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Suijs, J.P.M.

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Jeroen Suijs[1,2]

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Abstract

This paper generalizes the results of Suijs, De Waegenaere and Borm (1998) to arbitrary risks. It provides Pareto optimal allocations and shows that the zero utility premium calculation principle yields a core-allocation.

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[1]CentER for Economic Research, Tilburg University, PO Box 90153, 5000 LE, Tilburg, The Netherlands.

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1 Introduction

Within the bounds of their possibilities, individuals generally try to eliminate the risks resulting from their social and economic activities as much as possible. In this regard, a common way of doing so is by means of risk sharing and/or insurance. Firms, for instance, might prefer to cooperate in joint ventures when investing in risky projects, while investors perpetrate risk sharing by investing their capital in a diversified portfolio of financial assets.

In insurance, an insurance company contracts to bear (part of) an individual's risk in exchange for a fixed payment, the insurance premium. For actuarial scientists, this insurance premium has been and still is a much examined research topic. The main question they address is, what is a reasonable premium for the risk that is insured, for the insurance premium should be acceptable with respect to the opposite interests of both the insurer and the insured.

Classical actuarial theory mainly considers this problem from the insurer's point of view. In determining a reasonable insurance premium, it distinguishes between risk arising from the 'life' sector and risk arising from the 'non life' sector. For the first, there is a profusion of statistical data on the expected remaining life available, which makes the calculation of an appropriate premium relatively easy. For the latter, however, things are a bit more complicated. In 'non life' insurance the risk is not always easy to capture in a statistical framework. Therefore, several premium calculation principles have been developed to serve this purpose, see for instance Goovaerts, De Vylder and Haezendonck (1984).

These calculation principles, however, only take into account a part of the insurer's side of the deal. More precisely, they consider whether the premium is high enough to cover the risk. Competition arising from the presence of other insurers on the one hand, and the interests of the insured, on the other hand, are mostly ignored. It is, of course, better to consider all these aspects in an insurance deal, since the premium should not only be high enough to compensate the insurer for bearing the individual's risk, it should also be low enough so that an individual is willing to insure his risk (or a part of it) for this premium. The economic models for (re)insurance markets, which were developed from the 1960's on (cf. Borch (1962a) and Bühlmann (1980), (1984)), consider indeed the interests of both the insurers and the insureds. These models incorporate the possibility to study problems concerning fairness, Pareto optimality and market equilibrium. Bühlmann (1980), for example, shows that the Esscher calculation principle results in a Pareto optimal outcome. For an overview of economic models in insurance see Borch (1990).

Game theory is also used to model the interests of all parties in an insurance problem. Examples can be found in Borch (1962b), Lemaire (1991), Alegre and Mercè Claramunt (1995), and Suijs et al. (1998). The latter uses stochastic cooperative games to model individual insurance as well as reinsurance by insurance companies. The results they obtain, however, only hold for exponentially distributed losses. This paper generalizes Suijs et al. (1998) in the sense that it allows arbitrary random losses. We determine Pareto optimal allocations and show that the zero utility principle for calculating premiums (see Goovaerts et al. (1984)) yields a core allocation.

2 Stochastic Cooperative games

Let us first recall some of the definitions concerning stochastic cooperative games as introduced by Suijs, Borm, De Waegenaere and Tijs (1999). A stochastic cooperative game is described by a tuple $\Gamma = (N, \{\mathcal{X}_S\}_{S\subset N}, \{\succsim_i\}_{i\in N})$, where N is the set of agents, $\mathcal{X}_S$ the nonempty set of random payoffs coalition S can obtain, and $\succsim_i$ the preference relation of agent i over the set $L^1(\mathbb{R})$ of stochastic payoffs with finite expectation. We assume that for each agent the preferences are complete, transitive and continuous[1]. The class of all cooperative games with stochastic payoffs with agent set N is denoted by $SG(N)$.

An allocation of a stochastic payoff $X_S \in \mathcal{X}_S$ to coalition S is described by a pair $(d, r) \in \mathbb{R}^S \times \mathbb{R}^S$ such that $\sum_{i\in S} d_i \leq 0$ and $\sum_{i\in S} r_i = 1$ and $r_i \geq 0$ for all $i \in S$. The payoff to agent $i \in S$ according to the allocation (d, r) equals $d_i + r_i X_S$. The set of all allocations for coalition S is denoted by $Z_\Gamma(S)$.

The core of a stochastic cooperative game is defined as follows. Let $\Gamma \in SG(N)$ and $(d_i + r_i X_N)_{i\in N}) \in Z_\Gamma(N)$. Then the allocation $(d_i + r_i X_N)_{i\in N}$ is a core allocation for the game Γ if for each coalition S there is no allocation $(\tilde{d}_i + \tilde{r}_i X_N)_{i\in S} \in Z_\Gamma(S)$ such that $\tilde{d}_i + \tilde{r}_i X_S \succ_i d_i + r_i X_S$ for all $i \in S$. The set of all core allocations for Γ is denoted by $C(\Gamma)$.

Next, consider preferences $\{\succsim_i\}_{i\in N}$ such that for each $i \in N$ there exists a function $m_i : L^1(\mathbb{R}) \to \mathbb{R}$ satisfying

(M1) for all $X, Y \in L^1(\mathbb{R})$: $X \succsim_i Y$ if and only if $m_i(X) \geq m_i(Y)$;

(M2) for all $X \in L^1(\mathbb{R})$ and all $d \in \mathbb{R}$: $m_i(d + X) = d + m_i(X)$.

The interpretation is that $m_i(X)$ equals the amount of money m for which agent i is indifferent between receiving the amount $m_i(X)$ with certainty and receiving the stochastic payoff X. The amount $m_i(X)$ is called the certainty equivalent of X. Condition (M1) states that agent i weakly prefers one stochastic payoff to another one if and only if the certainty equivalent of the former is greater than or equal to the certainty equivalent of the latter. Condition (M2) states that the certainty equivalent is linearly separable in the deterministic amount of money d. The class of all stochastic coopertive games satisfying conditions (M1) and (M2) is denoted by $MG(N)$.

Example 1 Consider the preferences based on a utility function of the form $U(t) = \beta e^{-\alpha t}$, $(t \in \mathbb{R})$, where $\beta < 0$ and $\alpha > 0$. The certainty equivalent of $X \in L^1(\mathbb{R})$ can be defined by

[1] The preferences $\succsim$ are continuous if for all $X \in L^1(\mathbb{R})$ the sets $\{Y \in L^1(\mathbb{R}) | Y \succsim X\}$ and $\{Y \in L^1(\mathbb{R}) | Y \precsim X\}$ are closed.

