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# Core Concepts for Share Vectors

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**Abstract**

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities* –or simply a TU-game. A *value mapping* for TU-games is a mapping that assigns to every game a set of vectors each representing a distribution of the payoffs. A value mapping is *efficient* if to every game it assigns a set of vectors which components all sum up to the worth that can be obtained by all players cooperating together.

An approach to efficiently allocating the worth of the 'grand coalition' is using *share mappings* which assign to every game a set of *share vectors* being vectors which components sum up to one such that every component is the corresponding players' share in the total payoff that is to be distributed among the players. In this paper we discuss a class of share mappings containing the *(Shapley) share-core*, the *Banzhaf share-core* and the *Large Banzhaf share-core*. We provide characterizations of this class of share mappings and show how they are related to the corresponding *share functions* being functions that assign to every TU-game exactly one share vector.

**Keywords:** TU-Game, Share vector, Core, Reduced Game

# 1 Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a *cooperative game with transferable utilities*, or simply a TU-game, being a pair $(N, v)$, where $N = \{1, \ldots, n\}$ is a finite set of players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function* on $N$ such that $v(\emptyset) = 0$. For any coalition $E \subset N$, $v(E)$ is the worth of coalition $E$, i.e. the members of coalition $E$ can obtain a total payoff of $v(E)$ by agreeing to cooperate. We denote the collection of all TU-games by $\mathcal{G}$.

A solution of an $n$-person TU-game is an $n$-dimensional vector representing a distribution of payoffs. A *value function* on a subset $\mathcal{C}$ of $\mathcal{G}$ is a function that assigns a solution to any game in $\mathcal{C}$. A value function $f$ is efficient on $\mathcal{C}$ if for any game in $\mathcal{C}$ the total payoff it assigns to the players is equal to the worth $v(N)$ of the 'grand coalition', i.e. if $\sum_{i \in N} f_i(N, v) = v(N)$ for all $(N, v) \in \mathcal{C}$. An example of an efficient value function is the *Shapley value* (Shapley (1953)). An example of a value function that is not efficient is the *Banzhaf value* (Banzhaf (1965)) which is characterized in, e.g., Lehrer (1988) and Haller (1994). Since the Banzhaf value is not efficient it is not adequate in allocating the worth $v(N)$ of the 'grand coalition'. In order to allocate $v(N)$ according to the Banzhaf value van den Brink and van der Laan (1998a) characterized the *normalized Banzhaf value* which distributes the worth $v(N)$ proportional to the Banzhaf values of the players.

An alternative approach to efficiently allocating the worth $v(N)$ of the 'grand coalition' is the concept of *share functions* as introduced in van der Laan and van den Brink (1998). A *share vector* for game $(N, v) \in \mathcal{G}$ is an $n$-dimensional real vector $x \in \mathbb{R}^n$ such that $\sum_{i \in N} x_i = 1$. Here $x_i$ is player $i$'s share in the total payoff that is to be distributed among the players. A *share function* on $\mathcal{C} \subset \mathcal{G}$ is a function $\rho$ that assigns to every $(N, v) \in \mathcal{C}$ exactly one share vector $\rho(N, v) \in \mathcal{S}^n := \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1\}$.

For a game with $v(N) \neq 0$, the share vector of the game corresponding to the Shapley value of the game is the *Shapley share vector*, which is obtained by dividing the Shapley value of each player by the sum of the Shapley values of all players (being equal to $v(N)$ since the Shapley value is efficient). For a subset $\mathcal{C}$ of $\mathcal{G}$ such that $v(N) \neq 0$ for any $(N, v) \in \mathcal{C}$, the *Shapley share function* on $\mathcal{C}$ is defined to be the function assigning the Shapley share vector to each $(N, v) \in \mathcal{C}$. Similarly, the *Banzhaf share function* on $\mathcal{C}$ assigns to any game in the subset the *Banzhaf share vector*, which is obtained by dividing the Banzhaf value by the sum of payoffs over all players (or equivalently dividing the normalized Banzhaf value by $v(N)$).

In this paper we will apply the idea of share vectors to set-valued solution concepts for TU-games. A *value mapping* on a subset $\mathcal{C}$ of $\mathcal{G}$ is a mapping $M$ that assigns a set of solutions $M(N, v) \subset \mathbb{R}^n$ to any game in $(N, v) \in \mathcal{C}$. A value mapping $M$ is efficient if $\sum_{i \in N} y_i = v(N)$ for every $y \in M(N, v)$ and $(N, v) \in \mathcal{C}$. A well-known efficient value mapping is the *Core*-mapping which assigns to every TU-game $(N, v)$ its Core. Analogously to share functions, we can define a *share mapping* on $\mathcal{C} \subset \mathcal{G}$ being a mapping $M$ on $\mathcal{C}$ that assigns to every $(N, v) \in \mathcal{C}$ a set of share vectors $M(N, v) \subset \mathcal{S}^n$. Again, by definition a share mapping $M$ is efficient in the sense that all shares sum up to one for every share vector in $M(N, v)$. We define the *share-core mapping* on $\mathcal{G}$ by the mapping $C$ which assigns to a game $(N, v)$ satisfying $v(N) \neq 0$ the set of share vectors $x$ for which the vector $y$ given by $y_i = x_i v(N)$, $i \in N$ is in the Core of $(N, v)$.

The set $C(N, v)$ is called the *share-core* of $(N, v)$.

As is well-known, for so-called convex games the Core is not empty and the Shapley value is the barycenter of the Core (see Shapley (1971) and Ichiishi (1981)). Analogously the Shapley share vector is closely related to the share-core. In the following the share-core will therefore also be called the *Shapley share-core* and the share-core mapping the *Shapley share-core mapping*. We generalize the (Shapley) share-core mapping in a similar way as the Shapley value function is generalized in van der Laan and van den Brink (1998). In this way we obtain a class of share mappings which also contains the so-called *Banzhaf share-core mapping* and *Large Banzhaf share-core mapping*. We show that the Large Banzhaf share-core has certain appealing properties. In particular, each monotone game has a non-empty Large Banzhaf share-core and there exist monotone games for which it consists of exactly one element.

The class of share mappings will be characterized by applying a modified version of Davis and Maschler's reduced game property as used by Peleg (1986) in characterizing the Core[1]. Recall that another famous reduced game property is the Hart and Mas-Colell reduced game property as introduced in Hart and Mas-Colell (1988, 1989) in characterizing the Shapley value. In Dragan (1996) an alternative reduced game property is used for characterizing the Banzhaf value. A modification of these properties has been used in van den Brink and van der Laan (1999) to characterize a class of share functions containing the Shapley and Banzhaf share functions.

