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Surjective Function Theorems

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Surjective Function Theorems[1]

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Abstract: Given the unit sphere S^n, we prove the following theorem and several extensions: For any continuous function $f : S^n \mapsto S^n$, if f has no fixed point in S^n, then f must be surjective. Furthermore, there exists $x^* \in S^n$ such that $f(x^*) = -f(-x^*)$.

Keywords: Surjective function, fixed point, antifixed point, antipodal point, sphere, manifold.

1 Main Results

A function $f : D \mapsto I$ is said to be *surjective* (or f is said to map D *onto* I) if every element of I is the image of some element of D under the function f, i.e., $f(D) = I$. f is said to have *a fixed point* (*an antipodal point*) in D if there exists $x \in D$ such that $f(x) = x$ $(f(x) = -f(-x))$. In this **Note** we present several surjective function theorems which are not only interesting on their own but also fundamental. Our approach is a topological one. Now we introduce some notation: Let $n \geq 2$ denote any integer number, $\mathbb{R}^n$ the n-dimensional Euclidean space, and $x \cdot y = \sum_i x_i y_i$ the inner product of vectors x and y. We write $x \in \mathbb{R}^{n+1}$ by $x = (x_0, x_1, \cdots, x_n)$ or $x = (x_0, x_1, \cdots, x_n)^\top$. Furthermore, define $B^{n+1} = \{x \in \mathbb{R}^{n+1} \mid x \cdot x \leq 1\}$ (i.e. the $(n+1)$-dimensional unit ball), $B^n = \{x \in B^{n+1} \mid x_0 = 0\}$, $S^n = \{x \in \mathbb{R}^{n+1} \mid x \cdot x = 1\}$ (i.e. the n-dimensional unit sphere), $S^{n-1} = \{x \in S^n \mid x_0 = 0\}$, $e_0 = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, and $\mathbf{0} = (0, \cdots, 0) \in \mathbb{R}^{n+1}$.

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Theorem 1.1 *For any continuous function $f : S^n \mapsto S^n$, if f has no fixed point in S^n, then f must be surjective. Furthermore, there exists $x^* \in S^n$ such that $f(x^*) = -f(-x^*)$.*

Proof: Suppose to the contrary that f is not surjective. Then there would exist some $y^* \in S^n$ which is not in $f(S^n)$. Without loss of generality, we may assume that $y^* = e_0$. Thus, it follows from the continuity of f on the compact set S^n that there exists a positive δ such that $f_0(x) \leq 1 - \delta$ for all $x \in S^n$. Let $C = \{x \in S^n \mid x_0 \leq 1 - \delta\}$ and $C^b = \{x \in S^n \mid x_0 = 1 - \delta\}$. We have that $f(S^n) \subset C$. Define the function $g : C \mapsto B^n$ by

$$g(x) = \frac{1}{2-\delta}\sqrt{\frac{1+x_0}{1-x_0}}(0, x_1, \cdots, x_n)^\top.$$

Its inverse $h = g^{-1} : B^n \mapsto C$ is given by for $x \neq 0$

$$h_i(x) = \begin{cases} (2-\delta)\sqrt{x \cdot x} - 1, & if \quad i = 0; \\ (2-\delta)x_i\sqrt{\frac{2-(2-\delta)\sqrt{x \cdot x}}{(2-\delta)\sqrt{x \cdot x}}}, & if \quad i = 1, \cdots, n, \end{cases}$$

with $h(0) = -e_0$. It is easy to see that $h(x)$ converges to $-e_0$ as x goes to 0. Thus, both g and h are continuous functions. It may be difficult to figure out how g is constructed. Geometrically, we can visualize the idea in the sphere S^2 being imagined as the surface of the earth. Given any point P on the arctic circle C^b, there is a unique longitude line passing through P which links both the north pole e_0 and the south pole $-e_0$. This longitude intersects the equator line S^{n-1} uniquely at one point, say Q. Clearly, the section of the longitude line between P and the south pole is homeomorphic to the straight line between the core $\mathbf{0}$ and Q. Function g maps P to Q and the south pole to the core.

Consequently, we obtain a continuous function $g \circ f \circ h$ mapping from the convex and compact set B^n into itself. By using Brouwer's fixed point theorem, we know there exists $z^* \in B^n$ such that $z^* = g \circ f \circ h(z^*)$. Setting $x^* = h(z^*)$, we obtain $x^* = f(x^*) \in C$. This contradicts the hypothesis that f has no fixed point in S^n.

Now we prove the last part. Since f has no fixed point, it follows from Corollary 4 (c) of Whittlesey [3] or Milnor [2] that f must have an antipodal point. We complete the proof. □

In term of equation theory, the above theorem states that for each $y \in S^n$, the equation $f(x) = y$ has a solution.