$m(X) = U^{-1}(E(U(X)))$. It is easy to check that m satisfies condition (M1). For condition (M2), let $X \in L^1(\mathbb{R})$ and $d \in \mathbb{R}$. Then $U^{-1}(t) = -\frac{1}{\alpha}\ \log\left(\frac{t}{\beta}\right)$ and

$$\begin{aligned} m(d+X) &= U^{-1}(E(U(d+X))) \\ &= -\tfrac{1}{\alpha}\ \log\left(\tfrac{1}{\beta}\int \beta e^{-\alpha(d+t)} dF_X(t)\right) \\ &= -\tfrac{1}{\alpha}\ \log\left(e^{-\alpha d}\tfrac{1}{\beta}\int \beta e^{-\alpha t} dF_X(t)\right) \\ &= d - \tfrac{1}{\alpha}\ \log\left(\tfrac{1}{\beta}(\int \beta e^{-\alpha t} dF_X(t))\right) \\ &= d + m(X). \end{aligned}$$

The following theorem regarding nonemptiness of the core is due to Suijs and Borm (1999).

Theorem 2 Let $\Gamma \in MG(N)$. Then $C(G) \neq \emptyset$ if and only if $C(v_\Gamma) \neq \emptyset$, where

$$v_\Gamma(S) \ = \ \max\left\{\sum_{i\in S} m_i(\hat{d}_i + \hat{r}_i\hat{X}) |\ (\hat{d}_i + \hat{r}_i\hat{X})_{i\in S} \in Z_\Gamma(S)\right\},$$

for all $S \subset N$.

3 Insurance Games

For modeling insurance problems we use a slightly modified version of a stochastic cooperative game as introduced in Suijs et al. (1999). We show that by cooperating, individuals and insurers can redistribute their risks and, consequently, improve their welfare. First, we need to specify the agents that participate in the game. An agent can be one of two types, either an individual person or an insurer. The set of individual persons is denoted by N_P and the set of insurers is denoted by N_I. Hence, the agents of the game are denoted by the set $N_I \cup N_P$.

Again, all agents are assumed to be risk averse expected utility maximizers with utility function $U_i(t) = \beta_i e^{-\alpha_i t}$, $(t \in \mathbb{R})$, with $\beta_i < 0$, $\alpha_i > 0$ for all $i \in N_I \cup N_P$. By changing the signs of the parameters β_i and α_i the utility function becomes convex, and, as a consequence, the agent will be risk loving. Regarding the situations where one or more risk neutral/loving insurers are involved we confine ourselves to a brief discussion later on.

Next, let $-X_i$ with $X_i \in L^1(\mathbb{R}_+)$ describe the future random losses of agent $i \in N_I \cup N_P$. For an individual $i \in N_P$, the variable $-X_i$ describes the random losses that could occur to this individual. They include, for example, the monetary damages caused by cars, bikes, fires, or people. For an insurance company $i \in N_I$, the variable $-X_i$ describes the random losses corresponding to its insurance portfolio. We assume that the random losses $-X_i$, ($i \in N_I \cup N_P$), are mutually independent.

Now, let us focus on the possibilities that occur when agents decide to cooperate. Therefore, consider a coalition S of agents. If the members of S decide to cooperate, the total loss $X_S \in L^1(\mathbb{R})$ of coalition S equals the sum of the individual losses of the members of S, i.e., $X_S = -\sum_{i \in S} X_i$. Subsequently, the loss X_S has to be allocated to the members of S.

In Suijs et al. (1999) an allocation of the random payoff X_S to the members of coalition S is described by a pair (d, r) satisfying $\sum_{i \in S} d_i \leq 0$, $\sum_{i \in S} r_i = 1$, and $r_i \geq 0$ fo rall $i \in S$. Given an allocation (d, r), agent $i \in S \cap N_P$ receives $d_i + r_i X_S$. Applying this definition to insurance games, however, raises a problem. For X_S not only consists of the future random losses of agent i, but also of the future random losses of all other individuals $j \in S \cap N_P$. Hence, if agent i receives $d_i + r_i X_S$ he receives (part of) the random losses of his fellow agents $j \in S \cap N_P$. Furthermore, this means that an agent $j \in S \cap N_P$ transfers (part of) his random losses to agent i, or, put in other words, agent j insures (part of) his random losses at agent i. But this is rather unusual; agents only make insurance deals with insurance companies and not with other individuals. So, we need to modify our definition of an allocation so as to incorporate transfers of random losses from individuals to insurance companies only. The option we choose for is to replace the vector $r \in \Delta^S$ by a matrix $R \in \mathbb{R}^{S \times S}$. An element r_{ij} then represents the fraction of agent j's random loss that he transfers to agent i. Then by imposing the right conditions on R we can guarantee that individuals cannot transfer any risks among each other.

For explaining an allocation of the loss X_S in more detail, we distinguish between the following three cases. In the first case, coalition S consists of insurers only. So, $S \subset N_I$. Such a coalition is assumed to allocate the loss X_S in the following way. First, a coalition S allocates a fraction $r_{ij} \in [0, 1]$ of the loss X_j of insurer $j \in S$ to insurer $i \in S$. So, insurer i bears a total loss of $\sum_{j \in S} r_{ij} X_j$, where $r_{ij} \in [0, 1]$ and $\sum_{k \in S} r_{kj} = 1$. This is called proportional (re)insurance. This part of the allocation of X_S for coalition S is described by a matrix $R \in \mathbb{R}_+^{S \times S}$, where r_{ij} represents the fraction insurer i bears of insurer j's loss X_j. Second, the insurers are allowed to make deterministic transfer payments. This means that each insurance company $i \in S$ also receives an amount $d_i \in \mathbb{R}$ such that $\sum_{j \in S} d_j \leq 0$. These transfer payments can be interpreted as the aggregate premium insurers have to pay for the actual risk exchanges.

In the second case, coalition S consists of individual persons only. So, $S \subset N_P$. Then the gains of cooperation are assumed to be nil. That is, we do not allow any risk exchanges between the persons themselves. For, that is what the insurers are for in the first place. As a result, the only allocations (d, R) of X_S which are allowed are of the form $r_{ii} = 1$ for all $i \in S$ and $r_{ij} = 0$ for all $i, j \in S$ with $i \neq j$.

In the third and last case, coalition S consists of both insurers and individual persons. So, $S \subset N_I \cup N_P$. Now cooperation can take place in two different ways. First, insurers are allowed to exchange (parts of) their portfolios with other insurers, and, second, individual persons may transfer (parts of) their risks to insurers. Again, individual persons are not allowed to exchange risks with each other. Moreover, we assume that insurers cannot transfer (parts of) their portfolios to individuals.

Summarizing we can say that there are several restrictions on allocations. To be more precise, denote by S_I the set of insurers of coalition S, i.e., $S_I = S \cap N_I$, and by S_P the set of individuals of coalition S, i.e., $S_P = S \cap N_P$. Then an allocation $(d, R) \in \mathbb{R}^S \times \mathbb{R}_+^{S \times S}$ is feasible for the coalition S if for all $i \in S_P$ and all $j \in S$ with $i \neq j$ it holds that $r_{ij} = 0$ and $\sum_{i \in S} r_{ij} = 1$ for all $j \in S$. Furthermore, given an allocation (d, R) of the random loss X_S, agent $i \in S$ receives $d_i + \sum_{j \in S} r_{ij} X_j \ = \ d_i + R_i X^S$, where R_i denotes the i-th row of R and $X^S = (-X_i)_{i \in S}$.

