Delta_S(G)$ of a one-person coalition S is equal to its dividend per capita, namely $R(S)$. For coalitions S with more than one player we take a subset T of S. The dividend $\Delta_T(G)$ is divided equally among the players in T. Player $j \in T$ receives the amount $\alpha_j(\Delta_T(G)/|T|, R(S))R(S)$, which is equivalent for him to $\Delta_T(G)/|T|$. The dividend of coalition S is all that remains of $R(S)$ after the dividends of the subcoalitions T have been divided. The following lemma shows that the dividend $\Delta_S(G)$ is closely related to the dividend per capita $d_S(G)$.

Lemma 3.1 $\Delta_S(G) = |S|d_S(G)$ *for all games* $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ *and any coalition* S.

Proof. Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be a cooperative game with random payoffs and let S be a coalition. We show by induction that $\Delta_S(G) = |S|d_S(G)$.

If $|S| = 1$ then $\Delta_S(G) = R(S) = d_S(G) = |S|d_S(G)$. Now assume that $\Delta_T(G) = |T|d_T(G)$ for all coalitions T with $1 \leq |T| \leq k$, $k < |N|$. Let S be a coalition with $|S| = k + 1$. We obtain

$$\begin{aligned}\Delta_S(G) &= \left[1 - \sum_{T\subsetneq S}\sum_{j\in T}\alpha_j(\Delta_T(G)/|T|, R(S))\right] R(S) \\ &= \left[1 - \sum_{T\subsetneq S}\sum_{j\in T}\alpha_j(d_T(G), R(S))\right] R(S) \\ &= |S|d_S(G)\end{aligned}$$

where the second equality follows from induction and the third equality from the definition of the dividend per capita $d_S(G)$. □

The selector vector $M^\gamma(G)$ is defined by $M_i^\gamma(G) = m_i^\gamma(G)R(N)$, $i \in N$, $\gamma \in \Gamma(N)$, where

$$m_i^\gamma(G) = \sum_{S:\gamma(S)=i} \alpha_i(\Delta_S(G), R(N)).$$

The *selector value* Φ^s is the average of these selector vectors,

$$\Phi_i^s(G) = \left[|\Gamma(N)|^{-1} \sum_{\gamma\in\Gamma(N)} m_i^\gamma(G)\right] R(N),$$

for all $i \in N$.

A first remark on these definitions is that a marginal vector need not be a selector vector, as opposed to the case for cooperative TU games. Secondly, notice that $M^\gamma(G)$ need not be an efficient allocation of $R(N)$ even if G is a game where all the functions f^i are linear. The example below illustrates this.

Example 3.2 Consider the game $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ where $N = \{1, 2, 3\}$ and the payoffs are $R(\{1\}) = R(\{2\}) = 0$, $R(\{3\}) = 1$, $R(\{1, 2\}) = 2$, $R(\{1, 3\}) = 3$, $R(\{2, 3\}) = 1$ and $R(N) \sim U([3, 7])$, that is, $R(N)$ is uniformly distributed over the interval $[3, 7]$. We see that $\mathcal{S} = \{\{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, N\}$ and $\mathcal{A} = \{pR(S)|S \in \mathcal{S},\ p \in \mathbb{R}\}$ by definition.

Let $\beta_1 = \beta_3 = 1/2$ and $\beta_2 = 1/4$. Recall from section 2 that $u_{\beta_i}^X = \sup\{t \in \mathbb{R}|\Pr\{X \leq t\} \leq \beta_i\}$ is the β_i-quantile of the random variable X. All the players $i \in N$ have quantile preferences, thus $f_S^i(t) = t/u_{\beta_i}^{R(S)}$ for all $i \in N$, $S \in \mathcal{S}$, $t \in \mathbb{R}$. From this we obtain the maps α_i for all $i \in N$.

The dividends of the various coalitions are $\Delta_{\{1\}}(G) = \Delta_{\{2\}}(G) = 0$, $\Delta_{\{3\}}(G) = 1$, $\Delta_{\{1,2\}}(G) = 2$, $\Delta_{\{1,3\}}(G) = 2$, $\Delta_{\{2,3\}}(G) = 0$ and $\Delta_N(G) = -R(N)/20$. Consider the selector function γ defined by $\gamma(\{i\}) = i$, $i \in N$, $\gamma(\{1, 2\}) = \gamma(\{1, 3\}) = \gamma(N) = 1$ and $\gamma(\{2, 3\}) = 2$. Then

$$\begin{aligned}m_1^\gamma(G) &= \alpha_1(\Delta_{\{1\}}(G), R(N)) + \alpha_1(\Delta_{\{1,2\}}(G), R(N)) + \alpha_1(\Delta_{\{1,3\}}(G), R(N)) \\ &\quad + \alpha_1(\Delta_N(G), R(N)) \\ &= 0 + 2/5 + 2/5 - 1/20 = 3/4,\end{aligned}$$

$m_2^\gamma(G) = \alpha_2(\Delta_{\{2\}}(G), R(N)) + \alpha_2(\Delta_{\{2,3\}}(G), R(N)) = 0 + 0 = 0$ and for player 3 $m_3^\gamma(G) = \alpha_3(\Delta_{\{3\}}(G), R(N)) = 1/5$. The corresponding selector vector $M^\gamma(G) = (3/4, 0, 1/5)R(N)$ is not an efficient allocation of $R(N)$. In fact, all the selector vectors in this example are not efficient but the selector value is an efficient allocation of $R(N)$ (this is a corollary of theorem 4.4). $\diamond$

4 Properties and characterizations on subclasses of games

In this section we present properties of the solution concepts that we introduced in the previous section. For two subclasses of games where the three solution concepts coincide we provide characterizations of these solutions.

