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Voluntary Contribution to Multiple Public Projects

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Abstract

The problem of financing a set of public goods (facilities, projects) by private contributions is studied. The corresponding cooperative game, the *realization game*, is shown to be convex. For the noncooperative setting we study a realization scheme that induces a strategic game. This *contribution game* is shown to be best-response equivalent with a coordination game in which the payoff to all players is the utilitarian collective welfare function, i.e., the sum of the utility functions of the players. Several equilibrium properties are derived: no money is wasted in an equilibrium; a player whose necessary projects are not all realized does not contribute. Strategy profiles maximizing utilitarian welfare are strong Nash equilibria of the contribution game. Each strong Nash equilibrium corresponds to a core element of the realization game in a natural way. It is shown that there is a one-to-one correspondence between the set of strong Nash equilibria of the contribution game and the largest set of core elements of the realization game, that is consistent with maximizing the number of players with non-zero payoffs. It is precisely the subset of the core according to which rewards zero indicate null players.

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Keywords: public goods, cooperative games, coordination games, potential games, utilitarian welfare function

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1 Introduction

The object of this paper is to study problems of private provision of a collection of public facilities, or, as we call them, projects. The projects are considered pure public goods: once a project has been built, all players can use it. Specifically, in the contribution problem there are finitely many players. Each of these players is interested in a subset of the finite set of projects. Realization of these is necessary for him to derive a benefit: if and only if a superset of them is realized, he receives a reward. Associated with each project are its costs.

We focus on two decision making processes, differing in the degree of cooperation in the decision making process. In the cooperative situation, in presence of the possibility to enforce general agreement, we focus on the naturally related cooperative transferable utility game, i.e. the *realization game*. The game is determined by the associating each coalition of players to the aggregate profits that it is capable of generating itself independent from the others, just by making an optimal choice between the feasible combinations of projects. In the noncooperative mode, i.e. in absence of the possibility making binding agreements, an additional component, the *realization scheme*, determines the strategic *contribution game*. The players are assumed to submit a contribution independently of the other players, and given the profile of contributions the realization scheme determines which projects are realized, and consequently also the individual payoffs. The strategy space of each player is his set of possible contributions. This set is taken to be the interval from zero (inclusive) to a player's reward (exclusive), meaning that each player contributes a nonnegative amount, but strictly less than his reward. The payoff function of a player is a player's reward if his projects are realized, minus his contribution.

Both games are similar to those studied in Young (1998), who focuses on setting access charges to public facilities and publicly regulated monopolies. However, the assumption of exclusion is explicitly used and therefore the results do not apply to our case of pure public goods. We only allow for partial exclusion in the weak sense that a player can be excluded from a project only if the player is not interested in the project, i.e. the player should be indifferent between being excluded and getting service for free. Due to this weak exclusive characteristic the projects under consideration differ from *local* or *club* goods (see, e.g., Cornes and Sandler (1996)). Up to now, not much has been said about the realization function: after all contributions have been made, what projects will be built? In fact, many possible realization functions come to mind. But considering that the players behave noncooperatively to subsidize public goods, it is of obvious significance to investigate whether a realization scheme can be defined that induces the contributors to play the contribution game, perhaps without them being aware of it, in the interest of the collective player set. In this paper, a simple measure of collective welfare is used: the sum of the individual player's payoff functions, often referred to as the classical utilitarian collective welfare function (cf. Moulin (1988)).

It is indeed possible to define a realization scheme in such a way that the contribution game is best-response equivalent with a coordination game in which each contributor's payoff is this utilitarian welfare function. In terms of Monderer and Shapley (1996), this realization scheme makes the contribution game an ordinal potential game, where one of the ordinal potential functions is the utilitarian welfare function. The realization scheme takes into account that each contributor is willing to pay only for projects he is interested in and that the money allocated to a project is never more than its costs. Remaining contributions in excess of the costs of the realized projects go to waste. Under these restrictions, there may still be several ways to allocate as much of the contributions as possible to the projects. Our realization scheme builds only those projects that are completely financed by each such maximal allocation. To make the realization scheme more precise, the model uses maximal flows and minimum cuts in certain flow networks.

The existence of Nash equilibria of the contribution game is established and several of its properties are studied. Given a profile of contributions, there may be players whose projects are not realized. These players contribute zero if the profile is a Nash equilibrium. Moreover, the contributions in a Nash equilibrium exactly suffice to pay for the projects of the players making a positive contribution, so no money goes to waste.

Now that it has been established that the players at least implicitly act in the interest of utilitarian welfare and that the game has a nonempty collection of Nash equilibria, one can derive that there is a Nash equilibrium maximizing utilitarian welfare. Hence, single players have no incentive to deviate since the profile is a Nash equilibrium, and the entire player set has no incentive to deviate since the profile maximizes utilitarian welfare. But one can show more. Such strategy profiles are in fact strong Nash equilibria of the contribution game: there is no coalition of players with an incentive to deviate from a strategy profile maximizing utilitarian welfare.

In particular this means that each strong Nash equilibrium defines a pre-imputation of the cooperative realization game, and, as will be shown, it determines a core element. There exists a strong relation between the concept of the core and the concept of strong Nash equilibrium: there is a 1-1 correspondence between the set of strong Nash equilibria of the contribution game and the payoffs in the core except those that that give zero payoff to non-null players. We note that a similar result was shown in Young (1998) for the excludable case.

Summarizing, by choosing a particular realization scheme, one can guarantee that the players of a noncooperative contribution game act in common interest, in the sense that maximizing a player's payoff function given the strategy profile of his opponents is equivalent with maximizing utilitarian welfare given the strategy profile of his opponents. Not only do the players act in common interest, but there exist profiles maximizing utilitarian welfare, which turn out to be strong Nash equilibria of the contribution game and core elements of the realization game. Our last theorem explores the relationship between strong Nash equilibria and core elements in detail.

2 The cooperative realization game

In this section the model is specified and some preliminary results are provided. First we need some additional notation. For a set X its power set is identified with 2^X, i.e. the set of all mappings $f : X \to \{0,1\}$. In this fashion $S \subseteq X$ is identified with $f \in 2^X$ if $f(i) = 1$ if and only if $i \in S$.

Definition A *realization problem* is represented by an ordered tuple $\mathcal{R} = (N, M, m, \omega, c)$, where

(i) N is the finite set of players;

(ii) M is the finite set of public goods or projects;

(iii) $m = (m_i)_{i \in N} \in (2^M)^N$ specifies the set of projects required by each player: player $i \in N$ needs the projects in $m_i \subseteq M$;

(iv) $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_{++}$ specifies the reward to each player $i \in N$ if (a superset of) all projects in m_i are realized; it expresses the individual needs for projects;

(v) $c = (c_j)_{j \in M} \in \mathbb{R}^M_{++}$ specifies for each project $j \in M$ the costs c_j to build this project;

The projects are considered to be public goods: once a project has been built, all players can make use of it.

A *cooperative game* (von Neumann and Morgenstern (1944)) is an ordered pair (N, v) where N is the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function* relating each coalition $S \subseteq N$ to a real number $v(S)$ that is interpreted as the total of profits that S is able to generate through internal cooperation. Moreover, it is assumed that $v(\emptyset) = 0$. We will write $\mathcal{G}$ for the class of all cooperative games. Each realization problem corresponds to a cooperative game in a natural way. The value of a coalition of players $S \subseteq N$ is the total of net benefits that it is able to collect by realization of the right combination of projects. That is, to each realization problem $\mathcal{R} = (N, M, m, \omega, c)$ the related cooperative game $(N, v_{\mathcal{R}})$ is defined through

$$v_{\mathcal{R}}(S) = \max_{\tilde{m} \subseteq m} \left\{ \sum_{i \in S; m_i \subseteq \tilde{m}} \omega_i \Leftrightarrow \sum_{j \in \tilde{m}} c_j \right\} \text{ for all } S \subseteq N. \quad (2.1)$$

Throughout this paper we will refer to $(N, v_{\mathcal{R}})$ as the *cooperative realization game* for $\mathcal{R}$.

Example 2.1 Let $\mathcal{R} = (N, M, m, w, c)$ be the realization problem as is defined through $N = \{1, 2, 3\}, M = \{p, q, r\}, m_1 = \{p\}, m_2 = \{p, q\}, m_3 = \{q, r\}, \omega = (10, 10, 20)$ and $c = (9, 5, 10)$. Then the values of the different coalitions of the corresponding 3-player cooperative realization game $(N, v_{\mathcal{R}})$ are listed in the table below. ◁

Theorem 2.2 *The cooperative realization game $(N, v_{\mathcal{R}})$ is convex, i.e. for all $i \in N$ and*

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	N
$v_{\mathcal{R}}(S)$	0	1	0	5	6	6	6	16

Figure 2.1: The values for $v_{\mathcal{R}}$.