Example 1 Let $N_I = \{1, 2\}$, $N_P = \{3, 4\}$ and consider the coalition $S = \{1, 3, 4\}$. Then $X_S = -X_1 - X_3 - X_4$. A feasible allocation for S is the following. Let $d = (3, -2, -1)$ and $r_{11} = 1$, $r_{13} = \frac{1}{2}$, $r_{33} = \frac{1}{2}$, $r_{14} = \frac{1}{5}$ and $r_{44} = \frac{4}{5}$. Then insurer 1 receives

$$d_1 + R_1 X^S \quad = \quad 3 - X_1 - \tfrac{1}{2} X_3 - \tfrac{1}{5} X_4),$$

individual 3 receives

$$d_3 + R_3 X^S \quad = \quad -2 - \tfrac{1}{2} X_3,$$

and individual 4 receives

$$d_4 + R_4 X^S \quad = \quad -1 - \tfrac{4}{5} X_4.$$

So, individuals 3 and 4 pay a premium of 2 and 1, respectively, to insurer 1 for the insurance of their losses.

In conclusion, an insurance game Γ with agent set $N_I \cup N_P$ is described by the tuple $(N_I \cup N_P, \{\mathcal{X}_S\}_{S \subset N}, \{U_i\}_{i \in N_I \cup N_P})$, where N_I is the set of insurers, N_P the set of individuals, $\mathcal{X}_S = \{\sum_{i \in S} -X_i\}$ the random loss for coalition S, and U_i the utility function for agent $i \in N_I \cup N_P$. The class of all such insurance games with insurers N_I and individuals N_P is denoted by $IG(N_I, N_P)$.

3.1 Pareto Optimal Distributions of Risk

Since the preferences of each agent are described by means of an exponential utility function, we can confine ourselves to considering certainty equivalents. In this model, we define the certainty equivalent by $m_i(X) \ = \ U_i^{-1}(E(U_i(X)))$ provided that the expected utility exists, of course.[2] From Example 1 it follows that the certainty equivalent for exponential utility functions satisfies the conditions (M1) and (M2). Hence, the results stated in Suijs and Borm (1999) on certainty equivalents apply. One of these results concerns the Pareto optimality of an allocation. For insurance games this result reads as follows.

Proposition 2 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d_i + R_i X^S)_{i \in S} \in Z_\Gamma(S)$ is Pareto optimal for coalition S if and only if

$$\begin{aligned} &\sum_{i\in S} m_i(d_i + R_i X^S) \\ &\quad = \max\left\{\sum_{i\in S} m_i(\tilde{d}_i + \tilde{R}_i X^S) \mid (\tilde{d}_i + \tilde{R}_i X^S)_{i\in S} \in Z_\Gamma(S)\right\}. \end{aligned} \tag{1}$$

So, an allocation is Pareto optimal for coalition S if and only if this allocation maximizes the sum of the certainty equivalents. To determine these allocations, we first need to calculate the certainty equivalent of an allocation $(d_i + R_i X^S)_{i \in S} \in Z_\Gamma(S)$ for agent $i \in S$. Therefore, let $S \subset N_I \cup N_P$ and $(d_i + R_i X^S)_{i \in S} \in Z_\Gamma(S)$. The random loss coalition S has to allocate equals $X_S = \sum_{i \in S} X_i$. Given a feasible allocation $(d_i + R_i X^S)_{i \in S} \in Z_\Gamma(S)$, the random payoff to agent $i \in S$ equals

$$d_i + R_i X^S = d_i - \sum_{j\in S} r_{ij} X_j$$

if $i \in S_I$ and

$$d_i + R_i X^S = d_i - r_{ii} X_i$$

if $i \in S_P$. Consequently, we have that the certainty equivalent of $(d_i + R_i X^S)_{i \in S}$ equals

$$m_i(d_i + R_i X^S) = \begin{cases} d_i + \frac{1}{\alpha_i}\log\left(\int_0^\infty e^{\alpha_i r_{ii} t} dF_{X_i}(t)\right)^{-1}, & \text{if } i \in S_P, \\ d_i + \sum_{j\in S}\frac{1}{\alpha_i}\log\left(\int_0^\infty e^{\alpha_i r_{ij} t} dF_{X_j}(t)\right)^{-1}, & \text{if } i \in S_I. \end{cases} \tag{2}$$

The sum of the certainty equivalents then equals

$$\begin{aligned} \sum_{i\in S} m_i(d_i + R_i X^S) &= \sum_{i\in S} d_i + \sum_{i\in S_P}\frac{1}{\alpha_i}\log\left(\int_0^\infty e^{\alpha_i r_{ii} t} dF_{X_i}(t)\right)^{-1} \\ &\quad + \sum_{i\in S_I}\sum_{j\in S}\frac{1}{\alpha_i}\log\left(\int_0^\infty e^{\alpha_i r_{ij} t} dF_{X_j}(t)\right)^{-1}. \end{aligned} \tag{3}$$

[2]Throughout this paper we assume that the utility functions and random payoffs are such that the expected utility exist, and that we may interchange the order of integration and differentiation.

Since $\sum_{i\in S} d_i = 0$ for Pareto optimal allocations, we have for these allocations that the sum of the certainty equivalents is independent of the vector of transfer payments d. Intuitively, this is quite clear. For since $\sum_{h\in S} d_h \leq 0$, an increase in d_i for agent i implies that d_j decreases for at least one other agent j. Consequently, Pareto optimality is solely determined by the choice of the allocation risk exchange matrix R of the random losses. In fact, the next theorem shows that there is a unique allocation risk exchange matrix R^* inducing Pareto optimality.