We conclude the paper by introducing the concept of *marginal share vectors* and show that the share vector induced by a share function in this class is equal to the average of the corresponding marginal share vectors. After generalizing the concept of convex games, we show that on this subset of games there is a one-to-one relation between the class of share mappings introduced in this paper and the class of share functions as given in van den Brink and van der Laan (1999), in the same way as the Shapley share vector is related to the Shapley share-core for convex games.

The paper is organized as follows. In Section 2 we state some preliminaries on TU-

[1] Tadenuma (1992) usues an alternative reduced game property in characterizing the Core. A general approach to characterizing the core using reduced game properties is given by Funaki and Yamamoto (1997).

games and briefly discuss the class of share functions as characterized by van der Laan and van den Brink (1998). In Section 3 we introduce the class of share mappings and give some properties of the corresponding solution sets of share vectors. In Section 4 we discuss some examples of share mappings in this class, such as the Shapley share-core mapping and the (Large) Banzhaf share-core mapping. We also show the usefulnes of the share-core mappings introduced by applying them to the special class of weighted majority voting games. In Section 5 we state the modified version of Davis and Maschler's reduced game and characterize the class of share mappings as the unique class of share mappings satisfying non-emptyness, and modified versions of the axioms of the Davis-Maschler reduced game property, superadditivity and individual rationality. Finally, in Section 6 we discuss marginal share vectors, use these in generalizing the concept of convex games, and show a relation between share mappings and corresponding share functions.

## 2 Preliminaries on TU-games and share functions

In this section we give some preliminary concepts and definitions on cooperative games. For given $N$ and nonempty $T \subset N$ the *unanimity game* $(N, u^T)$ is given by $u^T(E) = 1$ if $T \subset E$ and $u^T(E) = 0$ otherwise, $E \subset N$.[2] In the sequel we denote $|E|$ for the number of elements of the set $E$. From Harsanyi (1959) we know that the characteristic function $v$ of a game $(N, v)$ can be expressed as a linear combination of the characteristic functions of the unanimity games $(T, u^T)$, $T \subset N$, by $v = \sum_{T\subset N} \Delta_v(T)u^T$ with $\Delta_v(T)$ the *dividend* of coalition $T \subset N$ given by $\Delta_v(T) = \sum_{E\subset T}(-1)^{(|T|-|E|)}v(E)$.

A TU-game $(N, v)$ is called *monotone* if $v(E) \leq v(F)$ for all $E \subset F \subset N$ and it is called *convex* if for every pair $E,\ F \subset N$ it holds that $v(E\cup F)+v(E\cap F) \geq v(E)+v(F)$. Observe that any unanimity game is monotone and convex. For a given game $(N, v) \in \mathcal{G}$ and given $T \subset N$, the *restriction* of $(N, v)$ to $T$ is denoted by the *subgame* $(T, v_T)$ and is given by $v_T(E) = v(E)$ for all $E \subset T$. The class $\mathcal{C} \subset \mathcal{G}$ is called *subgame closed* if for every $(N, v) \in \mathcal{C}$ and every $T \subset N$ it holds that $(T, v_T) \in \mathcal{C}$. Examples of subgame

[2]Note that we ignore the unanimity games $(N, u^{\emptyset})$. In the paper, when we speak about unanimity games we mean unanimity games $(N, u^T)$ with $T \neq \emptyset$.

closed classes of games are the class of all games $\mathcal{G}$, the class of all monotone games, and the class of all convex games. Note that a class of games with a fixed player set is not subgame closed.

A game $(N,v)$ is called a *null game* if $v = v^0$ with $v^0(E) = 0$ for all $E \subset N$. Now, let $\mu: \mathcal{G} \to \mathbb{R}$ be a function assigning a real value to any game $(N,v) \in \mathcal{G}$. The function $\mu: \mathcal{G} \to \mathbb{R}$ is *positive* on $\mathcal{C} \subset \mathcal{G}$ if $\mu(N,v) > 0$ for all $(N,v) \in \mathcal{C}$, and it is called *zero* on $\mathcal{C} \subset \mathcal{G}$ if $\mu(N,v) = 0$ for all $(N,v) \in \mathcal{C}$. By $\mathcal{G}_\mu^+ \subset \mathcal{G}$, respectively $\mathcal{G}_\mu^0 \subset \mathcal{G}$, we denote the class of games on which $\mu$ is positive, respectively zero. Moreover, we define $\mathcal{G}_\mu = \mathcal{G}_\mu^+ \cup \mathcal{G}_\mu^0$, i.e. $\mu(N,v) \geq 0$ for all $(N,v) \in \mathcal{G}_\mu \subset \mathcal{G}$. We call a function $\mu: \mathcal{G} \to \mathbb{R}$ *additive* on $\mathcal{C}$ if for every pair of games $(N,v),(N,w) \in \mathcal{C}$ such that[3] $(N, v+w) \in \mathcal{C}$ it holds that $\mu(N,v+w) = \mu(N,v) + \mu(N,w)$. A function $\mu: \mathcal{G} \to \mathbb{R}$ is *linear* on $\mathcal{C}$ if it is additive on $\mathcal{C}$ and for every $(N,v) \in \mathcal{C}$ and $c \in \mathbb{R}$ such that $(N,cv) \in \mathcal{C}$ it holds that $\mu(N,cv) = c\mu(N,v)$. Finally, we call $\mu: \mathcal{G} \to \mathbb{R}$ *symmetric* on $\mathcal{C}$ if for every $(N,v) \in \mathcal{C}$, every pair of symmetric players[4] $i,j$ in $(N,v)$ and every $E \subset N,\ E \supset \{i,j\}$, such that the subgames $(E \setminus \{i\}, v_{E\setminus\{i\}})$ and $(E \setminus \{j\}, v_{E\setminus\{j\}})$ are in $\mathcal{C}$, it holds that $\mu(E \setminus \{i\}, v_{E\setminus\{i\}}) = \mu(E \setminus \{j\}, v_{E\setminus\{j\}})$.

We now recall some well-known value functions for cooperative games that are mentioned in the introduction. The Shapley value (Shapley (1953)) is the value function $Sh$ given by

$$Sh_i(N,v) = \sum_{\substack{E \subset N \\ E \ni i}} \frac{(|E|-1)!(n-|E|)!}{n!} m_E^i(N,v) \text{ for all } i \in N,$$

where $m_E^i(N,v) = v(E) - v(E \setminus \{i\})$ is the *marginal contribution* of player $i$ to coalition $E \subset N$ in $(N,v) \in \mathcal{G}$. As mentioned in the introduction the Shapley value is an efficient value function.