Let $\mathcal{G}^N$ be a set of games $(N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ with player set N. A solution concept Ψ on $\mathcal{G}^N$

(i) is called *efficient* if for all $G \in \mathcal{G}^N$, $\Psi(G) = pR(N)$ for some $p \in \Delta^*(N)$.

(ii) is called *symmetric* if for all $G \in \mathcal{G}^N$, for all $i, j \in N$ such that $\alpha_i = \alpha_j$ and $R(S \cup \{i\}) = R(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$ we have $\Psi_i(G) = \Psi_j(G)$.

(iii) satisfies *anonymity* if for all $G \in \mathcal{G}^N$ and for all $\sigma \in \Pi(N)$ we have $\Psi(G^\sigma) = \sigma^*(\Psi(G))$ where $G^\sigma = (N, (R^\sigma(S))_{S\in\mathcal{S}^\sigma}, \mathcal{A}^\sigma, (\alpha_i^\sigma)_{i\in N})$, $R^\sigma(\sigma(U)) = R(U)$, $\mathcal{S}^\sigma = \{\sigma(S) | S \in \mathcal{S}\}$, $\mathcal{A}^\sigma = \{pR^\sigma(S) | p \in \mathbb{R},\ S \in \mathcal{S}^\sigma\}$, $\alpha^\sigma_{\sigma(i)} = \alpha_i$ and $(\sigma^*(pR(N)))_{\sigma(i)} = p_i R(N)$ for $i \in N$ and $p \in \mathbb{R}^N$.

(iv) satisfies the *null player property* if for all $G \in \mathcal{G}^N$, for all $i \in N$ such that $R(\{i\}) = 0$ and $R(S) = R(S \setminus \{i\})$ for all coalitions $S \neq \{i\}$ we have $\Psi_i(G) = 0$.

The three solution concept satisfy most of these properties.

Lemma 4.1 *The marginal value Φ^m and the dividend value Φ^d are efficient, symmetric, and they satisfy anonymity and the null player property. The selector value Φ^s is symmetric and satisfies anonymity and the null player property.*

Proof. We only show the efficiency of Φ^d. The remainder of the proof is left to the reader.

Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be a game with random payoffs. The dividend per capita of coalition N is by definition

$$d_N(G) = |N|^{-1} \left[1 - \sum_{T \subsetneq N} \sum_{j\in T} \alpha_j(d_T(G), R(N)) \right] R(N).$$

By lemma 2.3 we have

$$\alpha_i(d_N(G), R(N)) = |N|^{-1} \left[1 - \sum_{T \subsetneq N} \sum_{j\in T} \alpha_j(d_T(G), R(N)) \right]$$

for all $i \in N$. Summing both sides over N gives

$$\sum_{j\in N} \alpha_j(d_N(G), R(N)) = 1 - \sum_{T \subsetneq N} \sum_{j\in T} \alpha_j(d_T(G), R(N)).$$

Rearranging terms leads to

$$\sum_{T \subset N} \sum_{j \in T} \alpha_j(d_T(G), R(N)) = 1. \tag{4.3}$$

Hence,

$$\sum_{i \in N} \sum_{S: i \in S} \alpha_i(d_S(G), R(N)) = \sum_{S \subset N} \sum_{i \in S} \alpha_i(d_S(G), R(N)) = 1$$

where the last equality follows from (4.3). We conclude that Φ^d is an efficient allocation of $R(N)$. $\square$

We introduce another property based on its counterpart for TU games as in Young (1985).

(v) A solution concept Ψ on $\mathcal{G}^N$ satisfies *strong monotonicity* if for all $i \in N$ and for all games $G, G' \in \mathcal{G}^N$ such that[4] $M_i^\sigma(G) \succsim_i M_i^\sigma(G')$ for all $\sigma \in \Pi(N)$ we have $\Psi_i(G) \succsim_i \Psi_i(G')$.

Now we have the following result.

Lemma 4.2 *The marginal value Φ^m satisfies strong monotonicity on the class of all games G where f^i is a linear function for all $i \in N$.*

Proof. Let $G = (N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N})$ and $G' = (N, (Q(S))_{S \in \mathcal{S}'}, \mathcal{A}', (\alpha_i)_{i \in N})$ be games where all the functions f^i are linear. Let $i \in N$ be such that

$$M_i^\sigma(G) \succsim_i M_i^\sigma(G') \tag{4.4}$$

for all permutations σ. If $R(N) = Q(N) = 0$ then $\Phi_i^m(G) = \Phi_i^m(G') = 0$ because Φ^m is an allocation of the payoff for the grand coalition, which equals zero. Obviously, $\Phi_i^m(G) \succsim_i \Phi_i^m(G')$ because $0 \succsim_i 0$.

Next, consider the situation where $R(N) \neq 0$ and $Q(N) \neq 0$. By definition (4.4) equals

$$\alpha_i(Y_i^\sigma, R(N))R(N) \succsim_i \alpha_i(Y_i'^\sigma, Q(N))Q(N)$$

for all permutations σ where variables without (with) an accent refer to the game G (G'). Applying statement 2 of lemma 2.3 gives

$$\alpha_i(Y_i^\sigma, R(N))/f_N^i(1) \geq \alpha_i(Y_i'^\sigma, Q(N))/f_N'^i(1)$$

for all $\sigma \in \Pi(N)$. This implies that

$$(n!)^{-1} \sum_{\sigma \in \Pi(N)} \alpha_i(Y_i^\sigma, R(N))/f_N^i(1) \geq (n!)^{-1} \sum_{\sigma \in \Pi(N)} \alpha_i(Y_i'^\sigma, Q(N))/f_N'^i(1)$$

and once again by statement 2 of lemma 2.3 we get

$$\begin{aligned} \Phi_i^m(G) &= \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} \alpha_i(Y_i^\sigma, R(N))\right] R(N) \\ &\succsim_i \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} \alpha_i(Y_i'^\sigma, Q(N))\right] Q(N) = \Phi_i^m(G'). \end{aligned}$$

[4] We assume w.l.o.g. that the domains of the preference relations $\succsim_i$ and of the functions α_i, $i \in N$, can be extended to include all possible individual payoffs in both games.