Theorem 3 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d_i + R_i^* X^S)_{i\in S} \in Z_\Gamma(S)$ is Pareto optimal for S if and only if

$$r_{ij}^* = \begin{cases} \dfrac{\frac{1}{\alpha_i}}{\sum_{h\in S_I} \frac{1}{\alpha_h}} & \text{, if } i, j \in S_I, \\ \dfrac{\frac{1}{\alpha_i}}{\sum_{h\in S_I\cup\{j\}} \frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

PROOF: We have to show that R^* is the unique solution of

$$\begin{aligned} \max \sum_{i\in S_P} \tfrac{1}{\alpha_i} \log \left(\int_0^\infty e^{\alpha_i r_{ii} t} dF_{X_i}(t)\right)^{-1} \\ + \sum_{i\in S_I}\sum_{j\in S} \tfrac{1}{\alpha_i} \log \left(\int_0^\infty e^{\alpha_i r_{ij} t} dF_{X_j}(t)\right)^{-1} \end{aligned}$$

$$\begin{array}{rrcll} \text{s.t.:} & r_{jj} + \sum_{i\in S_I} r_{ij} & = & 1, & \text{for all } j \in S_P, \\ & \sum_{i\in S_I} r_{ij} & = & 1, & \text{for all } j \in S_I, \\ & r_{ii} & \geq & 0, & \text{if } i \in S_P, \\ & r_{ij} & \geq & 0, & \text{if } i \in S_I \text{ and } j \in S. \end{array}$$

Lemma 1 with $c = \alpha_i$ and $x = r_{ij}$ for all relevant combinations of $i, j \in S$, implies that the objective function is strictly concave. Hence, it is sufficient to prove that R^* is a solution of this maximization problem. The Karush-Kuhn-Tucker conditions[3] tell us that this is indeed

[3] The Karush-Kuhn-Tucker conditions read as follows:

$$\begin{array}{rcl} \text{If } f(x) & = & \max_y f(y) \\ & & \text{s.t. } \; g_k(y) \leq 0, \quad k \in K \\ & & \qquad g_l(y) = 0, \quad l \in L \end{array}$$

then there exist $\nu_k \geq 0$ ($\forall k \in K$) and $\lambda_l \in \mathbb{R}$ ($\forall l \in L$) such that

$$\begin{array}{rcl} \nabla f(x) & = & \sum_{k\in K} \nu_k \cdot \nabla g_k(x) + \sum_{l\in L} \lambda_l \cdot \nabla g_l(x) \\ \nu_k \cdot g_k(x) & = & 0, \text{ for all } k \in K. \end{array}$$

Moreover, if f is strictly concave and g_k ($k \in K$), g_l ($l \in L$) are convex then the reverse of the statement also holds and the maximum is unique.

the case if there exists $\lambda_j \in \mathbb{R}$ ($j \in S$), $\nu_{jj} \geq 0$ ($j \in S_P$) and $\nu_{ij} \geq 0$ ($i \in S_I,\ j \in S$) such that

$$\begin{aligned}
-\frac{\int_0^\infty te^{\alpha_j r^*_{jj} t} dF_{X_j}(t)}{\int_0^\infty e^{\alpha_j r^*_{jj} t} dF_{X_j}(t)} &= \lambda_j - \nu_{jj}, \quad \text{for all } j \in S_P, \\
-\frac{\int_0^\infty te^{\alpha_i r^*_{ij} t} dF_{X_j}(t)}{\int_0^\infty e^{\alpha_i r^*_{ij} t} dF_{X_j}(t)} &= \lambda_j - \nu_{ij}, \quad \text{for all } i \in S_I \text{ and all } j \in S, \\
\nu_{jj} r_{jj} &= 0, \qquad \text{for all } j \in S_P, \\
\nu_{ij} r_{ij} &= 0, \qquad \text{for all } i \in S_I \text{ and all } j \in S.
\end{aligned}$$

Substituting r^*_{ij} gives $\nu_{ij} = 0$ for all relevant combinations of $i, j \in S$ and

$$\begin{aligned}
\lambda_j &= -\frac{\int_0^\infty te^{t/\alpha(S_I \cup \{j\})} dF_{X_j}(t)}{\int_0^\infty e^{t/\alpha(S_I \cup \{j\})} dF_{X_j}(t)}, \quad \text{for all } j \in S_P, \\
\lambda_j &= -\frac{\int_0^\infty te^{t/\alpha(S_I)} dF_{X_j}(t)}{\int_0^\infty e^{t/\alpha(S_I)} dF_{X_j}(t)}, \qquad \text{for all } i \in S_I,
\end{aligned}$$

with $\alpha(S) = \sum_{i \in S} \frac{1}{\alpha_i}$. Consequently, R^* is the optimal solution. □

So, for a Pareto optimal allocation of a loss X_j within S one has to distinguish between two cases. In the first case the index j refers to an insurer and in the second case j refers to an individual. When X_j is the loss of insurer $j \in S_I$, the loss is allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S. When X_j is the loss of individual $j \in S_P$, the loss is allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S and individual j himself. Note that by the feasibility constraints nothing is allocated to the other individuals. Furthermore, remark that if only reinsurance of the insurance portfolios is considered, that is, $N_P = \emptyset$ then the Pareto optimal allocation coincides with the Pareto optimal allocation of (re)insurance markets discussed in Bühlmann (1980).

The determination of the allocation risk exchange matrix is, of course, only one part of the allocation. We still have to determine the vector of transfer payments d, that is, the premiums that have to be paid. Although an allocation $(d_i + R^*_i X^S)_{i \in S}$ may be Pareto optimal for any choice of d, not every d is satisfactory from a social point of view. An insurer will not agree with insuring the losses of other agents if he is not properly compensated, that is, if he does not receive a fair premium for the insurance. Similarly, insurance companies and individuals only agree to insure their losses if the premium they have to pay is reasonable. Consequently, there is a conflict of interests; both insurance companies and individuals want to pay a low premium for insuring their own losses, while insurance companies want to receive a high premium for bearing the losses of other agents.

3.2 The Zero Utility Premium Calculation Principle

Premium calculation principles indicate how to determine the premium for a certain risk. In the past, various of these principles were designed, for example, the net premium principle, the expected value principle, the standard deviation principle, the Esscher principle, and the zero utility principle (cf. Goovaerts et al. (1984). In this section we focus on the zero utility principle. A premium calculation principle determines a premium $\pi_i(X)$ for individual i for bearing the risk X. The zero utility principle assigns a premium $\pi_i(X)$ to X such that the utility level of individual i, who bears the risk X, remains unchanged when the wealth w_i of this individual changes to $w_i + \pi_i(X) - X$. Since individuals are expected utility maximizers this means that the premium $\pi_i(X)$ satisfies $U_i(w_i) = E(U_i(w_i + \pi_i(X) - X))$. Note that the premium of the risk X depends on the individual who bears this risk and his wealth w_i.