Corollary 1.2 *For any continuous function $f : S^n \mapsto S^n$, if $x \cdot f(x) \leq 0$ for every $x \in S^n$, then f must be surjective.*

Proof: Suppose to the contrary that f has a fixed point $x \in S^n$. Then $0 < x \cdot x = x \cdot f(x) \leq 0$ which is impossible. □

The above corollary is not so simple as it might appear. In fact we will show that it is at least as powerful as the classic Brouwer's fixed point theorem is. We prove Brouwer theorem via Corollary 1.2: If $f : B^n \mapsto B^n$ is a continuous function, then there exists $x \in B^n$ such that $f(x^*) = x^*$. Suppose to the contrary that Brouwer theorem is false. Then it holds $f(x) - x \neq 0$ for every $x \in B^n$. Let $g(x) = f(x) - x$. Then

$$x \cdot g(x) = x \cdot (f(x) - x) = x \cdot f(x) - x \cdot x = x \cdot f(x) - 1 < 0$$

for all $x \in B^n \cap S^n = S^{n-1}$. We define the function $h : S^n \mapsto S^n$ as follows: For $x \in S^n$, let $y(x) = (0, x_1, \cdots, x_n)^\top$. Obviously, $y(x) \in B^n$. If $y(x) \cdot g(y(x)) \leq 0$, define

$$h(x) = \frac{g(y(x))}{\sqrt{g(y(x)) \cdot g(y(x))}}.$$

If $y(x) \cdot g(y(x)) > 0$, it is easy to see that $\sum_{i=1}^n x_i^2 < 1$ and $x_0 \neq 0$. Define

$$z_0 = -\frac{y(x) \cdot g(y(x))}{x_0}, \quad z = (z_0, 0, \cdots, 0)^\top \in \mathbb{R}^{n+1},$$

and

$$h(x) = \frac{g(y(x)) + z}{\sqrt{(g(y(x)) + z) \cdot (g(y(x)) + z)}}.$$

Since $y(x) \cdot g(y(x))$ is a continuous function in x on a compact set, then there exists some $\delta > 0$ such that

$$y(x) \cdot g(y(x)) \leq -\delta$$

for all $x \in S^{n-1}$. Furthermore, there exists a positive ϵ such that $y(x) \cdot g(y(x)) \geq 0$ implies $| x_0 | \geq \epsilon$. Now it is readily verified that h is a continuous function and $x \cdot h(x) \leq 0$ for all

$x \in S^n$. So all conditions of Corollary 1.2 are met. Then h must be surjective. But it is impossible since there does not exist any $x \in S^n$ such that $h(x) = e_0$. □

A function $f : D \mapsto I$ is said to have *an antifixed point* in D if there exists $x \in D$ such that $f(x) = -x$.

Corollary 1.3 *For any continuous function $f : S^n \mapsto S^n$, if f has no antifixed point in S^n, then f must be surjective. Furthermore, there exists $x^* \in S^n$ such that $f(x^*) = -f(-x^*)$.*

Proof: Let $g(x) = -f(x)$ for all $x \in S^n$. Clearly, g is continuous on S^n, $g(S^n) \subset S^n$, and g has no fixed point in S^n. It follows from Theorem 1.1 that g is surjective and hence so is f. □

Corollary 1.4 *For any continuous function $f : S^n \mapsto S^n$, if $x \cdot f(x) \geq 0$ for every $x \in S^n$, then f must be surjective.*

Proof: Suppose to the contrary that f has an antifixed point $y \in S^n$. Then $0 > -y \cdot y = f(y) \cdot y \geq 0$ which is impossible. By Corollary 1.3 f is surjective. □

The coming result is Theorem 3 of Whittlesey [3]. Here we give an alternative proof which is conceptually simpler than Whittlesey's.

Theorem 1.5 *For any continuous function $f : S^n \mapsto S^n$, if $f(x) = -f(-x)$ for every $x \in S^n$, then f must be surjective.*

Proof: Suppose to the contrary that f is not surjective. Then there would exist some $y^* \in S^n$ which is not in $f(S^n)$. Without loss of generality, we may assume that $y^* = e_0$. By the assumption we know that there does not exist any $x \in S^n$ such that $f(x) = -e_0$, either. Now we define the function $g : S^n \mapsto B^n$ by

$$g_i(x) = \begin{cases} 0, & if \quad i = 0; \\ \frac{f_i(x)}{\sqrt{\sum_{j=1}^n f_j^2(x)}}, & if \quad i = 1, \cdots, n. \end{cases}$$