Similar reasoning shows that this result also holds if $R(N) = 0$ and $Q(N) \neq 0$ or if $R(N) \neq 0$ and $Q(N) = 0$. $\square$

The following example shows that this result need not hold if one of the functions f^i is not linear.

Example 4.3 Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ and $G' = (N, (Q(S))_{S\in\mathcal{S}'}, \mathcal{A}', (\alpha_i)_{i\in N})$ be two games with $N = \{1, 2\}$. Variables with accents refer to the game G'. The payoffs are such that $R(\{1\}) \sim_1 1/10R(N)$, $R(\{2\}) \sim_1 4/5R(N)$, $Q(\{1\}) \sim_1 1/10Q(N)$ and $Q(\{2\}) \sim_1 4/5Q(N)$. Let σ_i denote the permutation with $\sigma_i(1) = i$ and $\sigma_i(2) = 3 - i$, $i = 1, 2$. The marginal vectors are $M^{\sigma_1}(G) = (1/10, 9/10)R(N)$, $M^{\sigma_2}(G) = (1/5, 4/5)R(N)$, $M^{\sigma_1}(G') = (1/10, 9/10)Q(N)$ and $M^{\sigma_2}(G') = (1/5, 4/5)Q(N)$. Then the marginal values are $\Phi^m(G) = (3/20, 17/20)R(N)$ and $\Phi^m(G') = (3/20, 17/20)Q(N)$.

We concentrate on player 1. Let f^1_N and f'^1_N be surjective, continuous and strictly increasing functions such that

$$f^1_N(0) = 0, f^1_N(9) = 1/10, f^1_N(11) = 3/20, f^1_N(20) = 1/5,$$
$$f'^1_N(t) = t/80,\ t \in \mathbb{R}$$

From item 1 of assumption 2.2 and from $f^1_N(9) = 1/10$, $f'^1_N(8) = 1/10$ and $9 > 8$ it follows that

$$M^{\sigma_1}_1(G) = 1/10R(N) \succ_1 1/10Q(N) = M^{\sigma_1}_1(G').$$

Similarly we obtain $M^{\sigma_2}_1(G) \succ_1 M^{\sigma_2}_1(G')$. Hence, for player 1 we have $M^{\sigma}_1(G) \succ_1 M^{\sigma}_1(G')$ for all permutations σ. Once again by assumption 2.2 and by $f'^1_N(12) = 3/20$, $f^1_N(11) = 3/20$ and $12 > 11$ we get

$$\Phi^m_1(G') = 3/20Q(N) \succ_1 3/20R(N) = \Phi^m_1(G).$$

We conclude that the marginal value does not satisfy strong monotonicity. $\diamond$

The selector value and the dividend value are equal for games where all the players $i \in N$ have linear functions f^i.

Theorem 4.4 *If G is a game where all the players have linear functions f^i then the selector value and the dividend value coincide.*

Proof. Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be a game where f^i is a linear function for all $i \in N$. From lemma 3.1 we know that $\Delta_S(G) = |S|d_S(G)$ for all coalitions S. By the linearity of f^i and by statement 1 of lemma 2.3 we have $\alpha_i(\Delta_S(G), R(N)) = |S|\alpha_i(d_S(G), R(N))$. We conclude that

$$\begin{aligned}\Phi^s_i(G) &= \left[|\Gamma(N)|^{-1} \sum_{\gamma\in\Gamma(N)} \sum_{S:\gamma(S)=i} \alpha_i(\Delta_S(G), R(N))\right] R(N) \\ &= \left[|\Gamma(N)|^{-1} \sum_{\gamma\in\Gamma(N)} \sum_{S:\gamma(S)=i} |S|\alpha_i(d_S(G), R(N))\right] R(N)\end{aligned}$$

$$
\begin{aligned}
&= \left[|\Gamma(N)|^{-1} \sum_{S \subset N} |S| \alpha_i(d_S(G), R(N)) \cdot |\{\gamma \in \Gamma(N) : \ \gamma(S) = i\}| \right] R(N) \\
&= \left[|\Gamma(N)|^{-1} \sum_{S : i \in S} |S| \alpha_i(d_S(G), R(N)) \cdot |\Gamma(N)|/|S| \right] R(N) \\
&= \left[\sum_{S : i \in S} \alpha_i(d_S(G), R(N)) \right] R(N) = \Phi_i^d(G),
\end{aligned}
$$

where the fourth equality follows from $\gamma(S) \in S$. □

A corollary of this theorem is that Φ^s is an efficient solution for any game G where all the functions f^i are linear.

Denote by GLI^N the set of games G with player set N where all the players have identical linear functions f^i. The marginal, dividend and selector value coincide on this class of games.

Theorem 4.5 *For all $G \in GLI^N$ we have $\Phi^m(G) = \Phi^d(G) = \Phi^s(G)$.*

Proof. Let $G = (N, (R(S))_{S \in \mathcal{S}}, \mathcal{A}, (\alpha_i)_{i \in N}) \in GLI^N$. From theorem 4.4 we know that $\Phi^d(G) = \Phi^s(G)$. It remains to show that $\Phi^m(G) = \Phi^d(G)$.

If $R(N) = 0$ then $\Phi_i^m(G) = \Phi_i^d(G) = 0$ for all $i \in N$ because these values are multiples of $R(N) = 0$.