Now, let us return to insurance games and utilize the zero utility principle to determine the allocation transfer payments $d \in \mathbb{R}^{N_I \cup N_P}$. At first this might seem difficult since the zero utility principle requires initial wealths w_i which do not appear in our model of insurance games. The exponential utility functions, however, yield that the zero utility principle is independent of these initial wealths w_i. To see this, let $\Gamma \in IG(N_I, N_P)$ be an insurance game. Since utility functions are exponential we can rewrite the expression $U_i(w_i) = E(U_i(w_i + \pi_i(X) - X))$ as follows

$$w_i \;=\; U_i^{-1}(E(U_i(w_i + \pi_i(X) - X))) \;=\; w_i + \pi_i(X) + U_i^{-1}(E(U_i(-X))).$$

Hence, $\pi_i(X) = -U_i^{-1}(E(U_i(-X))) = -m_i(-X)$ which indeed is independent of the wealth w_i. Furthermore, we can calculate the premium that agents receive for the risk they bear. For this, recall that for the Pareto optimal allocation risk exchange matrix R^* we have that

$$r^*_{ij} = \begin{cases} \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I} \frac{1}{\alpha_h}} & \text{, if } i, j \in S_I, \\ \dfrac{\frac{1}{\alpha_i}}{\sum_{h \in S_I \cup \{j\}} \frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

Since the risk that insurer i bears equals $\sum_{j \in N_I \cup N_P} r^*_{ij} X_j$, the premium he should receive for this according to the zero utility principle equals

$$\pi_i(\sum_{j \in N_I \cup N_P} r^*_{ij} X_j) \;=\; -m_i(-\sum_{j \in N_I \cup N_P} r^*_{ij} X_j)$$

Note that due to the mutual independence of $(X_i)_{i \in N_I \cup N_P}$, the zero utility principle satisfies additivity, that is, $\pi_i(\sum_{j \in N_I \cup N_P} r^*_{ij} X_j) = \sum_{j \in N_I \cup N_P} \pi_i(r^*_{ij} X_j)$. As a consequence, we let the premium that individual $i \in N_P$ has to pay for insuring his loss at insurer j equal the zero utility premium that this insurer wants to receive for bearing this risk. Hence, individual i pays insurer j an amount $\pi_j(r^*_{ji} X_i) = -m_j(-r^*_{ji} X_i)$. Because individuals are not allowed to bear (part of) the risk of any other individual/insurer he does not receive any premium. So in aggregate he pays

$$\sum_{j \in N_I} \pi_j(r^*_{ji} X_i).$$

Similarly, the premium that insurer i has to pay for reinsuring the fraction r^*_{ji} of his own portfolio X_i at insurer j, equals the premium that insurer j wants to receive for bearing this risk, that is, $\pi_j(r^*_{ji}X_i) = -m_j(-r^*_{ji}X_i)$. Then the premium insurer i receives in aggregate equals

$$\sum_{j\in N_I\cup N_P} \pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I} \pi_j(r^*_{ji}X_i).$$

Since

$$\sum_{i\in N_I}\left(\sum_{j\in N_I\cup N_P} \pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I} \pi_j(r^*_{ji}X_i)\right) - \sum_{i\in N_P}\sum_{j\in N_I} \pi_j(r^*_{ji}X_i) \;=\; 0,$$

the zero utility principle yields an allocation transfer payments vector d^0 where

$$\begin{aligned} d^0_i &= \sum_{j\in N_I\cup N_P} \pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I} \pi_j(r^*_{ji}X_i) \\ &= \;-\sum_{j\in N_I\cup N_P} m_i(-r^*_{ij}X_j) \;+\; \sum_{j\in N_I} m_j(-r^*_{ji}X_i) \end{aligned} \tag{4}$$

for all $i \in N_I$ and

$$d^0_i = -\sum_{j\in N_I} \pi_j(r^*_{ji}X_i) = \sum_{j\in N_I} m_j(-r^*_{ji}X_i) \tag{5}$$

for all $i \in N_P$.

Theorem 4 Let $\Gamma \in IG(N_I, N_P)$. If d^0 is the vector of transfer payments determined by the zero utility premium calculation principle and R^* is the Pareto optimal risk exchange matrix then $(d^0_i + R^*_i X^N)_{i\in N} \in \mathcal{C}(\Gamma)$.

PROOF: By Theorem 2 it suffices to show that $(m_i(d^0_i - R^*_i X^N))_{i\in N} \in C(v_\Gamma)$. Hence, we must show that $\sum_{i\in S} m_i(d^0_i - R^*_i X^N) \geq v_\Gamma(S)$ for all $S \subset N$. So, let us start by determining $v_\Gamma(S)$.

$$\begin{aligned} v_\Gamma(S) &= \max\left\{\sum_{i\in S} m_i(d_i + R_iX^N)\;\middle|\; (d_i + R_iX^N)_{i\in N} \in Z_\Gamma(N)\right\} \\ &= \sum_{j\in S_P} \tfrac{1}{\alpha_j}\log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{j\})}dF_{X_j}(t)\right)^{-1} \\ &\quad + \sum_{i\in S_I}\sum_{j\in S} \tfrac{1}{\alpha_i}\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-1} \\ &= \sum_{j\in S_P} \tfrac{1}{\alpha_j}\log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{j\})}dF_{X_j}(t)\right)^{-1} \\ &\quad + \sum_{i\in S_I}\sum_{j\in S_P} \tfrac{1}{\alpha_i}\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-1} \end{aligned}$$

$$
\begin{aligned}
&= \; + \sum_{i\in S_I}\sum_{j\in S_I} \tfrac{1}{\alpha_i}\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-1} \\
&= \sum_{j\in S_P}\sum_{i\in S_I}\cup\{j\}\tfrac{1}{\alpha_j}\log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{j\})}dF_{X_j}(t)\right)^{-1} \\
&\quad + \sum_{j\in S_I}\sum_{i\in S_I}\tfrac{1}{\alpha_i}\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-1} \\
&= \sum_{j\in S_P}\alpha(S_I\cup\{j\})\log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{j\})}dF_{X_j}(t)\right)^{-1} \\
&\quad + \sum_{j\in S_I}\alpha(S_I)\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-1} \\
&= \sum_{j\in S_P}\log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{j\})}dF_{X_j}(t)\right)^{-\alpha(S_I\cup\{j\})} \\
&\quad + \sum_{j\in S_I}\log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}(t)\right)^{-\alpha(S_I)},
\end{aligned} \tag{6}
$$

where the second equality follows from Theorem 3.

Next, note that for $i \in N_I$ we have that

$$
\begin{aligned}
m_i(d_i^0 - R_i^* X^N) &= d_i^0 + \sum_{j\in N_I\cup N_P} m_i(-r_{ij}^* X_j) \\
&= -\sum_{j\in N_I\cup N_P} m_i(-r_{ij}^* X_j) + \sum_{j\in N_I} m_j(-r_{ji}^* X_i) \\
&\quad + \sum_{j\in N_I\cup N_P} m_i(-r_{ij}^* X_j) \\
&= \sum_{j\in N_I} m_j(-r_{ji}^* X_i) \\
&= \sum_{j\in N_I} \tfrac{1}{\alpha_j}\log\left(\int_0^\infty e^{t/\alpha(N_I)}dF_{X_j}\right)^{-1} \\
&= \log\left(\int_0^\infty e^{t/\alpha(N_I)}dF_{X_j}\right)^{-\alpha(N_I)}
\end{aligned}
$$

and for $i \in N_P$ that

$$
\begin{aligned}
m_i(d_i^0 + R_i^* X^N) &= d_i^0 + m_i(-r_{ii}^* X_i) \\
&= \sum_{j\in N_I} m_j(-r_{ji}^* X_i) + m_i(-r_{ii}^* X_i) \\
&= \sum_{j\in N_I\cup\{i\}} m_j(-r_{ji}^* X_i) \\
&= \sum_{j\in N_I\cup\{i\}} \tfrac{1}{\alpha_j}\log\left(\int_0^\infty e^{t/\alpha(N_I\cup\{i\})}dF_{X_j}\right)^{-1}
\end{aligned}
$$

$$= \log\left(\int_0^\infty e^{t/\alpha(N_I\cup\{i\})}dF_{X_j}\right)^{-\alpha(N_I\cup\{i\})},$$