A value function that is not efficient is the Banzhaf value (Banzhaf (1965)) being

[3] For a pair of games $(N,v),(N,w) \in \mathcal{G}$ the game $(N,v+w)$ is given by $(v+w)(E) = v(E) + w(E)$ for all $E \subset N$.

[4] Players $i,j \in N$ are *symmetric* in $(N,v) \in \mathcal{G}$ if $v(E \setminus \{i\}) = v(E \setminus \{j\})$ for all $E \subset N$ with $E \supset \{i,j\}$.

the value function $\beta$ given by

$$\beta_i(N,v) = \frac{1}{2^{n-1}} \sum_{\substack{E \subset N \\ E \ni i}} m^i_E(N,v) \text{ for all } i \in N.$$

In order to efficiently allocate $v(N)$ according to the Banzhaf value the *normalized Banzhaf value* $\overline{\beta}$ given by

$$\overline{\beta}(N,v) = \frac{v(N)}{\sum_{j \in N} \beta_j(N,v)} \beta(N,v)$$

can be used. Thus, the normalized Banzhaf value allocates $v(N)$ proportional to the Banzhaf values of the players.

A general approach to efficiently allocating payoffs in TU-games is using share functions which are introduced in van der Laan and van den Brink (1998). A *share function* on a set of games $\mathcal{C} \subset \mathcal{G}$ is a function $\rho$ that assigns to every game $(N,v) \in \mathcal{C}$ an $n$-dimensional real vector $\rho(N,v) \in \mathbb{R}^n$ such that the shares assigned to the players sum up to one for every game in $\mathcal{C}$, i.e. $\sum_{i \in N} \rho_i(N,v) = 1$ for all $(N,v) \in \mathcal{C}$. The $i^{th}$ component is the share of player $i \in N$ in the value to be distributed, e.g., in $v(N)$. Three properties that can be satisfied by such share functions are the following[5].

The first two properties are similar to the null player and symmetry properties for value functions. The share function $\rho$ satisfies the *null player property* on $\mathcal{C}$ if for every $(N,v) \in \mathcal{C}$ and every *null player* [6] $i$ in $(N,v)$ it holds that $\rho_i(N,v) = 0$. Share function $\rho$ satisfies *symmetry* on $\mathcal{C}$ if for every $(N,v) \in \mathcal{C}$ and every pair $i,j$ of symmetric players in $(N,v)$ it holds that $\rho_i(N,v) = \rho_j(N,v)$. Finally, for some function $\mu: \mathcal{G} \to \mathbb{R}$, the share function $\rho$ satisfies $\mu$-additivity on $\mathcal{C}$ if for every pair of games $(N,v), (N,w) \in \mathcal{C}$ such that $(N, v+w) \in \mathcal{C}$ it holds that $\mu(N,v+w)\rho(N,v+w) = \mu(N,v)\rho(N,v) + \mu(N,w)\rho(N,w)$. This last property is a generalization of the additivity property which is obtained by taking $\mu(N,v) = 1$ for all $(N,v) \in \mathcal{G}$. Although additivity is a reasonable property of value functions it does not make sense for share functions. However, a

[5] In van der Laan and van den Brink (1998) *efficient shares* (meaning that the components of $\rho_i(N,v)$ sum up to one for all $(N,v) \in \mathcal{C}$ is taken as a fourth axiom. In this paper we have taken this into our definition of a share function.

[6] Player $i \in N$ is a *null player* in $(N,v) \in \mathcal{G}$ if $v(E) = v(E \setminus \{i\})$ for all $E \subset N$.

share function that satisfies $\mu$-additivity for an additive $\mu$-function satisfies some kind of weighted additivity property in the sense that the shares assigned to the sum game of two games is a convex combination of the shares assigned to the two separate games. This can easily be seen by rewriting $\mu$-addivity for an additive $\mu$-function as $\rho(N, v + w) = \frac{\mu(N,v)}{\mu(N,v)+\mu(N,w)}\rho(N, v) + \frac{\mu(N,w)}{\mu(N,v)+\mu(N,w)}\rho(N, w)$. So, $\mu$ determines the weights of the games in this convex combination. What weights are appropriate depends on the application we have in mind.

The following theorem[7] characterizes a class of share functions on subclasses of games $\mathcal{C} \subset \mathcal{G}$ containing all positively scaled unanimity games $(N, \alpha u^T)$, $T \subset N$, $\alpha > 0$, i.e. $\alpha u^T(E) = \alpha$ if $T \subset E$, and $\alpha u^T(E) = 0$ otherwise. Examples of classes of games that contain all positively scaled unanimity games are the class of all games $\mathcal{G}$, the class of all monotone games, and the class of all convex games.

**Theorem 2.1 (van der Laan and van den Brink (1998))**
(i) *Let $\mu: \mathcal{G} \to \mathbb{R}$ be positive and symmetric on a subclass $\mathcal{C} \subset \mathcal{G}$ that contains all positively scaled unanimity games. Then there exists a unique share function $\rho^\mu$ on $\mathcal{C}$ satisfying the null player property, symmetry and $\mu$-additivity if and only if $\mu$ is additive on $\mathcal{C}$.*
(ii) *For given positive vectors $\omega^n \in \mathbb{R}^n_+$, $n \in \mathbb{N}$, let the function $\mu: \mathcal{G} \to \mathbb{R}$ be defined by $\mu(N, v) = \sigma^{\omega^n}(N, v)$, where $\sigma^{\omega^n}: \mathcal{G} \to \mathbb{R}$ is given by*

$$\sigma^{\omega^n}(N, v) = \sum_{i \in N} \sum_{E \ni i} \omega^n_{|E|} m^i_E(N, v).$$

*Then the share function $\rho^{\omega^n}$ on $\mathcal{G}^+_{\sigma^{\omega^n}}$ given by*

$$\rho^{\omega^n}_i(N, v) = \frac{\sum_{\substack{E \subset N \\ E \ni i}} \omega^n_{|E|} m^i_E(N, v)}{\sigma^{\omega^n}(N, v)} \text{ for every } i \in N,$$

*is the unique share function satisfying the null player property, symmetry, and $\sigma^{\omega^n}$-additivity on $\mathcal{G}^+_{\sigma^{\omega^n}}$.*

[7] In van der Laan and van den Brink (1998) results are stated more general for classes of games with fixed player set.

The second part of the theorem shows that any choice of positive weights on the marginal contributions (with equal weights assigned to coalitions of equal size) defines a share function satisfying the null player property, symmetry and $\sigma^{\omega^n}$-additivity on $\mathcal{G}^+_{\sigma^{\omega^n}}$. Note that all functions $\sigma^{\omega^n}$ are positive on all positively scaled unanimity games.