It follows from the assumption that $g(x) = -g(-x)$ for every $x \in S^n$. Clearly, g is a continuous function. This is impossible according to Borsuk-Ulam theorem (see [1, 4]) which says that if $l : S^n \mapsto B^n$ is a continuous function, then there exists $x \in S^n$ such that $l(x) = l(-x)$. □

It will be shown that Theorem 1.5 is actually equivalent to Borsuk-Ulam theorem. Suppose that the latter theorem is false. Let $g(x) = f(x) - f(-x)$. Then $g(x) \neq 0$ and $g(x) = -g(-x)$ for all $x \in S^n$. Define the function $h : S^n \mapsto S^n$ by

$$h(x) = \frac{g(x)}{\sqrt{g(x) \cdot g(x)}}.$$

Clearly, all conditions of Theorem 1.5 are satisfied. Then h must be surjective. But it is impossible since $h_0(x) = 0$ for all $x \in S^n$. □

Now we proceed to prove one more result. Define

$$S^n_+ = \{x \in S^n \mid x_0 > 0\} \text{ and } S^n_- = \{x \in S^n \mid x_0 < 0\}.$$

Theorem 1.6 *For any continuous function $f : S^n \mapsto S^n$, if it satisfies that*

$$f_0(0, x_1, \cdots, x_n) = 0$$

$$f_i(0, x_1, \cdots, x_n) = -f_i(0, -x_1, \cdots, -x_n), \; i = 1, \cdots, n$$

for all $(0, x_1, \cdots, x_n) \in S^{n-1}$, then either $S^n_+ \subset f(S^n)$ or $S^n_- \subset f(S^n)$ or both.

Proof: Suppose to the contrary that there exist two points $\bar{y} = (\bar{y}_0, \bar{y}_1, \cdots, \bar{y}_n) \in S^n_+$ and $\tilde{y} = (\tilde{y}_0, \tilde{y}_1, \cdots, \tilde{y}_n) \in S^n_-$ neither of which belongs to $f(S^n)$. We define $h : B^n \mapsto S^n$ by

$$h(0, x_1, \cdots, x_n) = (-\sqrt{1 - x \cdot x}, x_1, \cdots, x_n);$$

$h^{-1} : S^n \mapsto B^n$ by

$$h^{-1}(x_0, x_1, \cdots, x_n) = (0, x_1, \cdots, x_n);$$

$\bar{g} : B^n \setminus \{h^{-1}(\bar{y})\} \mapsto S^{n-1}$ with $\bar{g}(x)$ equal to the intersection point of the straight line going through x and $h^{-1}(\bar{y})$ on S^{n-1};

$\widehat{g} : B^n \setminus \{h^{-1}(\tilde{y})\} \mapsto S^{n-1}$ with $\widehat{g}(x)$ equal to the intersection point of the straight line going through x and $h^{-1}(\tilde{y})$ on S^{n-1}.

It is easy to verify that h, h^{-1}, $\bar{g}$, and $\widehat{g}$ are continuous functions. Now we construct the function $F : B^n \mapsto S^{n-1}$ by

$$F(0, x_1, \cdots, x_n) = \begin{cases} \bar{g} \circ h^{-1} \circ f \circ h(0, x_1, \cdots, x_n), & if \quad f_0 \circ h(0, x_1, \cdots, x_n) > 0, \\ \widehat{g} \circ h^{-1} \circ f \circ h(0, x_1, \cdots, x_n), & if \quad f_0 \circ h(0, x_1, \cdots, x_n) < 0, \\ f \circ h(0, x_1, \cdots, x_n), & if \quad f_0 \circ h(0, x_1, \cdots, x_n) = 0. \end{cases}$$

Note that $h^{-1}(x) = \bar{g}(x) = \widehat{g}(x) = x$ for all $x \in S^{n-1}$. Then for any $x \in B^n$ with $f_0 \circ h(x) = 0$, it holds that

$$\bar{g} \circ h^{-1} \circ f \circ h(x) = \widehat{g} \circ h^{-1} \circ f \circ h(x) = f \circ h(x).$$

From the continuity of h, h^{-1}, $\bar{g}$, and $\widehat{g}$, F is a continuous function and furthermore it satisfies that

$$F(0, x_1, \cdots, x_n) = f \circ h(0, x_1, \cdots, x_n) = f(0, x_1, \cdots, x_n) = \\ -f(0, -x_1, \cdots, -x_n) = -F(0, -x_1, \cdots, -x_n)$$

for all $(0, x_1, \cdots, x_n) \in S^{n-1}$. This contradicts an equivalent form of Borsuk-Ulam theorem which says that there does not exist any continuous function $l : B^n \mapsto S^{n-1}$ such that $l(x) = -l(-x)$ for every $x \in S^{n-1}$. □

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