$S \subset T \subseteq N \backslash \{i\}$ it holds

$$v_{\mathcal{R}}(S \cup \{i\}) \Leftrightarrow v_{\mathcal{R}}(S) \leq v_{\mathcal{R}}(T \cup \{i\}) \Leftrightarrow v_{\mathcal{R}}(T).$$

Proof. Let $S_i \subseteq S \cup \{i\}$ be such that $v_{\mathcal{R}}(S \cup \{i\}) = \omega(S_i) \Leftrightarrow c(m_{S_i})$ and let $T_0 \subseteq T$ be such that $v(T) = \omega(T_0) \Leftrightarrow c(m_{T_0})$. We have:

$$\begin{aligned} v_{\mathcal{R}}(T \cup \{i\}) \Leftrightarrow v_{\mathcal{R}}(T) &\geq \{\omega(T_0 \cup S_i) \Leftrightarrow c(m_{T_0 \cup S_i})\} \Leftrightarrow \{\omega(T_0) \Leftrightarrow c(m_{T_0})\} \\ &= \omega(S_i) \Leftrightarrow \omega(S_i \cap T_0) \Leftrightarrow \{c(m_{T_0 \cup S_i}) \Leftrightarrow c(m_{T_0} \backslash m_{S_i})\} \\ &\quad + \{c(m_{T_0}) \Leftrightarrow c(m_{T_0} \backslash m_{S_i})\} \\ &= \omega(S_i) \Leftrightarrow \omega(S_i \cap T_0) \Leftrightarrow c(m_{S_i}) + c(m_{S_i} \cap m_{T_0}) \\ &\geq v(S \cup \{i\}) \Leftrightarrow \{\omega(S_i \cap T_0) \Leftrightarrow c(m_{S_i \cap T_0})\} \\ &\geq v(S \cup \{i\}) \Leftrightarrow v(S). \end{aligned}$$

□

The convexity of cooperative realization games expresses that there is an incentive for the players to cooperate. Given the cooperation of the grand coalition the problem of allocating $v_{\mathcal{R}}(N)$ over the individual players remains. In case of an arbitrary cooperative game (N, v) a preferable allocation is stable in the sense that no coalition of players has an incentive to split off. The set of all those stable allocations is called the *core*, notation $core(N, v)$, and $core(N, v) := \{x \in \mathbb{R}^N \,|\, x(S) \geq v(S) \text{ for all } S \subseteq N, x(N) = v(N)\}$. By convexity of the game $(N, v_{\mathcal{R}})$ we have $core(N, v_{\mathcal{R}}) \neq \emptyset$ (cf. Shapley (1971)) for each realization problem $\mathcal{R}$.

The values $v_{\mathcal{R}}(S)$ can be calculated in polynomial time by determining minimum cuts of certain flow networks that will be defined subsequently. For $S \subseteq N$ construct a flow network , $_S$ as follows. The network , $_S$ has as node set V consisting of a source, a sink, S, and $m_S := \cup_{i \in S} m_i$. The nodes are called *So*, *Si*, *node*(i) $(i \in S)$, and *node*(j) $(j \in m_S)$. Moreover , $_S$ has as arc set A consisting of directed arcs. For each player $i \in S$ there is an arc a_i from the source *So* to player i's node *node*(i) with capacity $cap(i) = \omega_i$. When project $j \in m_S$ is an element of m_i, there is an arc a_{ij} from *node*(i) to *node*(j) with a capacity strictly larger than the individual benefits ω_i say for instance $cap(ij) = \omega_i + 1$. For each project $j \in m_S$ there is an arc a_j from *node*(j) to the sink *Si* with capacity $cap(j) = c_j$.

Example 2.3 The flow network , $_N$ corresponding to the realization problem as in Example 2.1 has the form of Figure 2.2. ◁

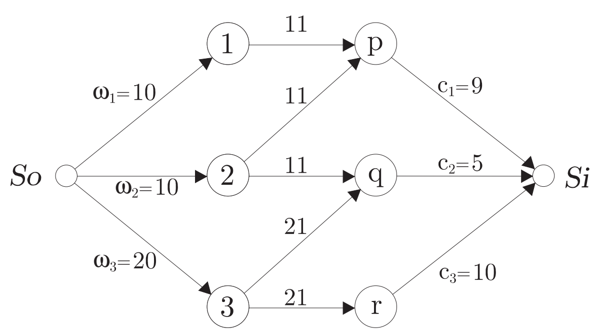

Figure 2.2: A flow network.

Theorem 2.7 shows how that the construction of precisely those projects that appear in some minimum cut of , $_S$ maximizes the aggregate payoffs for coalition S. Definitions concerning flows and cuts in a flow network (V, A) with a source and a sink are briefly reviewed. For a more detailed study, see for instance Rockafellar (1984). A flow is a function $f : A \to \mathbb{R}$ such that for each directed arc (i, j) from node i to node j, $f(i, j) \in [0, cap(i, j)]$, and flow is conserved at every node, except possibly at the source and the sink. One can understand a flow as an amount of water transported from the source, through the network, to the sink, without flooding the arcs. A cut is a set of arcs such that each positive flow from source to sink uses at least one of these arcs. Intuitively, it is called a cut because removal of the arcs in a cut would disconnect all possible channels for a positive flow. The maximal amount of flow in a network , $= (V, A)$ is denoted max flow(,). The capacity of a cut is the sum of the capacities of the arcs in this cut. A cut is minimum if there is no cut in the network with a smaller capacity. The capacity of a minimum cut of , is denoted min cut(,). The following results are often used.

Lemma 2.4 *In a network , $= (V, A)$,*

(i) max flow(,) = min cut(,).

(ii) *an arc is used to full capacity in each maximal flow if and only if it is in some minimum cut.*

The first part of the lemma is the famous max flow-min cut theorem of Ford and Fulkerson (1956). The proof of the second part is straightforward: an arc is used to full capacity in each maximal flow if and only if reducing its capacity reduces the value of the flow, if and only if the arc is in some minimum cut. Consider a flow network $,_S$ arising from some realization problem $\mathcal{R}$. Notice that the capacity of an arc a_{ij} with $i \in S, j \in m_S$ is chosen so large, that arcs of this type are never in a minimum cut of $,_S$. Thus, for every minimum cut C in a flow network $,_S$ there exist $S' \subseteq S$ and $T \subseteq M_S$ such that $C = \bigcup_{i \in S'} a_i \cup \bigcup_{j \in T} a_j$. With a slight abuse of notation, we will denote this cut C by (S', T) with $S' \subseteq S, T \subseteq m_S$. The set of minimum cuts of a flow network $,_S$ is denoted $MC(,_S)$. The following example illustrates these definitions.

Example 2.5 Consider a flow network that is similar to that in the previous example. Let $\omega_1 = 10, \omega_2 = 6, \omega_3 = 8$, and take $c_p = 9, c_q = 5, c_r = 10$. Then we obtain the flow network in Figure 2.3. The capacities of intermediary arcs are considered to be high and are omitted for notational convenience. There are infinitely many maximal flows by taking convex combinations of the maximal flows f and g defined as in Figure 2.5.