Examples of $\mu$ functions defined by a vector $\omega^n$ of weights are the function $\mu^S: \mathcal{G} \to \mathbb{R}$ given by $\mu^S(N,v) = v(N)$ (with $\omega^n_t = \frac{(t-1)!(n-t)!}{n!}$, $t = 1, \ldots, n$) and $\mu^B: \mathcal{G} \to \mathbb{R}$ given by $\mu^B(N,v) = \frac{1}{2^{n-1}} \sum_{E \subset N} (2|E| - n) v(E)$ (with $\omega^n_t = 2^{-(n-1)}$ for $t = 1, \ldots, n$). In van der Laan and van den Brink (1998) it is shown that the unique share function satisfying the properties stated in Theorem 2.1 with $\mu = \mu^S$ is the Shapley share function $\rho^S$ given by

$$\rho^S_i(N,v) = \frac{Sh_i(N,v)}{v(N)} \text{ for all } i \in N,$$

on the class of games $(N,v) \in \mathcal{G}$ with $v(N) \neq 0$, and the unique share function satisfying these properties with $\mu = \mu^B$ is the Banzhaf share function $\rho^B$ given by

$$\rho^B_i(N,v) = \frac{\beta_i(N,v)}{\sum_{j \in N} \beta_j(N,v)} = \frac{\overline{\beta}_i(N,v)}{\sum_{j \in N} \overline{\beta}_j(N,v)} = \frac{2^{n-1}\beta_i(N,v)}{\sum_{E \subset N} \sum_{i \in E} m^i_E(N,v)} \text{ for all } i \in N$$

on the class of games $(N,v)$ for which $\sum_{j \in N} \beta_j(N,v) \neq 0$. For other examples closely related to the Deegan-Packel value (see Deegan and Packel 1979) and the $\tau$-value (see Tijs 1981), we refer to van der Laan and van den Brink (1998)

# 3 Share mappings

A well-known efficient value mapping is the *Core*-mapping given by

$$Core(N,v) = \left\{ x \in \mathbb{R}^n | \sum_{i \in N} x_i(N,v) = v(N);\ \sum_{i \in E} x_i(N,v) \geq v(E),\ \forall E \subset N \right\}.$$

Analogously to share functions we can define a *share mapping* on $\mathcal{C} \subset \mathcal{G}$ being a mapping $M$ on $\mathcal{C}$ that assigns to every $(N,v) \in \mathcal{C}$ a set of share vectors $M(N,v) \subset \mathcal{S}^n = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1\}$. We define the *share-core mapping* on $\mathcal{G}$ by the mapping $C$ given by

$$C(N,v) = \left\{ x \in \mathcal{S}^n | v(N) \sum_{i \in E} x_i \geq v(E) \text{ for all } E \subset N \right\}.$$

In the following $C(N,v)$ is called the *share-core*. Clearly, if $v(N) \neq 0$, we have that $x \in C(N,v)$ if and only the vector $y$ given by $y_i = x_i v(N)$ is an element of $Core(N,v)$. Observe that this relation may not hold when $v(N) = 0$. In that case the Core becomes the set $Core(N,v) = \{x \in \mathbb{R}^n | \ \sum_{i \in N} x_i(N,v) = 0; \ \sum_{i \in E} x_i(N,v) \geq v(E), \ \forall E \subset N\}$, whereas $C(N,v) = \emptyset$ if there exists some $E \subset N$ with $v(E) > 0$, and $C(N,v) = \mathcal{S}^n$ if $v(E) \leq 0$ for all $E \subset N$. Observe that in this case a vector $y \in Core(N,v)$ may contain positive and negative components, whereas $C(N,v)$ is either empty or is equal to $\mathcal{S}^n$. However, if $v(N) = 0$ we have that for any $x \in C(N,v)$, it holds that player $i$ gets share $x_i$ in the zero worth $v(N)$, i.e. independent of the share vector $x$ each player $i$ gets a payoff $y_i = x_i v(N) = 0$ if $v(N)$ is to be distributed. Clearly, if we distribute $v(N) = 0$, the shares assigned to the players do not matter. Thus, $C(N,v) = \mathcal{S}^n$ seems to be reasonable, although it does not need to correspond to $Core(N,v)$.

Next, we generalize the concept of the share-core mapping to the concept of $\mu$-share-core mappings for functions $\mu: \mathcal{G} \to \mathbb{R}$ discussed in the previous section.

**Definition 3.1** *Let a function $\mu: \mathcal{G} \to \mathbb{R}$ be given. Then the $\mu$-**share-core mapping** on $\mathcal{G}$ is the mapping $C_\mu$ on $\mathcal{G}$ given by*

$$C_\mu(N,v) = \left\{ x \in \mathcal{S}^n | \mu(N,v) \sum_{i \in E} x_i \geq \mu(E, v_E) \text{ for all } E \subset N \right\}, \quad (N,v) \in \mathcal{G}.$$

*The set $C_\mu(N,v)$ of share vectors $x \in \mathcal{S}^n$ is called the $\mu$-**share-core** of the $n$-player game $(N,v)$.*

The definition says that the $\mu$-share-core of a game $(N,v)$ with $\mu(N,v) \neq 0$, consists of all share vectors $x$ such that the total share of every coalition $E \subset N$ is at least as high as the ratio of the $\mu$-value of the subgame corresponding to $E$ over the $\mu$-value of the original game $(N,v)$. If $\mu(N,v) = 0$ then each share vector in $\mathcal{S}^n$ belongs to the $\mu$-share-core of $(N,v)$ if $\mu(E, v_E) \leq 0$ for all $E \subset N$, while the $\mu$-share-core is empty if there is an $E \subset N$ with $\mu(E, v_E) > 0$. For the Shapley $\mu$-function given by $\mu^S(N,v) = v(N)$ for all $(N,v) \in \mathcal{G}$, the corresponding $\mu$-share-core is the share-core $C(N,v)$. Therefore we also call $C_{\mu^S}(N,v)$ the *Shapley share-core* of $(N,v)$, and $C_{\mu^S}$ the *Shapley share-core mapping*. The $\mu$-share-core of a game $(N,v)$ can be obtained as the Shapley share-core of the game $(N, v^\mu)$ defined by $v^\mu(E) = \mu(E, v_E)$ for all $E \subset N$. From this it follows immediately that the $\mu$-share-core is convex.