Then for $S \subset N_I \cup N_P$ it holds that

$$\begin{aligned}
\sum_{i\in S} m_i(d_i^0 + R_i^* X^N) &= \sum_{i\in S_I} \log\left(\int_0^\infty e^{t/\alpha(N_I)}dF_{X_j}\right)^{-\alpha(N_I)} \\
&\quad + \sum_{i\in S_P} \log\left(\int_0^\infty e^{t/\alpha(N_I\cup\{i\})}dF_{X_j}\right)^{-\alpha(N_I\cup\{i\})} \\
&\geq \sum_{i\in S_I} \log\left(\int_0^\infty e^{t/\alpha(S_I)}dF_{X_j}\right)^{-\alpha(S_I)} \\
&\quad + \sum_{i\in S_P} \log\left(\int_0^\infty e^{t/\alpha(S_I\cup\{i\})}dF_{X_j}\right)^{-\alpha(S_I\cup\{i\})} \\
&= v_\Gamma(S),
\end{aligned}$$

where the inequality follows from Lemma 2 with $c = 1$ and $x = \frac{1}{\alpha(S)}$. □

Example 5 In this example all monetary amounts are stated in thousands of dollars. Consider the following situation in automobile insurance with two insurance companies and three individual persons. So, $N_I = \{1, 2\}$ and $N_P = \{3, 4, 5\}$. The utility function of each agent can be described by $U_i(t) = e^{-\alpha_i t}$ with $\alpha_1 = 0.1$, $\alpha_2 = 0.167$, $\alpha_3 = 0.333$, $\alpha_4 = 0.125$ and $\alpha_5 = 0.2$, respectively. For evaluating random payoffs $X \in L^1(\mathbb{R})$, we focus on the corresponding certainty equivalent $m_i(X) = U_i^{-1}(E(U_i(X)))$, $i \in \{1, 2, 3, 4, 5\}$.

Each insurance company bears the risk of all the cars contained in its insurance portfolio. A car can be either one of two types. The first type corresponds to an average saloon car with a retail price of \$$20$, and generates relatively low losses. The second type corresponds to an exclusive sportscar with a retail price of \$$200$, and generates relatively high losses. More precisely, the monetary loss generated by a car is uniformly distributed between zero and its retail price. Thus the expected loss of a type 1 car and a type 2 car equal \$ 10 and \$ 100, respectively.

The insurance portfolio of insurer 1 consists of 900 cars of type 1 and 25 cars of type 2. For insurer 2 the portfolio consists of 400 cars of type 1 and 70 cars of type 2. The expected loss for insurer 1 then equals $900 \cdot 10 + 25 \cdot 100 =$ \$ 11500. The expected losses for insurer 2 equals \$ 11000. The individuals 3 and 4 each possess one car. Individual 3's car is of type 1 and individual 4's car is of type 2. Individual 5 possesses both cars. The expected losses are \$ 10, \$ 100, and \$ 110, respectively.

Next, let X_i denote the loss of agent i. If all agents cooperate, the Pareto optimal risk allocation matrix of the total random loss $X_1 + X_2 + X_3 + X_4 + X_5$ equals

$$R^* = \begin{bmatrix} \frac{5}{8} & \frac{5}{8} & \frac{10}{19} & \frac{5}{12} & \frac{10}{21} \\ \frac{3}{8} & \frac{3}{8} & \frac{6}{19} & \frac{3}{12} & \frac{6}{21} \\ 0 & 0 & \frac{3}{19} & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{12} & 0 \\ 0 & 0 & 0 & 0 & \frac{5}{21} \end{bmatrix},$$

Then the certainty equivalent of a Pareto optimal allocation $(d_i + R_i^* X^N)_{i \in N} \in Z_\Gamma(N)$ equals

$$\begin{aligned} m_1(d_1 + R_1^* X^N) &= d_1 - 18582, \\ m_2(d_2 + R_2^* X^N) &= d_2 - 11149, \\ m_3(d_3 + R_3^* X^N) &= d_3 - 2, \\ m_4(d_4 + R_4^* X^N) &= d_4 - 50, \\ m_5(d_5 + R_5^* X^N) &= d_5 - 39. \end{aligned}$$

The insurance premiums that the agents have to pay, are calculated according to the zero utility principle. This means that the aggregate premium which insurer 1 receives, equals

$$\begin{aligned} d_1^0 &= \pi_1\left(\tfrac{10}{16}X_2\right) + \pi_1\left(\tfrac{10}{19}X_3\right) + \pi_1\left(\tfrac{10}{24}X_4\right) + \pi_1\left(\tfrac{10}{21}X_5\right) - \pi_2\left(\tfrac{6}{16}X_1\right) \\ &= -m_1\left(-\tfrac{10}{16}X_2\right) - m_1\left(-\tfrac{10}{19}X_3\right) - m_1\left(-\tfrac{10}{24}X_4\right) \\ &\quad -m_1\left(-\tfrac{10}{21}X_5\right) + m_2\left(-\tfrac{6}{16}X_1\right) \\ &\approx 9739 + 6 + 62 + 78 - 5597 = 4288. \end{aligned}$$

Similarly, we get for insurer 2 and individuals 3, 4 and 5

$$\begin{aligned} d_2^0 &= 5597 + 3 + 37 + 47 - 9739 \;=\; -4055, \\ d_3^0 &= -3 - 6 \;=\; -9, \\ d_4^0 &= -37 - 62 \;=\; -99, \\ d_5^0 &= -47 - 78 \;=\; -125. \end{aligned}$$

With $d^0 = (4288, -4055, -9, -99, -125)$ being the aggregate insurance premiums we obtain that $(m_i(d_i^0 + R_i^* X^N))_{i \in N} = (-14294, -15204, -11, -149, -164)$. It is a straightforward exercise to check that $(-14294, -15204, -11, -149, -164)$ is a core allocation of the corresponding TU-game (N, v_Γ) presented in Table 1.