There is one minimum cut, namely $C = (S, T)$ with $S = \{3\}$ and $T = \{p, q\}$. Notice that

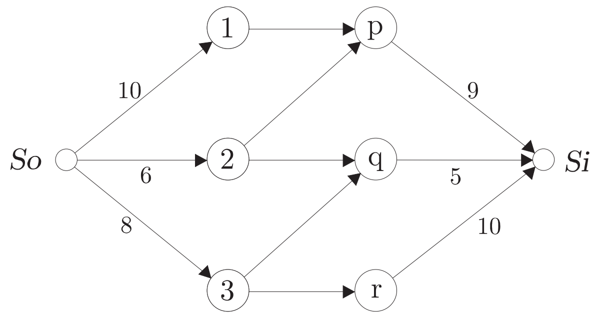

Figure 2.3: A flow network

arc	a_1	a_2	a_3	a_{1p}	a_{2p}	a_{2q}	a_{3q}	a_{3r}	a_p	a_q	a_r
f	8	6	8	8	1	5	0	8	9	5	8
g	9	5	8	9	0	5	0	8	9	5	8

Figure 2.4: Combining f and g gives uncountably many maxflows.

the maximal amount of flow from source to sink equals 22, which is exactly the capacity of the cut (S, T). ◁

Lemma 2.6 *Let $\mathcal{R}$ be a realization problem, and* , $_S$ *the corresponding flow network for a coalition of players S. If $C_1 = (S_1, T_1)$ and $C_2 = (S_2, T_2)$ are minimum cuts of* , $_S$*, then so are $C_3 = (S_1 \cap S_2, T_1 \cup T_2)$ and $C_4 = (S_1 \cup S_2, T_1 \cap T_2)$.*

Proof. Each directed path from the source to the sink is uniquely described by a pair (i, j) with $i \in N$, $j \in S$ and $j \in m_i$. By definition of a cut, for each such path (i, j) either $i \in S_k$ or $j \in T_k$ $(k = 1, 2)$. It follows easily that $i \in S_1 \cap S_2$ or $j \in T_1 \cup T_2$ and that$i \in S_1 \cup S_2$ or $j \in T_1 \cap T_2$. As a consequence, C_3 and C_4 are cuts. Moreover,

$$\begin{aligned}
\sum_{\ell \in C_1} cap(\ell) + \sum_{\ell \in C_2} cap(\ell) &= \sum_{i \in S_1} cap(i) + \sum_{j \in T_1} cap(j) + \sum_{i \in S_2} cap(i) + \sum_{j \in T_2} cap(j) \\
&= \sum_{i \in S_1 \cap S_2} cap(i) + \sum_{j \in T_1 \cup T_2} cap(j) + \\
&\quad + \sum_{i \in S_1 \cup S_2} cap(i) + \sum_{j \in T_1 \cap T_2} cap(j) \\
&= \sum_{\ell \in C_3} cap(\ell) + \sum_{\ell \in C_4} cap(\ell).
\end{aligned}$$

Since both C_1 and C_2 are minimum cuts, this implies that C_3 and C_4 are minimum cuts. □

Theorem 2.7 *Let $\mathcal{R} = (N, M, m, \omega, c)$ be a realization problem and $(N, v_{\mathcal{R}})$ the corresponding cooperative realization game. Let $S \subseteq N$ and let $(S_1, Q) \in MC(\,,_S)$. Then*

$$v_{\mathcal{R}}(S) = \sum_{i \in S, m_i \subseteq Q} \omega_i \Leftrightarrow \sum_{j \in Q} c_j. \tag{2.2}$$

Proof. Firstly, observe that every cut (U, V) of $,_S$ that is minimal with respect to inclusion is of the form $(\{i \in S | m_i \not\subseteq P\}, P)$ for some $P \subseteq M$. Hence

$$\sum_{i \in S, m_i \not\subseteq Q} cap(i) + \sum_{j \in Q} cap(j) = \min_{P \subseteq M} \left\{ \sum_{i \in S, m_i \not\subseteq P} cap(i) + \sum_{j \in P} cap(j) \right\}.$$

This gives

$$\begin{aligned}
\sum_{i \in S, m_i \subseteq Q} \omega_i \Leftrightarrow \sum_{j \in Q} c_j &= \sum_{i \in S} cap(i) \Leftrightarrow \left\{ \sum_{i \in S, m_i \not\subseteq Q} cap(i) + \sum_{j \in Q} cap(j) \right\} \\
&= \sum_{i \in S} \omega_i \Leftrightarrow \min_{P \subseteq M} \left\{ \sum_{i \in S, m_i \not\subseteq P} cap(i) + \sum_{j \in P} cap(j) \right\} \\
&= \sum_{i \in S} \omega_i + \max_{P \subseteq M} \left\{ \Leftrightarrow \sum_{i \in S, m_i \not\subseteq P} \omega_i \Leftrightarrow \sum_{j \in P} c_j \right\}
\end{aligned}$$

$$= \max_{P \subseteq M} \left\{ \sum_{i \in S, m_i \subseteq P} \omega_i \Leftrightarrow \sum_{j \in P} c_j \right\} = v_{\mathcal{R}}(S).$$

□

3 Noncooperative realization problems: contribution games

In the former section we discussed the cooperative model for realizing a set of projects. The collective of players is able to decide upon the optimal set of projects to be constructed. Here we consider the realization problem as a *contribution problem* where no binding agreements can be made and the different players have to decide individually how much they want to spend on having their projects realized. After the individual contributions have been made, an independent arbitrator is supposed to decide upon the projects to buy. This task involves a lot of ambiguity, since in general a profile of contributions can be associated with more than one feasible set of projects. Therefore the arbitrator makes use of a decision rule that we will call a *realization scheme*. A realization scheme relates each profile of contributions to an affordable combination of projects. A realization problem together with a realization scheme R is called a *contribution problem.* The arbitration procedure is not a black box: before the players make their bids known to the arbitrator the realization scheme is publicly announced.

It makes sense to require from the arbitrator that he puts forward a 'reasonable' realization scheme. For instance, it may be perceived as 'unfair' if the arbitrator decides to use the contribution of player 1 to buy other projects than he is interested in, especially if these are projects for zero contributors. Also the realization scheme should be such that it should give the players the right incentives. Those players which will profit a lot by having the desired set of projects should be pushed to contribute. Here we put forward a realization scheme that combines a number desirable features in this respect. A formal definition will be given subsequently, and requires some additional work. First we will define formally the strategic game that corresponds to the above noncooperative setting.

The contribution problem $\mathcal{C}$ induces a *contribution game*

$$G(\mathcal{C}) = (N, (X_i)_{i \in N}, (u_i)_{i \in N}),$$

where the strategy space of player $i \in N$, the set of possible contributions, is $X_i = [0, \omega_i)$. The realization scheme R is a mapping from the Cartesian product of the strategy spaces of the different players, i.e. $\prod_{i \in N} X_i$, to the set of all combinations of projects 2^M. Then for a given profile of individual contributions $x \in \prod_{i \in N} X_i$, $R(x)$ stands for the set of projects that will be realized. Player i's payoff function $u_i : X \to \mathbb{R}$ is defined, for each profile $x = (x_i)_{i \in N} \in X$ as

$$u_i(x) = \begin{cases} -x_i & \text{if} \quad m_i \not\subseteq R(x), \\ \omega_i - x_i & \text{if} \quad m_i \subseteq R(x). \end{cases}$$

That is: he gets his reward ω_i only if all of his projects are realized and his contribution x_i causes disutility.

By taking $X_i = [0, \omega_i)$, it is assumed that each player $i \in N$ contributes a nonnegative amount, but strictly less than his reward ω_i. This is not a very restrictive assumption: it makes no sense to contribute more than the benefit you can derive from the realization of your projects and contributing ω_i is weakly dominated by contributing 0. A slightly different approach, not influencing the results in the present paper, would be to endow each player $i \in N$ with an initial amount $e_i \in \mathbb{R}_+$ of money such that $e_i < \omega_i$ and to take $X_i = [0, e_i]$. We do not take this approach.

For notational convenience, let X be the Cartesian product of the strategy sets X_i, i.e. $X = \prod_{i \in N} X_i$. Moreover, in the same fashion we define for $i \in N$ and $S \subseteq N$:

$$X_{-i} := \prod_{k \in N \setminus \{i\}} X_k \text{ and } X_S := \prod_{i \in S} X_i.$$

Occasionally, notation like (x_i, x_{-i}) or $(x_{N \setminus S}, x_S)$ is used if the strategy of player i or coalition S needs stressing.

The promised realization scheme R is inspired by the techniques that we used to find the values of the characteristic function of the realization problem. We define in a similar way a flow network , (x) for each profile $x \in X$ of contributions. , (x) has a node set V consisting of a source, a sink, N, and M. The nodes are called So, Si, $node(i)$ $(i \in N)$, and $node(j)$ $(j \in M)$. , (x) has arc set A consisting of directed arcs. For each player $i \in N$ there is an arc a_i from the source So to player i's node $node(i)$ with capacity $cap(i) = x_i$. When project $j \in M$ is an element of m_i, there is an arc a_{ij} from $node(i)$ to $node(j)$ with a capacity strictly larger than any possible contribution by player i, for instance $cap(ij) = \omega_i + 1$. For each project $j \in M$ there is an arc a_j from $node(j)$ to the sink Si with capacity $cap(j) = c_j$. Notice that the underlying network (V, A) is the same for each , (x); only the capacities of the player arcs are different.