**Corollary 3.2** *For given $\mu: \mathcal{G} \to \mathbb{R}$ and game $(N, v) \in \mathcal{G}$, the $\mu$-share-core $C_\mu(N, v)$ is convex.*

The next lemma shows that a share vector in $C_\mu(N, v)$ is nonnegative for every game $(N, v)$ with $\mu(N, v) > 0$ in a subgame closed subclass $\mathcal{C} \subset \mathcal{G}_\mu$.

**Lemma 3.3** *For given $\mu: \mathcal{G} \to \mathbb{R}$, let $\mathcal{C} \subset \mathcal{G}_\mu$ be subgame closed and let $(N, v) \in \mathcal{C}$ be such that $\mu(N, v) > 0$. Then for every share vector $x \in C_\mu(N, v)$ it holds that $x_i(N, v) \geq 0$ for all $i \in N$.*

PROOF: Since $\mathcal{C}$ is subgame closed, we have that $(E, v_E) \in \mathcal{C}$ for all $E \subset N$. Since $\mathcal{C} \subset \mathcal{G}_\mu$ it follows that $\mu(E, v_E) \geq 0$ for every $E \subset N$. Let $x \in C_\mu(N, v)$. Taking $E = \{i\}$ it follows from $\mu(N, v)x_i \geq \mu(\{i\}, v_{\{i\}}) \geq 0$ and $\mu(N, v) > 0$ that $x_i \geq 0$ for all $i \in N$. $\square$

The following lemma appears to be useful when comparing with each other the corresponding share-cores of two functions $\mu$ and $\mu'$ on $\mathcal{G}$.

**Lemma 3.4** *For given $\mu: \mathcal{G} \to \mathbb{R}$ and $\mu': \mathcal{G} \to \mathbb{R}$, let $\mathcal{C} \subset \mathcal{G}_\mu \cap \mathcal{G}_{\mu'}$ be subgame closed, $(N, v) \in \mathcal{C}$ with $\mu(N, v) > 0$ and $\alpha > 0$. Then it holds that*

(i) *$C_\mu(N, v) \subset C_{\mu'}(N, v)$ if $\mu'(N, v)\mu(E, v_E) \geq \mu(N, v)\mu'(E, v_E)$ for all $E \subset N$;*

(ii) *$C_\mu(N, v) = C_{\mu'}(N, v)$ if $\mu'(E, v_E) = \alpha\mu(E, v_E)$ for all $E \subset N$.*

PROOF: (i) First, since $\mathcal{C} \subset \mathcal{G}_\mu \cap \mathcal{G}_{\mu'}$ and $\mathcal{C}$ is subgame closed, we have that $\mu(E, v_E) \geq 0$ and $\mu'(E, v_E) \geq 0$ for all $E \subset N$. Second, suppose that $x \in C_\mu(N, v)$. Then, according to Lemma 3.3 we have that $x_i(N, v) \geq 0$ for all $i \in N$. When $\mu(E, v_E) = 0$ it follows from $\mu'(N, v)\mu(E, v_E) \geq \mu(N, v)\mu'(E, v_E)$ and $\mu(N, v) > 0$ that $\mu'(E, v_E) = 0$, and hence $\mu'(N, v)\sum_{i \in E} x_i \geq 0 = \mu'(E, v_E)$. When $\mu(E, v_E) > 0$, it follows from $\mu'(N, v)\mu(E, v_E) \geq \mu(N, v)\mu'(E, v_E)$ that $\mu'(N, v)\sum_{i \in E} x_i \geq \frac{\mu(N,v)\mu'(E,v_E)}{\mu(E,v_E)} \sum_{i \in E} x_i \geq$

$\mu'(E, v_E)$, since $\frac{\mu(N,v)}{\mu(E,v_E)} \sum_{i \in E} x_i \geq 1$. Hence $\mu'(N, v) \sum_{i \in E} x_i \geq \mu'(E, v_E)$ for every $E \subset N$ and thus $x \in C_{\mu'}(N, v)$.
(ii) Since $\mu'(E, v_E) = \alpha\mu(E, v_E)$ it holds that $\mu'(N, v)\mu(E, v_E) = \alpha\mu(N, v)\mu(E, v_E) = \mu(N, v)\mu'(E, v_E)$, $E \subset N$. Since $\mu(N, v) > 0$ and hence also $\mu'(N, v) = \alpha\mu(N, v) > 0$ it follows from (i) that $C_\mu(N, v) = C_{\mu'}(N, v)$. □

# 4 Some examples and properties of $\mu$-share-cores

In this section we discuss some examples of $\mu$-share-cores. We have already discussed the Shapley share-core (being equal to the share-core $C(N, v)$ as mentioned in the introduction) that is generated by the function $\mu^S$. An alternative is the *Banzhaf share-core* that is generated by $\mu^B$. Comparing these two we conclude that the Banzhaf share-core is not always contained in the Shapley share-core, nor the other way around as the following example shows.

**Example 4.1** Let $(N, v) \in \mathcal{G}$ be given by $N = \{1, 2, 3\}$ and $v = u^{\{1,2\}} + u^{\{1,2,3\}}$. Then for $E \subset N$ it holds that

$$\mu^B(E, v_E) = \begin{cases} 0 & \text{if } E \in \{\{1\}, \{2\}, \{3\}, \{1, 3\},\ \{2, 3\}\} \\ 1 & \text{if } E = \{1, 2\} \\ \frac{7}{4} & \text{if } E = \{1, 2, 3\}. \end{cases}$$

From this it follows that[8]

$$C_{\mu^B}(N, v) = Conv \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{4}{7} \\ 0 \\ \frac{3}{7} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{4}{7} \\ \frac{3}{7} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

With $\mu^S(E, v_E) = v(E)$ it follows that

$$C_{\mu^S}(N, v) = Conv \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

[8] By *Conv* $A$ we denote the convex hull of $A \subset \mathbb{R}^n$.

Thus the Banzhaf share-core $C_{\mu^B}(N,v)$ is a strict subset of the Shapley share-core $C_{\mu^S}(N,v)$.

Next, consider the game $(N,v) \in \mathcal{G}$ given by $N = \{1,2,3\}$ and $v = u^{\{1,2\}} + u^{\{1,3\}} - u^{\{1,2,3\}}$. For this game it holds that

$$\mu^B(E, v_E) = \begin{cases} 0 & \text{if } E \in \{\{1\},\{2\},\{3\},\{2,3\}\} \\ 1 & \text{if } E \in \{\{1,2\},\{1,3\}\} \\ \frac{5}{4} & \text{if } E = \{1,2,3\}, \end{cases}$$

and it follows that

$$C_{\mu^B}(N,v) = Conv\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ 0 \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} \frac{3}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} \frac{4}{5} \\ \frac{1}{5} \\ 0 \end{pmatrix} \right\}.$$

From $\mu^S(E, v_E) = v(E)$ for every $E \subset N$ it follows that

$$C_{\mu^S}(N,v) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Thus, $C_{\mu^S}(N,v)$ is a strict subset of $C_{\mu^B}(N,v)$.