If $R(N) \neq 0$ then define $f = f^i$, $i \in N$. Also, define a corresponding cooperative TU game (N, v) by $v(S) = 0$ if $R(S) = 0$, $v(S) = 1/f_S(1)$ if $R(S) \neq 0$ for all coalitions S and $v(\emptyset) = 0$. Let $\sigma \in \Pi(N)$ be a permutation of N. We show by induction that $m^\sigma(G) = m^\sigma(v)/v(N)$. For $R(\{\sigma(1)\}) \neq 0$ it holds that

$$
\begin{aligned}
m_{\sigma(1)}^\sigma(G) &= \alpha_{\sigma(1)}(R(\{\sigma(1)\}), R(N)) \\
&= f_N(1)/f_{\{\sigma(1)\}}(1) \\
&= v(\{\sigma(1)\})/v(N) \\
&= m_{\{\sigma(1)\}}^\sigma(v)/v(N)
\end{aligned}
$$

where the second equality follows from statement 1 of lemma 2.3. For $R(\{\sigma(1)\}) = 0$ this result also holds because $\alpha_{\sigma(1)}(0, R(N)) = 0 = m_{\{\sigma(1)\}}^\sigma(v)/v(N)$. Further,

$$
\begin{aligned}
Y_{\sigma(2)}^\sigma &= \left[1 - \alpha_{\sigma(1)}(R(\{\sigma(1)\}), R(S_2^\sigma)) \right] R(S_2^\sigma)) \\
&= \left[1 - f_{S_2^\sigma)}(1)/f_{S_1^\sigma)}(1) \right] R(S_2^\sigma))
\end{aligned}
$$

and so,

$$
\begin{aligned}
m_{\sigma(2)}^\sigma(G) &= \alpha_{\sigma(2)}(Y_{\sigma(2)}^\sigma, R(N)) \\
&= \left[1 - f_{S_2^\sigma}(1)/f_{S_1^\sigma}(1) \right] f_N(1)/f_{S_2^\sigma)}(1) \\
&= f_N(1) \left[1/f_{S_2^\sigma}(1) - 1/f_{S_1^\sigma}(1) \right] \\
&= \left(v(S_2^\sigma) - v(S_1^\sigma) \right) / v(N) \\
&= m_{\sigma(2)}^\sigma(v)/v(N)
\end{aligned}
$$

for $R(S_2^\sigma)) \neq 0$. If $R(S_2^\sigma) = 0$ then also $R(S_1^\sigma) = 0$ and $m_{\sigma(2)}^\sigma(G) = 0 = m_{\sigma(2)}^\sigma(v)/v(N)$ holds for the same reason as above.

Now assume that $Y_{\sigma(i)}^\sigma = \left[1 - f_{S_i^\sigma}(1)/f_{S_{i-1}^\sigma}(1)\right] R(S_i^\sigma)$ for $i = 2, \ldots, k, k < n$. Using induction we obtain for $R(S_{k+1}^\sigma) \neq 0$

$$\begin{aligned}
&Y_{\sigma(k+1)}^\sigma \\
&= \left[1 - \sum_{i=1}^{k} \alpha_{\sigma(i)}(Y_{\sigma(i)}^\sigma, R(S_{k+1}^\sigma))\right] R(S_{k+1}^\sigma) \\
&= \left[1 - \alpha_{\sigma(1)}(Y_{\sigma(1)}^\sigma, R(S_{k+1}^\sigma)) - \sum_{i=2}^{k} \alpha_{\sigma(i)}(Y_{\sigma(i)}^\sigma, R(S_{k+1}^\sigma))\right] R(S_{k+1}^\sigma) \\
&= \left[1 - f_{S_{k+1}^\sigma}(1)/f_{S_1^\sigma}(1) - \sum_{i=2}^{k} [1 - f_{S_i^\sigma}(1)/f_{S_{i-1}^\sigma}(1)] f_{S_{k+1}^\sigma}(1)/f_{S_i^\sigma}(1)\right] R(S_{k+1}^\sigma) \\
&= \left[1 - f_{S_{k+1}^\sigma}(1)/f_{S_1^\sigma}(1) - \sum_{i=2}^{k} [f_{S_{k+1}^\sigma}(1)/f_{S_i^\sigma}(1) - f_{S_{k+1}^\sigma}(1)/f_{S_{i-1}^\sigma}(1)]\right] R(S_{k+1}^\sigma) \\
&= \left[1 - f_{S_{k+1}^\sigma}(1)/f_{S_k^\sigma}(1)\right] R(S_{k+1}^\sigma),
\end{aligned}$$

and so,

$$\begin{aligned}
m_{\sigma(k+1)}^\sigma(G) &= \alpha_{\sigma(k+1)}(Y_{\sigma(k+1)}^\sigma, R(N)) \\
&= \left[1 - f_{S_{k+1}^\sigma}(1)/f_{S_k^\sigma}(1)\right] f_N(1)/f_{S_{k+1}^\sigma)}(1) \\
&= f_N(1) \left[1/f_{S_{k+1}^\sigma}(1) - 1/f_{S_k^\sigma}(1)\right] \\
&= \left(v(S_{k+1}^\sigma) - v(S_k^\sigma)\right)/v(N) \\
&= m_{\sigma(k+1)}^\sigma(v)/v(N).
\end{aligned}$$

If $R(S_{k+1}^\sigma) = 0$ then $Y_{\sigma(k+1)}^\sigma = 0$ because it is a multiple of $R(S_{k+1}^\sigma)$. Also, $R(S_j^\sigma) = 0$ for all $j \leq k+1$ by assumption 2.1 and so $m_{\sigma(k+1)}^\sigma(G) = 0 = m_{\sigma(k+1)}^\sigma(v)/v(N)$. We have shown by induction that $m^\sigma(G) = m^\sigma(v)/v(N)$.