Example 3.1 In a contribution problem with player set $N = \{1, 2, 3\}$, project set $M = \{p, q, r\}$, and $m_1 = \{p\}, m_2 = \{p, q\}, m_3 = \{q, r\}$, the flow network , (x) given contributions $x = (x_1, x_2, x_3)$ has the form of Figure 3.1. ◁

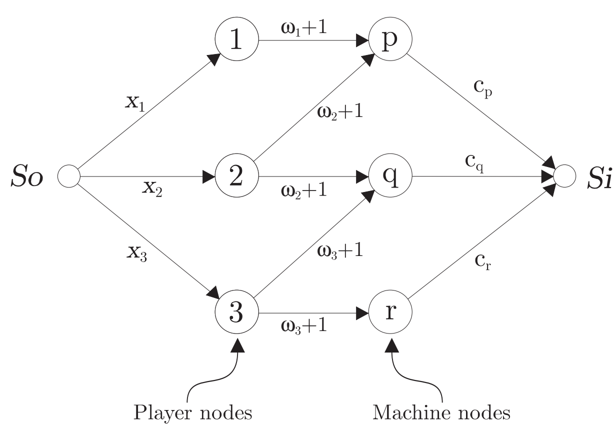

Figure 3.1: A flow network

Recall the definitions concerning flows and cuts in a flow network (V, A) with a source and a sink. Take a flow network , (x) arising from some contribution problem $\mathcal{C}$. The set of minimum cuts of , (x) will be denoted by $MC(x)$. Notice that the capacity of an arc a_{ij} with $i \in N, j \in M$ is chosen so large, that arcs of this type are never in a minimum cut of , (x). Thus, for every minimum cut $C \in MC(x)$ there exist $S \subseteq N$ and $T \subseteq M$ such that $C = (S, T)$.

What insight does the flow network , (x) defined above give us in the problem under consideration? Given the constraint that each player $i \in N$ is willing to contribute only to the cost of projects in his desired set m_i, a maximal flow f describes exactly

(i) how much of the total contribution $\sum_{i \in N} x_i$ can be used for the provision of the projects, namely max flow(, (x)),

(ii) which projects can be financed using this particular maximal flow, namely those with arcs used to maximum capacity, and

(iii) who contributes how much to the costs of these projects in the maximal flow f, namely player i contributes to project j the amount of flow through a_{ij}, $f(a_{ij})$.

Since selecting a particular maximal flow would strongly favour some of the players, the realization scheme R is defined as follows: in a contribution problem $\mathcal{C}$, for each profile $x \in X$ of contributions the set $R(x)$ of realized projects equals the set of projects used to maximal capacity by each maximal flow in , (x). By Lemma 2.4 (ii), this is equivalent with stating that a project is realized if and only if it is contained in some minimum cut of , (x). Formally, for all $x \in \prod_{i \in N} X_i$,

$$R(x) = \bigcup_{(S,T) \in MC(x)} T.$$

Many of the proofs use the fact that for each $x \in X$ there exists a minimum cut (S, T) in , (x) such that $R(x) = T$. This result follows immediately from the next lemma.

Lemma 3.2 *Let $\mathcal{C}$ be a contribution problem, $x \in X$ a profile of contributions, and , (x) the corresponding flow network. If $C_1 = (S_1, T_1)$ and $C_2 = (S_2, T_2)$ are minimum cuts, then $C_3 = (S_1 \cap S_2, T_1 \cup T_2)$ and $C_4 = (S_1 \cup S_2, T_1 \cap T_2)$ are also minimum cuts.*

Proof. See the proof of Lemma 2.6. □

The realization scheme R shares uses personalized contributions, i.e. each individual contribution x_i is used for projects in m_i. No player is subsidizing others at the cost of the realization of his own plan. Furthermore, each player i can freely dispose of the projects in m_i; all the projects will be realized in case he is willing to pay for the corresponding aggregate cost. The next sections will also show that in equilibrium the players together act on behalf of the desires of the society of players by maximizing utilitarian welfare.

4 Contribution games are potential games

Monderer and Shapley (1996) introduced several classes of potential games. A common feature of these classes is the existence of a real-valued function, called a potential, on the strategy space that incorporates the strategic possibilities of all players simultaneously.

In this section, contribution games are shown to be ordinal potential games; one of the ordinal potential functions is the classical utilitarian collective welfare function, defined as the sum of the individual players' utility functions. See Moulin (1988) for a survey of this and other welfare functions.

Definition 4.1 A game $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ is an *ordinal potential game* if there exists a function $P : \prod_{i \in N} X_i \to \mathbb{R}$ such that for each player $i \in N$, each strategy combination $x_{-i} \in X_{-i}$ of his opponents, and each $x_i, y_i \in X_i$:

$$u_i(y_i, x_{-i}) \Leftrightarrow u_i(x_i, x_{-i}) > 0 \Leftrightarrow P(y_i, x_{-i}) \Leftrightarrow P(x_i, x_{-i}) > 0.$$

The function P is called an (*ordinal*) *potential* of the game G.

In other words, if P is an ordinal potential function of the game G, the sign of the change in the payoff to a unilaterally deviating player matches the sign of the change in the value of P. In particular, this implies that the information concerning best responses of every player is incorporated into this potential function. The following result is clear.

Proposition 4.2 *Let $G = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ be a game with ordinal potential function P. Then $x \in X$ is a Nash equilibrium of G if and only if x is a Nash equilibrium of* *the game* $(N, (X_i)_{i \in N}, (P)_{i \in N})$*, the coordination game obtained by replacing all payoff functions by the ordinal potential function* P*.*

Ordinal potential games are characterized in Voorneveld and Norde (1997) using the notion of *weak improvement cycles*. Notice that an ordinal potential game has infinitely many different ordinal potential functions. The next theorem shows that the utilitarian welfare function is an ordinal potential function of a contribution game.

Theorem 4.3 *Let* $G(\mathcal{C})$ *be a contribution game. The utilitarian welfare function* $U : X \to \mathbb{R}$ *defined by* $U = \sum_{i \in N} u_i$ *is an ordinal potential of* $G(\mathcal{C})$*.*

Proof. Let $i \in N$, $x_{-i} \in X_{-i}$, and $x_i, y_i \in X_i$. Assume without loss of generality that $x_i < y_i$. For notational convenience, write $x = (x_i, x_{-i})$ and $y = (y_i, x_{-i})$. Discern three cases:

Case (i): $m_i \not\subseteq R(y)$.
Since some arcs that correspond with projects in m_i are not a member of any minimum cut of the flow network , (y), it must be that $a_i \in C$ for every $C \in MC(y)$. By decreasing *cap*(i) from y_i to x_i the collection of minimum cuts does not change. So $R(y) = R(x)$. This implies $u_i(y) \Leftrightarrow u_i(x) = U(y) \Leftrightarrow U(x) = x_i \Leftrightarrow y_i$.

Case (ii): $m_i \subseteq R(x)$. By Lemma 3.2, there exists a minimum cut (S, T) in the flow network , (x) such that $T = R(x)$. Since $m_i \subseteq T$, $a_i \notin S$. By increasing *cap*(i) from x_i to y_i, (S, T) remains a minimum cut; no new minimum cuts appear, although some may disappear. So $R(y) \subseteq R(x)$. Because $T \subseteq R(y)$ it follows that $R(y) = R(x)$ and that $u_i(y) \Leftrightarrow u_i(x) = U(y) \Leftrightarrow U(x) = x_i \Leftrightarrow y_i$.