In van der Laan and van den Brink (1998) the Banzhaf share function is characterized using $\mu^{\overline{B}}: \mathcal{G} \to \mathbb{R}$ given by $\mu^{\overline{B}}(N,v) = 2^{n-1}\mu^B(N,v) = \sum_{E \subset N}(2|E| - n)v(E)$. Although $\mu^B$ and $\mu^{\overline{B}}$ yield the same share function, they do not yield the same $\mu$-share-cores, as the following example shows.

**Example 4.2** Consider the first game $(N,v)$ of Example 4.1. We already determined $C_{\mu^B}(N,v)$. Further, with $\mu^{\overline{B}}(N,v) = 2^2\mu^B(N,v)$ it follows that

$$C_{\mu^{\overline{B}}}(N,v) = Conv\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3}{5} \\ \frac{2}{5} \end{pmatrix}, \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \\ 0 \end{pmatrix} \right\}.$$

Hence, $C_{\mu^{\overline{B}}}(N,v) \neq C_{\mu^B}(N,v)$.

For the game $(N, v)$ considered in Example 4.2 it holds that $C_{\mu^B}(N, v) \subset C_{\mu^{\overline{B}}}(N, v)$. The next theorem shows that this is always true for monotone games. Therefore we refer to the share-core generated by $\mu^{\overline{B}}$ as the *Large Banzhaf share-core*.

**Theorem 4.3** *If $(N, v) \in \mathcal{G}$ is monotone then $C_{\mu^B}(N, v) \subset C_{\mu^{\overline{B}}}(N, v)$.*

Proof: Let $(N, v) \in \mathcal{G}$ be monotone. If $v = v^0$ (i.e. $v^0(E) = 0$ for all $E \subset N$, see Section 2) then $\mu^B(E, v_E) = \mu^{\overline{B}}(E, v_E) = 0$ for all $E \subset N$, and hence by definition $C_{\mu^B}(N, v) = C_{\mu^{\overline{B}}}(N, v) = \mathcal{S}^n$. If $v \neq v^0$, then $\mu^B(N, v) > 0$ and $\mu^{\overline{B}}(N, v) > 0$. Moreover, for all $E \subset N$, we have that $\frac{\mu^{\overline{B}}(E, v_E)}{\mu^{\overline{B}}(N, v)} \leq \frac{\mu^{\overline{B}}(E, v_E)}{\mu^{\overline{B}}(N, v)} 2^{n-|E|} = \frac{\mu^B(E, v_E)}{\mu^B(N, v)}$. From part (i) of Lemma 3.4 it then follows that $C_{\mu^B}(N, v) \subset C_{\mu^{\overline{B}}}(N, v)$. $\square$

As shown in Example 4.1 for the Banzhaf share-core, also the Large Banzhaf share-core may contain elements not in the Shapley share-core and reversely. This is shown in the next example.

**Example 4.4** Let $(N, v) \in \mathcal{G}$ be given by $N = \{1, 2, 3\}$ and $v = u^{\{1,2\}} + u^{\{1,3\}}$. For every $E \subset N$ it holds that $\mu^S(E, v_E) = v(E)$ and

$$C_{\mu^S}(N, v) = Conv \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}.$$

Furthermore,

$$\mu^{\overline{B}}(E, v_E) = \begin{cases} 0 & \text{if } E \in \{\{1\}, \{2\}, \{3\}, \{2,3\}\} \\ 2 & \text{if } E \in \{\{1,2\}, \{1,3\}\} \\ 8 & \text{if } E = \{1,2,3\}, \end{cases}$$

and it follows that

$$C_{\mu^{\overline{B}}}(N, v) = Conv \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ 0 \\ \frac{3}{4} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{3}{4} \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 0 \end{pmatrix} \right\}.$$

Thus, $C_{\mu^S}(N, v)$ is a strict subset of $C_{\mu^{\overline{B}}}(N, v)$.

Next, consider the game $(N, v) \in \mathcal{G}$ given by $N = \{1, 2, 3, 4\}$ and

$$v(E) = \begin{cases} 0 & \text{if } |E| = 1 \\ 1 & \text{if } |E| = 2 \text{ or } E \subset \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\} \\ 2 & \text{if } E = \{1,2,3\} \\ 24 & \text{if } E = N. \end{cases}$$

From this it follows that

$$\mu^{\overline{B}}(E, v_E) = \begin{cases} 0 & \text{if } |E| = 1 \\ 2 & \text{if } |E| = 2 \\ 6 & \text{if } E \subset \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\} \\ 9 & \text{if } E = \{1,2,3\} \\ 130 & \text{if } E = N. \end{cases}$$

It is not difficult to verify that for this four player game the Large Banzhaf share-core $C_{\mu^{\overline{B}}}(N, v)$ is a strict subset of the Shapley share-core $C_{\mu^S}(N, v)$.

To give an example of a game for which the Large Banzhaf share-core is a strict subset of the Shapley share-core we need at least four players. The next theorem states that for games with at most three players and $v(E) \geq 0$ for all $E \subset N$, the Shapley share-core is always contained in the Large Banzhaf core.

**Theorem 4.5** *For every $(N, v) \in \mathcal{G}$ with $|N| \leq 3$ and $v(E) \geq 0$ for all $E \subset N$, it holds that $C_{\mu^S}(N, v) \subset C_{\mu^{\overline{B}}}(N, v)$.*

Proof: If $(N, v)$ is not monotone then there exist $E \subset F \subset N$ such that $v(E) > v(F) \geq 0$. Then $C_{\mu^S}(N, v) = \emptyset$ and thus $C_{\mu^S}(N, v) \subset C_{\mu^{\overline{B}}}(N, v)$. Next, suppose that $(N, v)$ is monotone. If $v = v^0$ then $C_{\mu^{\overline{B}}}(N, v) = C_{\mu^S}(N, v) = \mathcal{S}^n$. If $v \neq v^0$ then $\mu^S(N, v) > 0$ and we distinguish the following three cases:

(i) If $n = 1$ then $\mu^{\overline{B}}(N, v) = v(N) = \mu^S(N, v) > 0$. With part (i) of Lemma 3.4 it follows that the proposition is true in this case.