Now, let us take a closer look at the changes in insurer 1's utility when the allocation $(d_i^0 + R_i^* X^N)_{i \in N}$ is realized. In the initial situation insurer 1 bears the risk X_1 of his own insurance portfolio. The certainty equivalent of X_1 equals

$$\begin{aligned} m_1(X_1) &= 900 \cdot -10 \log\left(0.5(e^2 - 1)\right) + 25 \cdot -10 \log\left(0.05(e^{20} - 1)\right) \\ &\approx -14704. \end{aligned}$$

S	$v_\Gamma(S)$	S	$v_\Gamma(S)$	S	$v_\Gamma(S)$
$\{1\}$	-14704	$\{2,5\}$	-17730	$\{2,3,4\}$	-17725
$\{2\}$	-17551	$\{3,4\}$	-189	$\{2,3,5\}$	-17742
$\{3\}$	-14	$\{3,5\}$	-209	$\{2,4,5\}$	-17893
$\{4\}$	-174	$\{4,5\}$	-369	$\{3,4,5\}$	-383
$\{5\}$	-195	$\{1,2,3\}$	-29509	$\{1,2,3,4\}$	-29658
$\{1,2\}$	-29498	$\{1,2,4\}$	-29647	$\{1,2,3,5\}$	-29672
$\{1,3\}$	-14715	$\{1,2,5\}$	-29661	$\{1,2,4,5\}$	-29810
$\{1,4\}$	-14861	$\{1,3,4\}$	-14872	$\{1,3,4,5\}$	-15044
$\{1,5\}$	-14876	$\{1,3,5\}$	-14888	$\{2,3,4,5\}$	-17905
$\{2,3\}$	-17562	$\{1,4,5\}$	-15053	$\{1,2,3,4,5\}$	-29822
$\{2,4\}$	-17713				

Table 1:

To allocate the total risk in a Pareto optimal way, insurer 1 bears the fraction $r^*_{12} = \frac{10}{16}$ of the risk X_2 of insurer 2. For this risk he receives a premium $\pi_1(\frac{10}{16}X_2)$ as determined by the zero utility principle. From the definition of the zero utility calculation principle and the independence of X_1 and X_2, it follows that $m_1(-X_1 - \frac{10}{16}X_2 + \pi_1(\frac{10}{16}X_2)) \approx -14704$. So insurer 1's welfare does not change when he insures a part of the risk of insurer 2. A similar argument holds when he insures a part of the risks of the other agents. Hence

$$\begin{aligned} & m_1(-X_1 - \tfrac{10}{16}X_2 + \pi_1(\tfrac{10}{16}X_2) - \tfrac{10}{19}X_3 + \pi_1(\tfrac{10}{19}X_3) \\ & \quad -\tfrac{10}{24}X_4 + \pi_1(\tfrac{10}{24}X_4) - \tfrac{10}{21}X_5 + \pi_1(\tfrac{10}{21}X_5)) \;\approx\; -14704 \end{aligned}$$

The increase in insurer 1's welfare arises only from the risk $\frac{6}{16}X_1$ he transfers to insurers 2:

$$\begin{aligned} & m_1(-\tfrac{10}{16}X - \pi_2(\tfrac{6}{16}X_1) - \tfrac{10}{16}X_2 + \pi_1(\tfrac{10}{16}X_2) - \tfrac{10}{19}X_3 + p_1(\tfrac{10}{19}X_3) \\ & \quad -\tfrac{10}{24}X_4 + \pi_1(\tfrac{10}{24}X_4) - \tfrac{10}{21}X_5 + \pi_1(\tfrac{10}{21}X_5)) \\ = \;& m_1(d^0_1 + R^*_1 X^N) \;\approx\; -14294 \;>\; -14704. \end{aligned}$$

The phenomenon described above is subsistent in the definition of the zero utility principle. This means that the welfare of an insurer always remains the same when he bears the risk of someone else in exchange for the zero utility principle based premium. An increase in welfare only arises when he transfers (a part of) his own risk to someone else.

4 Subadditivity for Collective Insurances

In the insurance games defined in the previous section individual persons are not allowed to cooperate; they cannot redistribute the risk amongst themselves. Looking at the individuals' behavior in everyday life, this is a justified assumption. People who want to insure themselves against certain risks do so by contacting insurance companies, pension funds etc. We show, however, that when this restriction is abandoned then the mere fact that risk exchanges could take place between individuals implies that insurance companies have incentives to employ subadditive premiums. Whether or not such risk exchanges actually do take place is not important. As a consequence, collective insurances become cheaper for the individuals.

Let N_P be the set of individuals. A premium calculation principle π is called subadditive if for all subsets $S, T \subset N_P$ with $S \cap T = \emptyset$ it holds that $\pi(X_S) + \pi(X_T) \geq \pi(X_S + X_T)$. Here, X_S denotes the total loss of the coalition S. So, it is attractive for the individuals to take a collective insurance, since this reduces the total premium they have to pay.

Next, consider a game with agent set N_P only where the individuals are allowed to redistribute their risks. This situation can be described by an insurance game $\Gamma \in IG(N_P, \emptyset)$. So, the individuals N_P can now insure their losses among each other. Then we can associate with Γ the TU-game (N, v_Γ), with

$$v_\Gamma(S) = \max \left\{ \sum_{i \in S} m_i(d_i - R_i X^S) | \, (d_i - R_i X^S)_{i \in S} \in Z_\Gamma(S) \right\}$$

for all $S \subset N_P$. Note that this maximum is attained for Pareto optimal allocations $(d_i - R_i^* X^S)_{i \in S} \in Z_\Gamma(S)$ for coalition S. For this game, the value $v_\Gamma(S)$ can be interpreted as the maximum premium coalition S wants to pay for the insurance of the total risk X_S. To see this, suppose that the coalition S can insure the loss X_S for a premium $\pi(X_S)$ that exceeds the valuation of the risk X_S, that is, $-\pi(X_S) < v_\Gamma(S)$. Then for each allocation $y \in \mathbb{R}^S$ of the premium $-\pi(X_S)$ there exists an allocation $(\tilde{d}_i - R_i^* X^S) \in Z_\Gamma(S)$ such that $E(U_i(\tilde{d}_i - R_i^* X^S)) > U_i(y_i)$ for all $i \in S$. Indeed, let $(d_i - R_i^* X^S)_{i \in S} \in Z_\Gamma(S)$ be such that $\sum_{i \in S} m_i(d_i - R_i^* X^S) = v_\Gamma(S)$. Define

$$\tilde{d}_i = d_i - m_i(d_i - R_i^* X^S) + y_i + \tfrac{1}{\#S} \left(v_\Gamma(S) + \pi(X_S) \right),$$

for all $i \in S$. Since

$$\begin{aligned} \sum_{i \in S} \tilde{d}_i &= \sum_{i \in S} d_i - \sum_{i \in S} m_i(d_i - R_i X^S) + \sum_{i \in S} y_i + v_\Gamma(S) + \pi(X_S) \\ &= \textstyle\sum_{i \in S} d_i \leq 0 \end{aligned}$$

it follows that $(\tilde{d}_i - R_i X^S)_{i \in S} \in Z_\Gamma(S)$. Then by the linearity of m_i in $\tilde{d}_i$ (cf. expression (2)) we have for all $i \in S$ that

$$m_i(\tilde{d}_i - R_i^* X^S) = y_i + \tfrac{1}{\#S} \left(v_\Gamma(S) + \pi(X_S) \right) > y_i.$$

Hence, the members of S prefer the allocation $(\tilde{d}_i - R_i^* X^S)_{i \in S}$ of X_S to an insurance of X_S and paying the premium $\pi(X_S)$. Consequently, they will not pay more for the insurance of the risk X_S than the amount $-v_\Gamma(S)$. Now, it is a straightforward exercise to show that this maximum premium $-v_\Gamma(S)$ satisfies subadditivity, i.e. $-v_\Gamma(S) - v_\Gamma(T) \geq -v_\Gamma(S \cup T)$. For totally balancedness of insurance games implies superadditivity, i.e. $v_\Gamma(S) + v_\Gamma(T) \leq v_\Gamma(S \cup T)$ for all disjoint $S, T \subset N$.