Case (iii): $m_i \not\subseteq R(x)$ and $m_i \subseteq R(y)$.
In this case, $u_i(y) \Leftrightarrow u_i(x) = \omega_i \Leftrightarrow y_i + x_i > 0$. When player i spends the amount x_i, $a_i \in C$ for every $C \in MC(x)$. Let $z_i \in (x_i, y_i]$ be the smallest contribution of player i for which a_i is no longer in every minimum cut of the flow network , (x_{-i}, z_i). Case 1 shows that $R(x) = R(x_{-i}, t)$ for every $t \in (x_i, z_i)$. Case 2 shows that $R(y) = R(x_{-i}, t)$ for every $t \in [z_i, y_i]$. By increasing *cap*(i) from x_i to z_i, no minimum cut disappears, whereas some minimum cuts will appear, at least one of them not containing a_i. Therefore $R(x)$ is a

proper subset of $R(x_{-i}, z_i) = R(y)$ and as a consequence

$$U(y) \Leftrightarrow U(x) = \sum_{\ell : m_\ell \subseteq R(y)} \omega_\ell \Leftrightarrow y_i \Leftrightarrow \sum_{\ell : m_\ell \subseteq R(x)} \omega_\ell + x_i \geq \omega_i \Leftrightarrow y_i + x_i > 0.$$

This concludes the proof. □

Consequently, a contribution game is best-response equivalent with a coordination game where the payoff functions of the players are replaced by the utilitarian welfare function U. This is a significant insight: even though the players play a noncooperative game, utilitarian social welfare enters their game in the sense that, given the strategy profile of the opponents, a player maximizes his payoff if and only if he maximizes utilitarian welfare. Therefore, it is of interest to investigate the relation between equilibria of the contribution game and strategies that maximize social welfare.

5 Equilibria of contribution games

The existence of Nash equilibria of contribution games is established in the first theorem of this section. Two properties of Nash equilibria are derived: no money is wasted in an equilibrium and if a player is not satisfied since not all of his projects are realized, then he contributes nothing. These two properties are used to establish the existence of strategy profiles that maximize utilitarian welfare in a contribution game. Utilitarian welfare maximizing strategy profiles are proven to be strong Nash equilibria: no coalition of players has an incentive to deviate from such a profile.

Theorem 5.1 *Each contribution game $G(\mathcal{C})$ has a Nash equilibrium.*

Proof. The proof is based on an algorithm which is shown to terminate in finitely many steps with a Nash equilibrium of the game. Initially, set $k = 0$ and $x^0 = 0$: each player contributes zero. The general step of the algorithm is as follows. After k iterations, we are given a strategy profile x^k such that

$$\sum_{i \in N} x_i^k = \sum_{j \in R(x^k)} c_j = \max \text{ flow}(, (x^k)) = \min \text{ cut}(, (x^k)), \tag{5.1}$$

$$\{i \in N \mid x_i^k > 0, m_i \not\subseteq R(x^k)\} = \emptyset, \tag{5.2}$$

$$\{i \in N \mid m_i \subseteq R(x^{k-1})\} \subset \{i \in N \mid m_i \subseteq R(x^k)\} \text{ if } k \geq 1. \tag{5.3}$$

The profile $x^0 = 0$ trivially satisfies these conditions. Next, define

$$\begin{aligned} C^k &= \{i \in N \mid x_i^k > 0, m_i \subseteq R(x^k)\}, \\ F^k &= \{i \in N \mid x_i^k = 0, m_i \subseteq R(x^k)\}, \\ N^k &= \{i \in N \mid x_i^k = 0, m_i \not\subseteq R(x^k)\}. \end{aligned}$$

The algorithm stops after k iterations if $N^k = \emptyset$ or if $N^k \neq \emptyset$ and x^k is a Nash equilibrium of $G(\mathcal{C})$. If the algorithm does not stop, some player $i(k+1) \in N$ can improve by unilaterally changing his contribution. We claim that $i(k+1) \in N^k$. To prove this claim, notice that by (5.2) N is the union of the pairwise disjoint sets C^k, F^k, and N^k.

Clearly, $i(k+1) \notin F^k$, since players $i \in F^k$ achieve their payoff maximum ω_i by contributing nothing and therefore cannot possibly increase their payoff.

To show that $i(k+1) \notin C^k$, consider $h \in C^k$. By definition $x_h^k > 0$ and $m_h \subseteq R(x^k)$. Player h cannot benefit from increasing his contribution: Lemma 3.2 implies that there exists a minimum cut (S,T) in , (x^k) such that $T = R(x^k)$. Hence $a_h \notin S$. If h increases his contribution, (S,T) will remain a minimum cut. Hence h's projects will still be realized, but he contributes more, which decreases his payoff. Player h also cannot benefit from decreasing his contribution: Property (5.1) implies that each maximal flow f in , (x^k) uses every arc a_i with $i \in N$ to full capacity x_i^k and each arc a_j with $j \in R(x^k)$ to full capacity c_j. If player h decreases his contribution, say to λx_h^k with $\lambda \in [0,1)$, a maximal flow f' in the new flow network can be constructed as follows:

$$\text{For } i \in N: \quad f'(a_i) = \begin{cases} \lambda f(a_i) & \text{if } i = h, \\ f(a_i) & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i: \quad f'(a_{ij}) = \begin{cases} \lambda f(a_{ij}) & \text{if } i = h, j \in m_h, \\ f(a_{ij}) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M: \quad f'(a_j) = \sum_{i \in N : j \in m_i} f'(a_{ij}).$$

If $j \in m_h$ is such that $f(a_{ij}) > 0$, then $f'(a_j) < f(a_j) = c_j$, so j is not used to full capacity by the maximal flow f' in the new flow network: not all projects in m_h are used to full capacity by every maximal flow, so player h will lose his reward ω_h if he decreases his contribution, thus decreasing his payoff from $\omega_h \Leftrightarrow x_h^k > 0$ to something nonpositive, namely $\Leftrightarrow \lambda x_h^k$.

Consequently, $i(k+1) \in N^k$, which implies $x_{i(k+1)}^k = 0$. The fact that he can improve, means that $\sum_{j \in m_{i(k+1)} \setminus R(x^k)} c_j < \omega_{i(k+1)}$: he can pay the costs necessary to finance that part of his projects that is not realized in , (x^k). Set

$$x_i^{k+1} = \begin{cases} x_i^k & \text{if} \quad i \neq i(k+1), \\ \displaystyle\sum_{j \in m_i \setminus R(x^k)} c_j & \text{if} \quad i = i(k+1). \end{cases}$$

Notice that a maximal flow f in , (x^k) can easily be extended to a maximal flow in , (x^{k+1}) by adding a flow via player $i(k+1)$ that pays exactly for his projects in $m_{i(k+1)} \setminus R(x^k)$. Since such an extended maximal flow exactly finances the projects in $R(x^k) \cup m_{i(k+1)}$ and no others, it follows that $R(x^{k+1}) = R(x^k) \cup m_{i(k+1)}$. Increasing k by one, this also means that the input again satisfies (5.1) – (5.3), so that the general step can be repeated.

Two things remain to be shown: that the algorithm ends and that, if it ends after k iterations, x^k is indeed a Nash equilibrium of the game.

By construction, the algorithm ends after k iterations if $N^k \neq \emptyset$ and x^k is a Nash equilibrium, or if $N^k = \emptyset$. If $N^k = \emptyset$, x^k must be a Nash equilibrium, since it was shown above that any player that could benefit from unilateral deviation had to be a member of N^k. By (5.3), $|N^k| < |N^{k-1}|$ if $k \geq 1$, so the algorithm terminates. □

Now that the existence of Nash equilibria in contribution games has been established, it becomes interesting to study their properties. The next proposition makes clear that players whose project sets are not completely realized do not contribute anything in an equilibrium. Moreover, no money is wasted: in an equilibrium, the contributions of the players exactly suffice to pay for the realized projects.

Proposition 5.2 *Let $G(\mathcal{C})$ be a contribution game and $x \in \mathrm{NE}(G(\mathcal{C}))$.*

(i) *Let $i \in N$. If $m_i \not\subseteq R(x)$, then $x_i = 0$.*

(ii) $\sum_{j \in R(x)} c_j = \min \text{cut}(, (x)) = \max \text{flow}(, (x)) = x(N)$.