Similar reasoning as for the marginal vectors shows that $\alpha_i(d_S(G), R(N)) = d_S(v)/v(N)$ for all coalitions S and $i \in N$. Thus we have

$$\begin{aligned}
\Phi_i^m(G) &= \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} m_i^\sigma(G)\right] R(N) \\
&= \left[(n!)^{-1} \sum_{\sigma \in \Pi(N)} m_i^\sigma(v)/v(N)\right] R(N),
\end{aligned}$$

and by (3.1) and (3.2)

$$\begin{aligned}
\Phi_i^m(G) &= \phi_i(v)/v(N) R(N) \\
&= \left[\sum_{S: i \in S} d_S(v)/v(N)\right] R(N) \\
&= \left[\sum_{S: i \in S} \alpha_i(d_S(G), R(N))\right] R(N)
\end{aligned}$$

$$= \Phi_i^d(G)$$

for all players $i \in N$. □

Furthermore, there exists a characterization of these solution concepts on the class of games GLI^N. This characterization is based on a characterization of the Shapley value for cooperative TU games by Young (1985).

Theorem 4.6 *The marginal value Φ^m is the unique solution concept on GLI^N that satisfies efficiency, symmetry and strong monotonicity.*

Proof. From the lemmas 4.1 and 4.2 it follows that Φ^m satisfies efficiency, symmetry and strong monotonicity on GLI^N.

To show the uniqueness, let Ψ be a solution concept on GLI^N that satisfies efficiency, symmetry and strong monotonicity. Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N}) \in GLI^N$ and let (N, v) be the corresponding TU game as in the proof of lemma 4.5. By efficiency there is a $p \in \Delta^*(N)$ such that $\Psi(G) = pR(N)$. Define $\psi(v) = p_i v(N)$ for all $i \in N$. This ψ is a solution concept on the class of TU games $SG^N = \{(N, v) | v \geq 0,\ v(T) = 0 \Rightarrow v(S) = 0 \text{ for all } S \subset T\}$. Further, ψ satisfies efficiency, symmetry and strong monotonicity as defined for cooperative TU games by Young (1985). In theorem A.1 of the appendix we show that the Shapley value ϕ is the *unique solution on* SG^N that satisfies efficiency, symmetry and strong monotonicity. Hence, $\psi(v) = \phi(v)$ and $\Psi(G) = \phi(v)/v(N) \cdot R(N) = \Phi^m(G)$ if $v(N) \neq 0$. If $v(N) = 0$ then we have $R(N) = 0$. By efficiency and symmetry we have $\Psi_i(G) = 0 = \Phi_i^m(G)$ for all $i \in N$. □

We will now turn our attention to games with random payoffs that need not have linear functions f^i, $i \in N$. The subgame of $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ restricted to coalition T is denoted by $G_T = (T, (R(S))_{S\in\mathcal{S}_T}, \mathcal{A}_T, (\alpha_i)_{i\in T})$ with $\mathcal{S}_T = \{S \in \mathcal{S} | S \subset T\}$, $\mathcal{A}_T = \{pR(S) \in \mathcal{A} | S \subset T\}$. Let $\bar{\mathcal{G}}^N = \cup_{M\in 2^N\setminus\{\emptyset\}} \mathcal{G}^M$ be the class of cooperative games with random payoffs and player set N, and all of its subgames. A sixth property for solution concepts on $\bar{\mathcal{G}}^N$ is based on the balanced contributions property for cooperative TU games by Myerson (1980).

(vi) A solution concept Ψ on $\bar{\mathcal{G}}^N$ is said to have *balanced contributions* if for all games $G \in \mathcal{G}^N$, for all coalitions $T \subset N$ and for all $i, j \in T$, $i \neq j$, we have

$$\alpha_i(\Psi_i(G_T), R(T)) - \alpha_i(\Psi_i(G_{T\setminus\{j\}}), R(T))$$
$$= \alpha_j(\Psi_j(G_T), R(T)) - \alpha_j(\Psi_j(G_{T\setminus\{i\}}), R(T)).$$

We have the following results concerning two-person games.

Lemma 4.7 *If G is a two-person game then $\Phi^m(G) = \Phi^d(G) = \Phi^s(G)$, the three solution concepts coincide. These solutions have balanced contributions on $\bar{\mathcal{G}}^N$ with $|N| = 2$.*

Proof. Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be a two-person game with $N = \{1, 2\}$. If $R(N) = 0$ then $R(\{1\}) = R(\{2\}) = 0$ by assumption 2.1. For any of the two permutations σ we have

$$
\begin{aligned}
M^{\sigma}_{\sigma(1)} &= \alpha_{\sigma(1)}(R(\{\sigma(1)\}), R(N))R(N) = \alpha_{\sigma(1)}(0, 0)R(N) = 1 \cdot R(N)(= 0) \\
M^{\sigma}_{\sigma(2)} &= (1 - \alpha_{\sigma(1)}(0, 0))R(N) = 0 \cdot R(N)(= 0)
\end{aligned}
$$

and the average of these marginal vectors is $\Phi^m(G) = (1/2, 1/2)R(N) = (0, 0)$. In a similar way we can show that $\Phi^d(G) = \Phi^s(G) = (0, 0)$ because these are also multiples of $R(N) = 0$.