5 Remarks

In this paper (re)insurance problems are modelled as cooperative games with stochastic payoffs. In fact, we defined a game that dealt with both the insurance and the reinsurance problem simultaneously. We showed that there is only one allocation risk exchange matrix yielding a Pareto optimal distribution of the losses and that a core allocation results when insurance premiums are calculated according to the zero utility principle.

Recall that insurers do not benefit from insuring the risks of the individuals when utilizing the additive zero utility principle; this premium calculation principle yields the lowest premium for which insurers still want to exchange risks with the individuals (see Example 5). So, from a social point of view, it might be best to adopt a middle course and look for premiums where both insurers and individuals benefit from the insurance transaction. Interesting questions then remaining are: are these premiums additive or subadditive and do they yield core allocations?

An issue only briefly mentioned in this paper concerns the insurers' behavior. What if an insurer is risk neutral or risk loving instead of risk averse? Thus, there is at least one insurer whose utility function is linear or of the form $u_i(t) = \beta_i e^{-\alpha_i t}$ ($t \in \mathbb{R}$) with $\beta_i > 0$, $\alpha_i < 0$. Although the proofs are not provided here, most of the results presented in this paper still hold for these situations. This means that the corresponding games have nonempty cores and that the zero utility principle still yields a core allocation. The result that does change is the Pareto optimal allocation of the risk. The allocations that are Pareto optimal when all insurers are risk averse are not Pareto optimal anymore when one or more insurers happen to be risk loving. In fact, they are the worst possible allocations of the risk one can think of. In that case, allocating all the risk to the most risk loving insurer is Pareto optimal. This would actually mean that only one insurance company is needed, since other insurance companies will ultimately reinsure their complete portfolios at this most risk loving insurer.

Finally, it should be mentioned that the results presented in this paper still go through if we replace the assumption that the risks $(X_i)_{i \in N_P \cup N_i}$ are mutually independent by the assumption that the covariance matrix of $(X_i)_{i \in N_P \cup N_i}$ is negative definite.

6 Appendix: Proofs

Lemma 1 Let $c \in \mathbb{R} \backslash \{0\}$ and let F be a probability distribution function corresponding to a non-degenerate random variable. Then the function h_c defined by $h_c(x) = \log \left(\int_0^\infty e^{xct} dF(t) \right)^{-1}$ for $x \geq 0$, is strictly concave in x.

PROOF: Since

$$\frac{dh_c}{dx} = -c \frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)}$$

we obtain that

$$
\begin{aligned}
\frac{dh_c^2}{dx^2} &= -c^2 \frac{\int_0^\infty t^2 e^{xct} dF(t) \int_0^\infty e^{xct} dF(t) - \left(\int_0^\infty t e^{xct} dF(t)\right)^2}{\left(\int_0^\infty e^{xct} dF(t)\right)^2} \\
&= -c^2 \int_0^\infty \left(t - \frac{\int_0^\infty \tau e^{xc\tau} dF(\tau)}{\int_0^\infty e^{xc\tau} dF(\tau)} \right)^2 \frac{e^{xct}}{\int_0^\infty e^{xc\tau} dF(\tau)} dF(t) \\
&\leq 0.
\end{aligned}
$$

Hence, h_c is concave. The lemma then follows from the observation that the inequality is binding if and only if F corresponds to a degenerate random variable. □

Lemma 2 Let $c \in \mathbb{R} \backslash \{0\}$. Then the function $h_c(x) = \left(\int_0^\infty e^{xct} dF(t)\right)^{\frac{1}{x}}$, $x > 0$, is increasing in x.

PROOF: Note that

$$
\begin{aligned}
\frac{dh_c}{dx} &= \frac{d}{dx} e^{\frac{1}{x} \log\left(\int_0^\infty e^{xct} dF(t)\right)} \\
&= h_c(x) \left(\frac{-1}{x^2} \log \left(\int_0^\infty e^{xct} dF(t) \right) + \frac{c}{x} \frac{\left(\int_0^\infty t e^{xct} dF(t)\right)}{\left(\int_0^\infty e^{xct} dF(t)\right)} \right) \\
&= -\frac{h_c(x)}{x^2} \left(\log \left(\int_0^\infty e^{xct} dF(t) \right) - xc \frac{\left(\int_0^\infty t e^{xct} dF(t)\right)}{\left(\int_0^\infty e^{xct} dF(t)\right)} \right).
\end{aligned}
$$

Since $x > 0$ it is sufficient to show that

$$
\log \left(\int_0^\infty e^{xct} dF(t) \right) - xc \frac{\left(\int_0^\infty t e^{xct} dF(t)\right)}{\left(\int_0^\infty e^{xct} dF(t)\right)} \leq 0. \tag{7}
$$

First, note that

$$
\lim_{x \downarrow 0} \log \left(\int_0^\infty e^{xct} dF(t) \right) - xc \frac{\left(\int_0^\infty t e^{xct} dF(t)\right)}{\left(\int_0^\infty e^{xct} dF(t)\right)} = 0.
$$

Second, differentiating expression (7) to x yields

$$
\begin{aligned}
& c\frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)} - c\frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)} - xc \frac{d}{dx} \left(\frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)} \right) \\
&= -xc \frac{d}{dx} \frac{\int_0^\infty t e^{xct} dF(t)}{\int_0^\infty e^{xct} dF(t)} \\
&= -xc^2 \frac{\int_0^\infty t^2 e^{xct} dF(t) \int_0^\infty e^{xct} dF(t) - \left(\int_0^\infty t e^{xct} dF(t)\right)^2}{\left(\int_0^\infty e^{xct} dF(t)\right)^2} \\
&= -xc^2 \int_0^\infty \left(t - \frac{\int_0^\infty \tau e^{xc\tau} dF(\tau)}{\int_0^\infty e^{xc\tau} dF(\tau)} \right)^2 \frac{e^{xct}}{\int_0^\infty e^{xc\tau} dF(\tau)} dF(t) \\
&\leq 0.
\end{aligned}
$$

Thus, since (7) is a decreasing function in x taking the value zero in $x = 0$, it follows that (7) is negative for all $x > 0$. □

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