Proof. (i) Assume $m_i \not\subseteq R(x)$ and suppose that $x_i > 0$. By definition of R, $m_i \not\subseteq R(x)$ implies that there is no minimum cut (S, T) in $, (x)$ such that $m_i \subseteq T$. Hence, a_i is in each minimum cut of $, (x)$. Reducing i's contribution slightly does not change the set of minimum cuts and thus increases i's payoff, contradicting $x \in \text{NE}(G(\mathcal{C}))$. Hence $m_i \not\subseteq R(x)$ implies $x_i = 0$.
(ii). Obviously

$$\sum_{j \in R(x)} c_j \leq \min \text{cut}(, (x)) = \max \text{flow}(, (x)) \leq x(N).$$

By Lemma 3.2, $, (x)$ has a minimum cut (S, T) such that $R(x) = T$. If $m_i \subseteq T = R(x)$ and $x_i > 0$, then $a_i \notin S$. If $m_i \not\subseteq T = R(x)$, then $x_i = 0$ by Proposition 5.2(i). Hence S contains no arcs a_i with $cap(i) = x_i > 0$. Thus

$$\sum_{j \in R(x)} c_j = \min \text{cut}(, (x)) = \max \text{flow}(, (x)).$$

Suppose that

$$\sum_{j \in R(x)} c_j = \min \text{cut}(, (x)) = \max \text{flow}(, (x)) < \sum_{i \in N} x_i.$$

Then there exists an $i \in N$ with $x_i > 0$ such that a_i is not used to full capacity in some maximal flow in $, (x)$. According to Lemma 2.4, a_i is in no minimum cut. Then i can reduce his contribution slightly without affecting the set of minimum cuts and thus increase his payoff, contradicting $x \in \text{NE}(G(\mathcal{C}))$. □

The next proposition shows that a strategy profile maximizing utilitarian welfare U exists in each contribution game $G(\mathcal{C})$. Notice that $\arg\max_{x \in X} U(x)$ is a subset of $\text{NE}(G(\mathcal{C}))$; otherwise, some player could increase his payoff by deviating, but then the ordinal potential U would increase as well.

Proposition 5.3 *Let $G(\mathcal{C})$ be a contribution game and $U = \sum_{i \in N} u_i$. Then the utilitarian welfare function achieves its maximum:*

$$\text{argmax}_{x \in X} U(x) \neq \emptyset.$$

Proof. Observe that $\text{argmax}_{x \in X} U(x) = \text{argmax}\{U(x) \mid x \in \text{NE}(G(\mathcal{C}))\}$. Let $x \in \text{NE}(G(\mathcal{C}))$. By Proposition 5.2(ii):

$$\begin{aligned} U(x) := \sum_{i \in N} u_i(x) &= \sum_{i \in N : m_i \subseteq R(x)} \omega_i - \sum_{i \in N} x_i \\ &= \sum_{i \in N : m_i \subseteq R(x)} \omega_i - \sum_{j \in R(x)} c_j. \end{aligned}$$

There are finitely many projects, so the collection $\{R(x) \mid x \in \text{NE}(G(\mathcal{C}))\} \subseteq 2^M$ has finitely many elements. This implies that $\{U(x) \mid x \in \text{NE}(G(\mathcal{C}))\}$ also has finitely many elements. Consequently, the maximum is attained in this set, i.e. $\arg\max_{x \in X} U(x) \neq \emptyset$. $\square$

In a potential game, the collection of strategy profiles at which there is a potential achieving its maximum is called the potential maximizer. The potential maximizer is suggested as an equilibrium refinement tool by Monderer and Shapley (1996) and Peleg, Potters, and Tijs (1996) . In ordinal potential games, different potentials give rise to different maximizers (as opposed, for instance, to exact potential games). Hence the collection of strategies maximizing utilitarian welfare in a contribution game may be a proper subset of the potential maximizer of the game.

Strong Nash equilibria have been defined in Aumann (1959). In a game $(N, (X_i)_{i \in N}, (u_i)_{i \in N})$, a strategy combination $x \in X$ is a *strong Nash equilibrium* if for every $\emptyset \neq S \subseteq N$ and every $y_S \in X_S$ there exists an $i \in S$ such that $u_i(x) \geq u_i(x_{N \setminus S}, y_S)$. In other words, $x \in X$ is a strong Nash equilibrium if there is no coalition $\emptyset \neq S \subseteq N$ of players and no alternative strategy $y_i \in X_i \setminus \{x_i\}$ for the members $i \in S$ such that $u_i(x_{N \setminus S}, y_S) > u_i(x)$ for each player $i \in S$.

Strong Nash equilibria were defined to be strategy profiles for which there is no coalition of players that can deviate and make each of its members strictly better off. A slightly weaker definition would be to require that there is no coalition of players that can deviate and make each of its members not worse off and at least one of its members better off. In contribution games, however, the two definitions are equivalent, since each payoff function u_i satisfies

$$\forall x, y \in X : x_i \neq y_i \Rightarrow u_i(x) \neq u_i(y).$$

The set of strong Nash equilibria of a game G is denoted $\mathrm{SNE}(G)$. Although the set of Nash equilibria is nonempty in a wide class of noncooperative games, existence of strong Nash equilibria is much rarer. Existence of strong Nash equilibria in contribution games is established in the next theorem by showing that a strategy profile maximizing utilitarian welfare is a strong Nash equilibrium.

Theorem 5.4 *Let $G(\mathcal{C})$ be a contribution game and $U = \sum_{i \in N} u_i$. Then* $\underset{x \in X}{\operatorname{argmax}}\, U(x) \subseteq \mathrm{SNE}(G(\mathcal{C}))$*. Hence* $\mathrm{SNE}(G(\mathcal{C})) \neq \emptyset$*.*

Proof. As soon as the inclusion is established, existence of strong Nash equilibria follows from Proposition 5.3. Let $x \in \arg\max_{t \in X} U(t)$. Then individual players cannot profitably deviate from x, since $x \in \mathrm{NE}(G(\mathcal{C}))$. The entire player set N cannot profitably deviate from x, since x maximizes $\sum_{i \in N} u_i$. Suppose that $x \notin \mathrm{SNE}(G(\mathcal{C}))$. Then there exists a coalition $S \subset N$ with $1 < |S| < |N|$ and strategies $y_i \in X_i \backslash \{x_i\}$ for each $i \in S$ such that $u_i(x_{N \backslash S}, y_S) > u_i(x)$ for each $i \in S$. For notational convenience, define $y = (x_{N \backslash S}, y_S)$. Below it is shown that there is a strategy profile $z \in X$ such that $U(z) > U(x)$, contradicting $x \in \arg\max_{t \in X} U(t)$.

A player $i \in N$, in general, belongs to one of four types:

$$\begin{array}{lcl} \text{(type 1)} & x_i > 0 \quad , & m_i \subseteq R(x); \\ \text{(type 2)} & x_i = 0 \quad , & m_i \not\subseteq R(x); \\ \text{(type 3)} & x_i > 0 \quad , & m_i \not\subseteq R(x); \\ \text{(type 4)} & x_i = 0 \quad , & m_i \subseteq R(x). \end{array}$$

Since x is a Nash equilibrium, Proposition 5.2(i) implies that there are no players of the third type in , (x). If a player is of the fourth type, he achieves his payoff maximum ω_i without contributing: such players cannot belong to S. Hence, members of S are either of type 1 or of type 2.

Write $S = S_1 \cup S_2$ with $S_k = \{i \in S \mid i \text{ is of type } k\}, k = 1, 2$. The fact that the members of S deviate from x and improve their payoff implies

$$y_i < x_i \text{ and } m_i \subseteq R(y) \quad \text{if} \quad i \in S_1, \tag{5.4}$$

$$y_i > 0 \text{ and } m_i \subseteq R(y) \quad \text{if} \quad i \in S_2. \tag{5.5}$$

It is impossible that $S_2 = \emptyset$. To prove this, suppose, to the contrary, that all members of S are of type 1: $S = S_1$. By (5.4), there exists for each $i \in S = S_1$ a $\lambda_i \in [0, 1)$ such that $y_i = \lambda_i x_i$. Take any maximal flow f in , (x). Then a maximal flow f' in , (y) is obtained as follows:

$$\text{For } i \in N: \qquad f'(a_i) = \begin{cases} \lambda_i f(a_i) & \text{if } i \in S, \\ f(a_i) & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i: \quad f'(a_{ij}) = \begin{cases} \lambda_i f(a_{ij}) & \text{if } i \in S, j \in m_i, \\ f(a_{ij}) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M: \qquad f'(a_j) = \textstyle\sum_{i \in N : j \in m_i} f'(a_{ij}).$$

According to Proposition 5.2(ii), the contributions in , (x) are exactly sufficient to pay for the projects in $R(x)$. By definition of R: $f(a_j) = c_j$ for all $j \in R(x)$. If $i \in S$ pays part of the costs of $j \in m_i$ according to f, i.e., $f(a_{ij}) > 0$, then this flow decreases by a factor λ_i in , (y), so that $f'(a_j) < f(a_j) = c_j$. Hence a_j is not used to full capacity by the maximal flow f' in , (y), implying that $j \notin R(y)$. Then $m_i \not\subseteq R(y)$, contradicting (5.4). This completes the proof that $S_2 \neq \emptyset$.