(ii) If $n = 2$ then $\mu^{\overline{B}}(N,v) = 2v(N) > 0$. The condition of part (i) of Lemma 3.4 clearly is satisfied for $E = N$. For $E \subset N$ with $|E| = 1$ we have that $\frac{\mu^{\overline{B}}(E,v_E)}{\mu^{\overline{B}}(N,v)} = \frac{\sum_{F\subset E}(2|F|-|E|)v(F)}{\sum_{F\subset N}(2|F|-n)v(F)} = \frac{v(E)}{2v(N)} \leq \frac{v(E)}{v(N)} = \frac{\mu^S(E,v_E)}{\mu^S(N,v)}$ and the condition is satisfied.

(iii) If $n = 3$ then $\mu^{\overline{B}}(N,v) = \sum_{F\subset N}(2|F|-n)v(F) = -\sum_{\substack{F\subset N\\|F|=1}} v(F) + \sum_{\substack{F\subset N\\|F|=2}} v(F)$ $+3v(N)$. Monotonicity of $(N,v)$ implies that $\sum_{\substack{F\subset N\\|F|=2}} v(F) - \sum_{\substack{F\subset N\\|F|=1}} v(F) \geq 0$. So, $\mu^{\overline{B}}(N,v) > 0$. Further, it follows for $E \subset N$ with $|E| = 1$ that $\frac{\mu^{\overline{B}}(E,v_E)}{\mu^{\overline{B}}(N,v)} = \frac{v(E)}{\mu^{\overline{B}}(N,v)} \leq \frac{v(E)}{v(N)} = \frac{\mu^S(E,v_E)}{\mu^S(N,v)}$, and for $E \subset N$ with $|E| = 2$ that $\frac{\mu^{\overline{B}}(E,v_E)}{\mu^{\overline{B}}(N,v)} = \frac{2v(E)}{\mu^{\overline{B}}(N,v)} \leq \frac{v(E)}{v(N)} = \frac{\mu^S(E,v_E)}{\mu^S(N,v)}$. Hence the condition of part (i) of Lemma 3.4 is satisfied. □

It is well-known that the Core, and hence also the Shapley share-core, can be empty, even for monotone games. However, for monotone games the Large Banzhaf share-core has the nice property of being not empty. To prove the next theorem, remark that a monotone game $(N,v)$ is convex if for every $E \subset F \subset N$ and every $i \in E$ it holds that $v(F) - v(F \setminus \{i\}) \geq v(E) - v(E \setminus \{i\})$.

**Theorem 4.6** *If $(N,v) \in \mathcal{G}$ is monotone it holds that $C_{\mu^{\overline{B}}}(N,v) \neq \emptyset$.*

Proof: If $v = v^0$ then $C_{\mu^{\overline{B}}}(N,v) = \mathcal{S}^n$. For $v \neq v^0$ (and thus $\mu^{\overline{B}}(N,v) > 0$) define the game $(N,w)$ by $w(E) = \frac{\mu^{\overline{B}}(E,v_E)}{\mu^{\overline{B}}(N,v)}$ for every $E \subset N$. Monotonicity of $(N,v)$ implies that for every $E \subset F \subset N$ and every $i \in E$ it holds that

$$\mu^{\overline{B}}(F,v_F) - \mu^{\overline{B}}(F\setminus\{i\}, v_{F\setminus\{i\}})$$

$$= \sum_{H\subset F}\sum_{j\in H}(v(H) - v(H\setminus\{j\})) - \sum_{H\subset F\setminus\{i\}}\sum_{j\in H}(v(H) - v(H\setminus\{j\}))$$

$$= \sum_{\substack{H\subset F\\H\ni i}}\sum_{j\in H} v(H) - v(H\setminus\{j\}))$$

$$\geq \sum_{\substack{H \subset E \\ H \ni i}} \sum_{j \in H} (v(H) - v(H \setminus \{j\})) = \mu^{\overline{B}}(E, v_E) - \mu^{\overline{B}}(E \setminus \{i\}, v_{E \setminus \{i\}}),$$

and hence also $w(F) - w(F \setminus \{i\}) \geq w(E) - w(E \setminus \{i\})$. Thus $(N, w)$ is a convex game and so $Core(N, w) \neq \emptyset$. Hence, by definition of the game $(N, w)$ it follows that

$$\begin{aligned}
C_{\mu^{\overline{B}}}(N, v) &= \{x \in \mathcal{S}^n \mid \mu^{\overline{B}}(N, w) \sum_{i \in E} x_i \geq \mu^{\overline{B}}(E, w_E),\ E \subset N\} \\
&= \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = 1 \text{ and } \sum_{i \in E} x_i \geq \frac{\mu^{\overline{B}}(E, v_E)}{\mu^{\overline{B}}(N, v)},\ E \subset N\} \\
&= \{y \in \mathbb{R}^n \mid \sum_{i \in N} y_i = w(N) \text{ and } \sum_{i \in E} y_i \geq w(E),\ E \subset N\} \\
&= Core(N, w) \neq \emptyset.
\end{aligned}$$

□

Having showed that the Large Banzhaf share core is not empty for monotone games, we now consider the question how 'small' the Large Banzhaf share-core can be on the class of monotone games. The next example shows that there exist monotone games for which the Large Banzhaf share-core contains exactly one element.

**Example 4.7** Consider the monotone game $(N, v)$ given by $v(E) = \alpha$, $E \subset N$, for some $\alpha > 0$. According to Theorem 4.6 the Large Banzhaf share-core $C_{\mu^{\overline{B}}}(N, v)$ is not empty. Suppose that $x \in C_{\mu^{\overline{B}}}(N, v)$. For every $E \subset N$ it holds that $\mu^{\overline{B}}(E, v_E) = \alpha|E|$, and thus $x$ must satisfy $\sum_{i \in E} x_i \geq \frac{\mu^{\overline{B}}(E, v_E)}{\mu^{\overline{B}}(N, v)} = \frac{|E|}{n}$. Considering all $E \subset N$ with $|E| = 1$ this yields that $x_i \geq \frac{1}{n}$ for all $i \in E$. Since $\sum_{i \in N} x_i = 1$ it must hold that $x_i = \frac{1}{n}$ for all $i \in E$. Hence, $C_{\mu^{\overline{B}}}(N, v)$ contains only one element.