Now assume that $R(N) \neq 0$. Let $\sigma_1(1) = 1$, $\sigma_1(2) = 2$, $\sigma_2(1) = 2$ and $\sigma_2(2) = 1$. The corresponding marginal vectors are

$$
\begin{aligned}
M^{\sigma_1}(G) &= (\alpha_1(R(\{1\}), R(N)), 1 - \alpha_1(R(\{1\}), R(N)))R(N) \\
M^{\sigma_2}(G) &= (1 - \alpha_2(R(\{2\}), R(N)), \alpha_2(R(\{2\}), R(N)))R(N)
\end{aligned}
$$

and the marginal value equals

$$
\begin{aligned}
\Phi^m(G) = \frac{1}{2}(&1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)), \\
&1 - \alpha_1(R(\{1\}), R(N)) + \alpha_2(R(\{2\}), R(N)))R(N).
\end{aligned}
$$

The dividends per capita are $d_{\{i\}}(G) = R(\{i\})$, $i = 1, 2$, for the one-person coalitions and for the grand coalition $d_N(G) = \frac{1}{2}(1 - \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))R(N)$. Therefore

$$
\begin{aligned}
\Phi^d_1(G) &= \Big(\alpha_1(d_{\{1\}}(G), R(N)) + \alpha_1(d_N(G), R(N))\Big) R(N) \\
&= \left(\alpha_1(R(\{1\}), R(N)) + \frac{1}{2}\,(1 - \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))\right) R(N) \\
&= \frac{1}{2}\,(1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))\, R(N) = \Phi^m_1(G)
\end{aligned}
$$

and similarly $\Phi^d_2(G) = \Phi^m_2(G)$.

There are only two selector functions, namely γ_1 and γ_2 defined by $\gamma_1(\{i\}) = \gamma_2(\{i\}) = i$, $i \in N$, $\gamma_1(N) = 1$ and $\gamma_2(N) = 2$. The dividends are $\Delta_{\{i\}}(G) = R(\{i\})$, $i = 1, 2$, and $\Delta_N(G) = (1 - \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))R(N)$. This leads to

$$
\begin{aligned}
m^{\gamma_1}_1(G) &= \alpha_1(\Delta_{\{1\}}(G), R(N)) + \alpha_1(\Delta_N(G), R(N)) \\
&= \alpha_1(R(\{1\}), R(N)) + 1 - \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)) \\
&= 1 - \alpha_2(R(\{2\}), R(N)),
\end{aligned}
$$

$m^{\gamma_1}_2 = \alpha_2(\Delta_{\{2\}}(G), R(N)) = \alpha_2(R(\{2\}), R(N))$ and so, $M^{\gamma_1}(G) = M^{\sigma_2}(G)$. Analogously, for selector function γ_2 we have $M^{\gamma_2}(G) = M^{\sigma_1}(G)$. We conclude that the selector value $\Phi^s(G)$, the average of the selector vectors, coincides with the marginal value $\Phi^m(G)$, the average of the marginal vectors.

Finally, we check balanced contributions for the grand coalition N. By efficiency $\Phi^m_i(G_{\{i\}}) = R(\{i\})$ for $i = 1, 2$. We have

$$
\begin{aligned}
&\alpha_1(\Phi^m_1(G), R(N)) - \alpha_1(\Phi^m_1(G_{\{1\}}), R(N)) \\
&= (1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))/2 - \alpha_1(R(\{1\}), R(N)) \\
&= (1 - \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N)))/2 \\
&= \alpha_2(\Phi^m_2(G), R(N)) - \alpha_2(\Phi^m_2(G_{\{2\}}), R(N)).
\end{aligned}
$$

We conclude that Φ^m has balanced contributions. □

Moreover, we have the following characterization, which is inspired by Hart and Mas-Colell (1989) and Ortmann (1998).

Theorem 4.8 *The marginal value Φ^m is the unique solution concept on $\bar{\mathcal{G}}^N$ with $|N| = 2$ that is efficient and has balanced contributions.*

Proof. Let $|N| = 2$. By definition, Φ^m is efficient and from lemma 4.7 it follows that Φ^m has balanced contributions on $\bar{\mathcal{G}}^N$.

To show the uniqueness, let Ψ be a solution concept on $\bar{\mathcal{G}}^N$ that is efficient and has balanced contributions. If $G \in \bar{\mathcal{G}}^N$ is a one-person game then $\Psi(G) = \Phi^m(G)$ because of efficiency.

Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be a two-person game. By efficiency there exists a vector $p = (p_1, p_2) \in \Delta^*(N)$ such that $\Psi(G) = (p_1, p_2)R(N)$. Next to this, Ψ has balanced contributions:

$$\begin{aligned}&\alpha_1(\Psi_1(G), R(N)) - \alpha_1(\Psi_1(G_{\{1\}}), R(N))\\ &= \alpha_2(\Psi_2(G), R(N)) - \alpha_2(\Psi_2(G_{\{2\}}), R(N)).\end{aligned}$$

By efficiency we have $\Psi_i(G_{\{i\}}) = R(\{i\})$ for $i \in N$. Together with $\Psi(G) = (p_1, p_2)R(N)$ this gives

$$p_1 - \alpha_1(R(\{1\}), R(N)) = p_2 - \alpha_2(R(\{2\}), R(N)).$$

Using $p_1 + p_2 = 1$ leads to

$$2p_1 = 1 + \alpha_1(R(\{1\}), R(N)) - \alpha_2(R(\{2\}), R(N))$$

from which we conclude that $\Psi = \Phi^m$. □

Of course, this characterization also holds for the dividend value and the selector value, as lemma 4.7 indicates. For three-person games, the three solution concepts can all be different, as the following example shows.