Define $V = \{i \in N \mid x_i > 0\}$ and the nonempty project set $M' = \bigcup_{i \in S_2} m_i \backslash R(x) = \bigcup_{i \in S_2} m_i \backslash \bigcup_{i \in V} m_i$.

Let f be a maximal flow in , (x). By definition of R, every arc a_j with $j \in R(x)$ is used to full capacity c_j by f. By Proposition 5.2(ii):

$$\sum_{i \in N} f(a_i) = \sum_{i \in N} x_i = \sum_{j \in R(x)} c_j = \sum_{j \in R(x)} f(a_j).$$

Let g be a maximal flow in , (y). By (5.5), $M' \subseteq R(y)$. By definition of R, every arc a_j with $j \in M'$ is used to full capacity by g:

$$\text{for } j \in M' : c_j = g(a_j).$$

Since $M' = \bigcup_{i \in S_2} m_i \backslash \bigcup_{i \in V} m_i$, the flow in arc a_j with $j \in M'$ is generated entirely by members of S_2: for $j \in M'$ we have

$$c_j = g(a_j) = \sum_{i \in S_2 : j \in m_i} g(a_{ij}).$$

The total flow through the arcs a_j with $j \in M'$ then equals

$$\sum_{j \in M'} c_j = \sum_{j \in M'} \sum_{i \in S_2 : j \in m_i} g(a_{ij})$$

and is generated entirely by the members of S_2. Given flow g, an arbitrary player $i \in S_2$ pays $\sum_{j \in M' \cap m_i} g(a_{ij})$ for the projects in M' . Summing over the players in S_2 yields

$$\sum_{j \in M'} c_j = \sum_{i \in S_2} \sum_{j \in M' \cap m_i} g(a_{ij}) = \sum_{j \in M'} \sum_{i \in S_2 : j \in m_i} g(a_{ij}).$$

Define a strategy profile $z \in X$ as follows:

$$z_i = \begin{cases} \sum_{j \in M' \cap m_i} g(a_{ij}) & \text{if } i \in S_2, \\ x_i & \text{otherwise.} \end{cases}$$

Combine flows f and g to a feasible flow h in , (z) as follows:

$$\text{For } i \in N: \quad h(a_i) = \begin{cases} z_i & \text{if } i \in S_2, \\ f(a_i) = x_i = z_i & \text{otherwise.} \end{cases}$$

$$\text{For } i \in N, j \in m_i: \quad h(a_{ij}) = \begin{cases} g(a_{ij}) & \text{if } i \in S_2, j \in M', \\ f(a_{ij}) & \text{otherwise.} \end{cases}$$

$$\text{For } j \in M: \quad h(a_j) = \begin{cases} g(a_j) = c_j & \text{if } j \in M', \\ f(a_j) = c_j & \text{if } j \in R(x) \\ f(a_j) = 0 & \text{otherwise.} \end{cases}$$

Notice that

$$\begin{aligned} \sum_{i \in N} z_i &= \sum_{i \in S_2} z_i + \sum_{i \in N \backslash S_2} z_i \\ &= \sum_{i \in S_2} \sum_{j \in M' \cap m_i} g(a_{ij}) + \sum_{i \in N \backslash S_2} x_i \\ &= \sum_{j \in M'} c_j + \sum_{j \in R(x)} c_j = \sum_{j \in M} h(a_j). \end{aligned}$$

Thus, h is a maximal flow in , (z) and $\{a_j \mid j \in R(x) \cup M'\}$ is a minimum cut of , (z). Hence $R(z) = R(x) \cup M'$. But then $u_i(z) \geq u_i(x)$ for each $i \in N \backslash S_2$ and

$u_i(z) = \omega_i \Leftrightarrow z_i > 0 = u_i(x)$ for each $i \in S_2$, implying $U(z) > U(x)$. This contradicts $x \in \arg\max_{t \in X} U(t)$. Conclude that x is indeed a strong Nash equilibrium. $\square$

The converse inclusion of Theorem 5.4 holds as well. The set of realized projects is the same in each strong Nash equilibrium and, as a consequence, every strong Nash equilibrium maximizes utilitarian welfare.

Theorem 5.5 *Let $G(\mathcal{C})$ be a contribution game and $U = \sum_{i \in N} u_i$. If $x, y \in \mathrm{SNE}(G(\mathcal{C}))$, then $R(x) = R(y)$. Hence $\mathrm{SNE}(G(\mathcal{C})) \subseteq \mathrm{argmax}_{t \in X}\, U(t)$.*

Proof. As soon as the implication is established, the inclusion of the set of strong Nash equilibria in the set of maximizers of utilitarian welfare can be shown as follows: let $x \in \mathrm{SNE}(G(\mathcal{C}))$ and $y \in \mathrm{argmax}_{t \in X} U(t)$. Then $y \in \mathrm{SNE}(G(\mathcal{C}))$ by Theorem 5.4 and $R(x) = R(y)$ by the implication. Then Proposition 5.2(ii) implies

$$\begin{aligned}
U(x) &= \sum_{i \in N} u_i(x) = \sum_{i \in N : m_i \subseteq R(x)} \omega_i \Leftrightarrow \sum_{i \in N} x_i \\
&= \sum_{i \in N : m_i \subseteq R(x)} \omega_i \Leftrightarrow \sum_{j \in R(x)} c_j = \sum_{i \in N : m_i \subseteq R(y)} \omega_i \Leftrightarrow \sum_{j \in R(y)} c_j \\
&= \sum_{i \in N : m_i \subseteq R(y)} \omega_i \Leftrightarrow \sum_{i \in N} y_i = \sum_{i \in N} u_i(y) \\
&= U(y) = \max_{t \in X} U(t).
\end{aligned}$$

So $x \in \mathrm{argmax}_{t \in X} U(t)$, as was to be shown.

To show the implication, let $x, y \in \mathrm{SNE}(G(\mathcal{C}))$ and suppose $R(x) \neq R(y)$. Without loss of generality, $R(y) \backslash R(x) \neq \emptyset$. Below it is shown that the coalition

$$D = \{i \in N \mid y_i > 0, m_i \cap \{R(y) \backslash R(x)\} \neq \emptyset\}$$

can profitably deviate from x, contradicting $x \in \mathrm{SNE}(G(\mathcal{C}))$.

By Proposition 5.2(i): if $i \in D$, then $m_i \not\subseteq R(x)$, so $x_i = 0$ and $u_i(x) = 0$.

Let f be a maximal flow in , (y). By definition of R, every arc a_j with $j \in R(y) \backslash R(x)$ is used to full capacity c_j by f. Since this flow is generated entirely by the members of D, one finds

$$\forall j \in R(y) \backslash R(x) \ : \ c_j = \sum_{i \in D : j \in m_i} f(a_{ij}).$$

Player $i \in D$ contributes $\sum_{j \in m_i \cap \{R(y) \backslash R(x)\}} f(a_{ij})$ to the projects in $R(y) \backslash R(x)$ in the maximal flow f. Define $z \in X$ by

$$z_i = \begin{cases} x_i & \text{if } i \notin D, \\ \sum_{j \in m_i \cap \{R(y) \backslash R(x)\}} f(a_{ij}) & \text{if } i \in D. \end{cases}$$

It will be shown that this deviation from x by the members of D will guarantee the realization of $R(x) \cup R(y)$, which is an improvement for the members of D. Let g be a maximal flow in , (x). A flow h in , (z) is defined as follows.

1) For $i \in N$ we define put $h(a_i) = z_i$.

2) For $i \in N, j \in m_i$ let

$$h(a_{ij}) = \begin{cases} f(a_{ij}) & \text{if } i \in D, j \in R(y) \backslash R(x), \\ g(a_{ij}) = 0 & \text{if } i \in D, j \notin R(y) \backslash R(x), \\ g(a_{ij}) & \text{otherwise.} \end{cases}$$

3) For $j \in M$ let

$$h(a_j) = \begin{cases} g(a_j) = 0 & \text{if } j \in M \backslash \{R(x) \cup R(y)\}, \\ \displaystyle\sum_{i \in D : j \in m_i} h(a_{ij}) = c_j & \text{if } j \in R(y) \backslash R(x), \\ g(a_j) = c_j & \text{if } j \in R(x) \backslash R(y). \end{cases}$$

Notice that h is a maximal flow in , (z) and max flow(, (z)) $= \sum_{i \in N} z_i =$ $= \sum_{j \in R(x) \cup R(y)} c_j =$ min cut(, (z)). Hence $R(z) = R(x) \cup R(y)$. Then for each $i \in D$: $u_i(z) = \omega_i \Leftrightarrow z_i > 0 = u_i(x)$, contradicting $x \in \mathrm{SNE}(G(\mathcal{C}))$. □

Since by Theorem 5.5 each strong Nash equilibrium induces maximal utilitarian welfare, the corresponding profile of individual net-payoffs defines a pre-imputation of the realization game. Next we will show that each of these pre-imputations are core allocations of the realization game. To be more precise, there is an one-to-one correspondence between the set of all strong Nash equilibria and the subset of the core of the realization game where players with zero payoffs must be null players. In other words, the set of strong Nash equilibria naturally corresponds to the largest subset of the core that maximizes the number of players with positive rewards. Recall that $i \in N$ is a *null player* for a cooperative game (N, v) if for all $S \subseteq N \backslash \{i\}$ it holds $v(S \cup \{i\}) = v(S)$. So a null player i is a dummy player with $v(\{i\}) = 0$ (see e.g. Shapley (1953)). With the set of all null players of a game (N, v) denoted by $null(N, v)$, we have the following.