Example 4.9 Let $G = (N, (R(S))_{S\in\mathcal{S}}, \mathcal{A}, (\alpha_i)_{i\in N})$ be the three-person game with $N = \{1, 2, 3\}$, $R(\{i\}) = 0$ for all $i \in N$, $R(S) = 1$ if $|S| = 2$ and $R(N)$ is uniformly distributed over the closed interval [3,7]. The players 1 and 3 have expectation preferences and for player 2 we have

$$f_S^2(t) = \begin{cases} t, & |S| = 2, \\ 2t/5, & S = N, t \le 0, \\ t^{1/6}/2, & S = N, t > 0. \end{cases}$$

For this game the four solution concepts are

$$\begin{aligned}\Phi^m(G) &= (19/60, 11/30, 19/60)R(N),\\ \Phi^d(G) &= (7/15 - (1/2)^{1/6}/3, 1/15 + 2(1/2)^{1/6}/3, 7/15 - (1/2)^{1/6}/3)R(N),\\ \Phi^s(G) &= (7/15 - (1/2)^{1/6}/3, 17/30 - (1/2)^{1/6}/3, 7/15 - (1/2)^{1/6}/3)R(N).\end{aligned}$$

Notice that the selector value is not efficient. Further, $\Phi_i^s(G) = \Phi_i^d(G)$ for $i = 1, 3$. This is due to the fact that both the players 1 and 3 have expectation preferences and so, linear functions f^i. The inequality $\Phi_2^s(G) \neq \Phi_2^d(G)$ comes from the preferences of player 2: $\Delta_{\{1,2\}}(G) = 1 \sim_2 1/2 \cdot R(N)$ and $d_{\{1,2\}}(G) = 1/2 \sim_2 (1/2)^{7/6} R(N)$. Therefore,

$$\alpha_2(\Delta_{\{1,2\}}(G), R(N)) = 1/2 < 2(1/2)^{7/6} = 2\alpha_2(d_{\{1,2\}}(G), R(N))$$

although $\Delta_{\{1,2\}}(G) = 2d_{\{1,2\}}(G)$. $\diamond$

A Appendix

In this appendix we provide a characterization of the Shapley value on the class of TU games $SG^N = \{(N, v) | v \geq 0,\ v(T) = 0 \Rightarrow v(S) = 0 \text{ for all } S \subset T\}$. This characterization is inspired by the characterization of the Shapley value on the class of superadditive games by Young (1985) and we use it in the proof of theorem 4.6.

Let C^N be a set of TU games with player set N and let ψ be a solution concept on C^N, that is, $\psi(v) \in \mathbb{R}^N$ for all $v \in C^N$. Then ψ satisfies

(a) *efficiency* if $\sum_{i \in N} \psi(v) = v(N)$ for all $v \in C^N$.

(b) *symmetry* if for all $i, j \in N$ such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N \setminus \{i, j\}$ (i and j are *symmetric players*) we have $\psi_i(v) = \psi_j(v)$ for all $v \in C^N$.

(c) *strong monotonicity* if for all $i \in N$ and for all games $v, w \in C^N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subset N$, $i \notin S$, we have $\psi_i(v) \geq \psi_i(w)$.

Theorem A.1 *The Shapley value ϕ is the unique solution on SG^N that satisfies efficiency, symmetry and strong monotonicity.*

In the proof of this theorem we need a lemma that we present below. First, we introduce some definitions. Let the game (N, u_T) be the so-called unanimity game defined by

$$u_T(S) = \begin{cases} 1, & S \supset T, \\ 0, & \text{otherwise,} \end{cases}$$

for all $S \subset N$. The unanimity games $\{(N, u_T) | T \in 2^N \setminus \{\emptyset\}\}$ form a basis of the class of all TU games with player set N and the unique linear expansion of v with respect to unanimity games is given by

$$v = \sum_{T \neq \emptyset} \Delta_T(v) u_T.$$

For $t \in \{1, \ldots, n\}$ define $\Delta_t(v) = \max_{T:|T|=t} \Delta_T(v)$. Let $v^1 = \sum_{T \neq \emptyset} \Delta_{|T|}(v) u_T$. Clearly, v^1 is symmetric, that is, $v^1(S_1) = v^1(S_2)$ for all coalitions S_1, S_2 such that $|S_1| = |S_2|$.

Let $\delta_T(v) = \Delta_{|T|}(v) - \Delta_T(v)$ (≥ 0). Now we can write

$$v = v^1 - \sum_{T \neq \emptyset} \delta_T(v) u_T.$$

Define the index $k(v) = |\{T | \delta_T(v) > 0\}|$. Suppose player $i \in N$ is such that

$$i \notin \cap_{T:\delta_T(v)>0} T$$

and define the game $w^i = v + \sum_{T\neq\emptyset, i\notin T} \delta_T(v) u_T$. The following lemma shows that $v \in SG^N$ implies $w^i \in SG^N$ and that $k(v) - 1$ is an upper bound of $k(w^i)$.

Lemma A.2

1. $v \in SG^N \Rightarrow w^i \in SG^N$
2. $k(w^i) \leq k(v) - 1$

Proof. Let $v \in SG^N$. Clearly, $w^i \geq 0$. It remains to show that

$$w^i(Q) = 0 \Rightarrow w^i(S) = 0 \text{ for all } S \subset Q. \tag{A.5}$$

Let Q be a coalition such that

$$w^i(Q) = v(Q) + \sum_{T\neq\emptyset, i\notin T} \delta_T(v) u_T(Q) = 0.$$

Because $v(Q)$, $\delta_T(v)$ and $u_T(Q)$ are all nonnegative numbers we have $v(Q) = 0$ and $u_T(Q) = 0$ for all coalitions T with $i \notin T$. From $v \in SG^N$ we get $v(S) = 0$ for all $S \subset Q$. Also, $u_T(Q) = 0$ implies that $Q \not\supset T$. But then $S \not\supset T$ for all $S \subset Q$ and so, $u_T(S) = 0$. We conclude that (A.5) is satisfied.