Theorem 5.6 *Let $\mathcal{R} = (N, M, m, \omega, c)$ be a realization problem. Let $(N, v_{\mathcal{R}})$ and $\mathcal{C}_{\mathcal{R}} = (N, (X_i)_{i \in N}, (u_i)_{i \in N})$ be the corresponding cooperative realization game and noncooperative contribution game respectively. Then there exists a one-to-one correspondence from the set of strong Nash-equilibria of $\mathcal{C}_{\mathcal{R}}$ to*

$$K := \{u \in core(N, v_{\mathcal{R}}) \mid u_i = 0 \textit{ implies } i \in null(N, v_{\mathcal{R}})\}.$$

Proof. Theorem 5.5 states that in each strong Nash-equilibrium the same subset of M is realized. Call this subset R. Define N_+ to be $\{i \in N \mid m_i \subseteq R\}$. Let $i \in N_+$. Then m_i is realized in any strong Nash-equilibrium y, so the reward ω_i of player i contributes to $U(y)$. Hence, $v_{\mathcal{R}}(N) > v_{\mathcal{R}}(N \backslash \{i\})$. Because of the convexity of $v_{\mathcal{R}}$, there exists a core-element u with $u_i = v_{\mathcal{R}}(N) - v_{\mathcal{R}}(N \backslash \{i\}) > 0$. Hence, $u_i > 0$ for every $i \in N_+$. If $i \notin N_+$, then ω_i does not contribute to $U(y)$ for any strong Nash-equilibrium and therefore $v_{\mathcal{R}}(N) = v_{\mathcal{R}}(N \backslash \{i\})$. So $u_i = 0$ for every core-element u. We conclude that $N_+ = \{i \in N \mid u_i > 0\}$ for any $u \in K$. Now let $u \in K$. We define $x(u)$ as follows:

$$x(u)_i = \begin{cases} \omega_i - u_i & \text{if } i \in N_+ \text{ (so } u_i > 0) \\ 0 & \text{if } i \notin N_+ \text{ (so } u_i = 0). \end{cases}$$

We prove that $x := x(u)$ is a strong Nash-equilibrium. Firstly we have to show that x_i is a strategy of player i, i.e. that $x_i \in [0, \omega_i)$. This is the case since for all $i \in N_+$, $u_i > 0$ and $u_i \leq v_{\mathcal{R}}(N) - v_{\mathcal{R}}(N \backslash \{i\}) \leq \omega_i$. We have:

$$\sum_{i \in N} x_i = \sum_{i \in N_+} \omega_i - \sum_{i \in N} u_i = \sum_{i \in N_+} \omega_i - v_{\mathcal{R}}(N) = \sum_{i \in N_+} \omega_i - (\sum_{i \in N_+} \omega_i - \sum_{j \in R} c_j) = \sum_{j \in R} c_j.$$

Hence, in order to prove that $U(x)$ is maximal (and thus, by Theorem 5.4, x is a strong Nash-equilibrium), we have to show that $R(x) = R$. That is, for every $j \in R$, there must exist a minimum cut in , (x) of which a_j is a member. We know a cut with capacity $\sum_{i \in N} x_i$ of which a_j is a member: take all arcs of the players i with $x_i = 0$ and all arcs of elements in R. Hence, it is sufficient to show that a minimum cut has (at least) capacity $\sum_{i \in N} x_i$. Let (S, Q) be a minimum cut (here $S \subseteq N$ and $Q \subseteq M$). Then $m_i \subseteq Q$ for all $i \in N \backslash S$. The total capacity of (S, Q) equals $\sum_{i \in S} x_i + \sum_{j \in Q} c_j$. We have:

$$\sum_{i \in S} x_i + \sum_{j \in Q} c_j \geq \sum_{i \in S} x_i + \sum_{j \in \bigcup_{i \in N \backslash S} m_i} c_j \geq \sum_{i \in S} x_i + \sum_{i \in N \backslash S} \omega_i \Leftrightarrow v_{\mathcal{R}}(N \backslash S) \geq$$

$$\geq \sum_{i \in S} x_i + \sum_{i \in N \backslash S} (\omega_i \Leftrightarrow u_i) \geq \sum_{i \in S} x_i + \sum_{i \in N \backslash S} x_i = \sum_{i \in N} x_i.$$

The first inequality holds because (S, Q) is a cut, the second one follows from the definition of $v_{\mathcal{R}}$ and the third one follows from the assumption that u is a core element.

Now let x be a strong Nash-equilibrium of $\mathcal{C}_{\mathcal{R}}$. Then $u = (u_i(x))_{i \in N}$ can be considered as an allocation of $v_{\mathcal{R}}$. Because, by Theorems 5.4 and 5.5, U takes its maximum at x, the allocation u is efficient, i.e. $\sum_{i \in N} u_i = v_{\mathcal{R}}(N)$ $(= \max_{y \in X} U(y))$.

Let $S \subseteq N$. In order to show that $u \in core(N, v_{\mathcal{R}})$, we must prove that $\sum_{i \in S} u_i \geq v_{\mathcal{R}}(S)$. Because, $u \geq 0$, we can assume that $v_{\mathcal{R}}(S) > 0$.

Let $S_+ \subseteq S$ be a *smallest* subcoalition of S such that $v_{\mathcal{R}}(S) = v_{\mathcal{R}}(S_+)$. Then: $v_{\mathcal{R}}(S) = \sum_{i \in S_+} \omega_i \Leftrightarrow \sum_{j \in Q} c_j$, in which $Q = \bigcup_{i \in S_+} m_i$. We prove that $Q \subseteq R$. Let $i \in S_+$ (since $v_{\mathcal{R}}(S_+) > 0$, $S_+ \neq \emptyset$). Then $v_{\mathcal{R}}(S_+) \Leftrightarrow v_{\mathcal{R}}(S_+ \backslash \{i\}) > 0$. By convexity, we get $v_{\mathcal{R}}(N) \Leftrightarrow v_{\mathcal{R}}(N \backslash \{i\}) > 0$. Hence, the grand coalition strictly benefits from the fact that i is one of its members, so ω_i contributes to the value of N. Therefore, $m_i \subseteq R$.

Because in an equilibrium no money is wasted and a coalition only pays for projects it needs, we have that $\sum_{i \in S_+} x_i \leq \sum_{j \in Q} c_j$.

Hence:

$$\sum_{i \in S} u_i \geq \sum_{i \in S_+} u_i = \sum_{i \in S_+} (\omega_i \Leftrightarrow x_i) \geq \sum_{i \in S_+} \omega_i \Leftrightarrow \sum_{j \in Q} c_j = v_{\mathcal{R}}(S_+) = v_{\mathcal{R}}(S).$$

We conclude that $u \in core(N, v_{\mathcal{R}})$. To show that $u \in K$, consider a player $i \in N$ with $u_i = 0$. Then $m_i \nsubseteq R$. Hence $v_{\mathcal{R}}(N) = v_{\mathcal{R}}(N \backslash \{i\})$. This gives that player i is a null player.

To prove the one-to-one correspondence, one has to prove that, (i), for each $y \in K$ we have: $u(x(y)) = y$ and , (ii), for each strong Nash-equilibrium y, we have: $x(u(y)) = y$. These two statements are straightforward, since $\{i \in N \mid u_i > 0\}$ =$\{i \in N \mid m_i \subseteq R(y)\}$ for every $u \in K$ and every strong Nash-equilibrium y. □

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