To show the second item, notice that

$$\begin{aligned} w^i &= v + \sum_{T\neq\emptyset, i\notin T} \delta_T(v) u_T \\ &= \sum_{T\neq\emptyset} \Delta_T(v) u_T + \sum_{T\neq\emptyset, i\notin T} \Big(\Delta_{|T|}(v) - \Delta_T(v)\Big) u_T \\ &= \sum_{T\neq\emptyset, i\in T} \Delta_T(v) u_T + \sum_{T\neq\emptyset, i\notin T} \Delta_{|T|}(v) u_T. \end{aligned}$$

Hence, for all coalitions T

$$\Delta_T(w^i) = \begin{cases} \Delta_T(v), & i \in T, \\ \Delta_{|T|}(v), & i \notin T. \end{cases}$$

It readily follows that

$$\Delta_t(w^i) = \max_{T:|T|=t} \Delta_T(w^i) = \Delta_t(v)$$

for all $t \in \{1, \ldots, n\}$. Furthermore,

$$\delta_T(w^i) = \Delta_{|T|}(w^i) - \Delta_T(w^i) = \Delta_{|T|}(v) - \Delta_T(w^i) = \begin{cases} \delta_T(v), & i \in T, \\ 0, & i \notin T. \end{cases}$$

Now we get

$$\begin{aligned} k(w^i) &= |\{T|\delta_T(w^i) > 0\}| \\ &= |\{T|\delta_T(v) > 0,\ i \in T\}| \\ &\leq |\{T|\delta_T(v) > 0\}| - 1 \\ &= k(v) - 1 \end{aligned}$$

where the inequality follows from $i \notin T$ for at least one coalition T with $\delta_T(v) > 0$. □

Now we can prove the characterization of the Shapley value on the class SG^N.

Proof of theorem A.1. It is obvious that the Shapley value ϕ satisfies efficiency, symmetry and strong monotonicity on SG^N.

Let $v \in SG^N$, then $v = \sum_{T \neq \emptyset} \Delta_T(v) u_T$. Define for $t = 1, \ldots, n$

$$\Delta_t(v) = \max_{T:|T|=t} \Delta_T(v), \text{ and } \delta_T(v) = \Delta_{|T|}(v) - \Delta_T(v)\ (\geq 0).$$

Let $v^1 = \sum_{T \neq \emptyset} \Delta_{|T|}(v) u_T$. Now we can write

$$v = v^1 - \sum_{T \neq \emptyset} \delta_T(v) u_T. \tag{A.6}$$

Define the index $k(v) = |\{T|\delta_T(v) > 0\}|$. Let g be a solution on SG^N that is efficient, symmetric and strongly monotonic. We show by induction on $k(v)$ that $g(v) = \phi(v)$.

If $k(v) = 0$ then $v = v^1$. Because v^1 is a symmetric game, all the players in N are symmetric. From efficiency and symmetry we obtain $g_i(v) = v(N)/n = \phi_i(v)$ for all $i \in N$.

Now assume that $g(v) = \phi(v)$ for all games $v \in SG^N$ with $k(v) \leq k - 1$, for some positive integer k. Let $v \in SG^N$ be a game with $k(v) = k$. Define $D = \cap_{T \neq \emptyset, \delta_T(v) > 0} T$.

First, let $i \in N \setminus D$. Define the game (N, w^i) by

$$w^i = v^1 - \sum_{T \neq \emptyset, i \in T} \delta_T(v) u_T.$$

According to (A.6) we can rewrite this to

$$w^i = v + \sum_{T \neq \emptyset, i \notin T} \delta_T(v) u_T.$$

By lemma A.2 $w^i \in SG^N$ and $k(w^i) \leq k(v) - 1 = k - 1$. Then

$$g(w^i) = \phi(w^i) \tag{A.7}$$

by induction.

Let coalition S be such that $i \notin S$. Then

$$\begin{aligned} v(S \cup \{i\}) - v(S) &= v^1(S \cup \{i\}) - \sum_{T \neq \emptyset} \delta_T u_T(S \cup \{i\}) - \left(v^1(S) - \sum_{T \neq \emptyset} \delta_T u_T(S) \right) \\ &= v^1(S \cup \{i\}) - v^1(S) - \sum_{T \neq \emptyset} \delta_T(v) \left(u_T(S \cup \{i\}) - u_T(S) \right) \\ &= v^1(S \cup \{i\}) - v^1(S) - \sum_{T \neq \emptyset, i \in T} \delta_T(v) \left(u_T(S \cup \{i\}) - u_T(S) \right) \\ &= w^i(S \cup \{i\}) - w^i(S). \end{aligned}$$

From strong monotonicity we obtain $g_i(v) = g_i(w^i)$ and $\phi_i(v) = \phi_i(w^i)$. Together with (A.7) this gives $g_i(v) = \phi_i(v)$.

Second, let $i, j \in D$ be two players and let $S \subset N \setminus \{i, j\}$. Then by definition

$$v(S \cup \{i\}) = v^1(S \cup \{i\}) - \sum_{T \neq \emptyset} \delta_T(v) u_T(S \cup \{i\}).$$

Because $u_T(S \cup \{i\}) = u_T(S \cup \{j\}) = 0$ for all coalitions T with $\delta_T(v) > 0$ and because $v^1(S \cup \{i\}) = v^1(S \cup \{j\})$, the game v^1 is symmetric, we get

$$v(S \cup \{i\}) = v^1(S \cup \{j\}) - \sum_{T \neq \emptyset} \delta_T(v) u_T(S \cup \{j\}) = v(S \cup \{j\}).$$

Any two players in D are symmetric players in v. By symmetry $g_i(v) = g_j(v)$ and $\phi_i(v) = \phi_j(v)$ for all $i, j \in D$. Together with efficiency and with $g_k(v) = \phi_k(v)$ for all $k \in N \setminus D$ this implies $g_i(v) = \phi_i(v)$ for all $i \in D$. □

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