We conclude this section by considering weighted majority voting games. A *weighted majority voting game* on $N$ is a game $(N, v) \in \mathcal{G}$ for which there exist numbers $s, s_1, \ldots, s_n \in \mathbb{N}$ such that $v(E) = 1$ if $\sum_{i \in E} s_i \geq s$, and $v(E) = 0$ otherwise, with $\frac{1}{2} \sum_{i \in N} s_i < s \leq \sum_{i \in N} s_i$. We call a coalition $E \subset N$ *winning* (respectively *losing*) if $v(E) = 1$ (respectively $v(E) = 0$). Clearly a weighted majority voting game is monotone and also *proper*, i.e. $v(N \setminus E) = 0$ if $v(E) = 1$. Moreover $v(N) = 1$ and so it is not a null game. It is well-known that the Core (and hence also the Shapley share-core) of a weighted mojority voting game is empty if the game does not have a *veto player*.[9] However, since any weighted majority voting game is monotone, it follows from Theorem 4.6 that the Large Banzhaf share-core is not empty on the class of these games. The Banzhaf share-core may be empty, but for non-emptyness the existence of a veto player is not required, as is shown in the next example.

**Example 4.8** Consider the weighted majority voting game $(N, v)$ on $N = \{1, 2, 3\}$ such that the winning coalitions are the ones that contain at least two players, i.e. $v(E) = 1$ if $|E| \geq 2$ and $v(E) = 0$ otherwise. Clearly, their is no veto player and so the Shapley share-core is empty. The Banzhaf and large Banzhaf $\mu$-functions are given by

$$\mu^B(E, v_E) = \begin{cases} 0 & \text{if } |E| = 1 \\ 1 & \text{if } |E| = 2 \\ \frac{3}{2} & \text{if } |E| = 3, \end{cases}$$

and $\mu^{\overline{B}}(E, v_E) = 2^{|E|-1}\mu^B(E, v_E)$. From this it follows that $C_{\mu^B}(N, v) = \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^\top\}$ and

$$C_{\mu^{\overline{B}}}(N, v) = Conv\left\{ \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} \right\}.$$

So, the Shapley share-core is empty, the large Banzhaf share-core is non-empty but quite large, and the Banzhaf share-core consists of a unique element. This last share-core seems very reasonable in this case.

# 5 Characterization of the $\mu$-share-core mapping

The traditional core for value vectors has been characterized by Peleg (1986) by using the concept of the *Davis-Maschler reduced game*. For a given game $(N, v) \in \mathcal{G}$ and

[9] A player $i$ is a veto player if $i \in E$ for all $E$ such that $v(E) = 1$.

payoff vector $y \in \mathbb{R}^n$, the DM-$y$-reduced game of the nonempty coalition $T \subset N$ is defined implicitly as the game $(T, v^{T,y})$ satisfying for every $E \subset T$

$$v^{T,y}(E) = \begin{cases} 0 & \text{if } E = \emptyset \\ v(N) - \sum_{i \in N \setminus E} y_i & \text{if } E = T \\ \max_{F \subset N \setminus T} \{v(E \cup F)) - \sum_{i \in F} y_i\} & \text{otherwise.} \end{cases}$$

To give a characterization of the $\mu$-share-core mapping for given function $\mu$ on $\mathcal{G}$ we first generalize this concept of the Davis-Maschler reduced game.

**Definition 5.1** *For given function $\mu: \mathcal{G} \to \mathbb{R}$, game $(N, v)$ in $\mathcal{G}$ and share vector $x \in \mathcal{S}^n$, the* **DM-$(\mu, x)$-reduced game** *of $(N, v) \in \mathcal{G}$ of the nonempty coalition $T \subset N$ is the game $(T, v^{T,\mu,x})$ that, for every $E \subset T$, satisfies*

$$\mu(E, v_E^{T,\mu,x}) = \begin{cases} 0 & \textit{if } E = \emptyset \\ \mu(N, v)\left(1 - \sum_{i \in N \setminus E} x_i\right) & \textit{if } E = T \\ \max_{F \subset N \setminus T} \left\{\mu(E \cup F, v_{(E \cup F)}) - \mu(N, v) \sum_{i \in F} x_i\right\} & \textit{otherwise.} \end{cases}$$

Since $x \in \mathcal{S}^n$, for $E = T$ the condition can be rewritten as

$$\mu(E, v_E^{T,\mu,x}) = \mu(T, v^{T,\mu,x}) = \mu(N, v) \sum_{i \in T} x_i. \tag{1}$$

From the definition above it follows straightforward that when the function $\mu$ is taken to be $\mu^S(N, v) = v(N)$, the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ for given share vector $x$ is equal to the standard Davis-Maschler reduced game $(T, v^{T,y})$ with the vector $y$ given by $y_i(N, v) = x_i v(N)$, $i \in N$. Although the Davis-Maschler reduced game is uniquely determined for every vector $y$ and every $T \subset N$, existence and uniqueness of the DM-$(\mu, x)$-reduced game is not guaranteed for arbitrary function $\mu$ on $\mathcal{G}$ and share vector $x$, as is illustrated in the following examples.

**Example 5.2** First, consider the game $(N, v) \in \mathcal{G}$ given by $N = \{1, 2, 3\}$, $v = u^{\{3\}} + u^{\{1,2\}}$ and take the share vector $x = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})^\top \in \mathcal{S}^n$. Let $\mu: \mathcal{G} \to \mathbb{R}$ be given by $\mu(N, v) = \sum_{i \in N} v(\{i\})$. So, $\mu(N, v) = 1$ and for $T = \{1, 2\}$, the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ must satisfy

$$\begin{aligned} \mu(\{i\}, v_{\{i\}}^{T,\mu,x}) &= \max\{\mu(\{i\}, v_{\{i\}}),\ \mu(\{i, 3\}, v_{\{i,3\}}) - \frac{1}{2}\} \\ &= \max\{0, 1 - \frac{1}{2}\} = \frac{1}{2},\ i = 1, 2 \end{aligned}$$

and $\mu(T, v^{\mu,T,x}) = \mu(N, v)\,(x_1 + x_2) = 1 \times \frac{1}{2} = \frac{1}{2}$.
Thus, it must hold that $\sum_{i\in\{1,2\}} v^{\mu,T,x}(\{i\}) = \mu(T, v^{\mu,T,x}) = \frac{1}{2}$ and $v^{\mu,T,x}(\{i\}) = \mu(\{i\}, v^{T,\mu,x}_{\{i\}}) = \frac{1}{2}$, $i = 1, 2$. Clearly, there is no game $(T, v^{T,\mu,x})$ that satisfies these conditions.

Next, consider the game $(N, v) \in \mathcal{G}$ given by $N = \{1, 2, 3\}$, $v = u^{\{3\}}$ and take again the share vector $x = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})^\top$. Now, let $\mu$ be given by $\mu(N, v) = \max_{E\subset N} v(E)$, so that $\mu(N, v) = 1$. For $T = \{1, 2\}$, the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ must satisfy

$$\begin{aligned}\mu(\{i\}, v^{\mu,T,x}_{\{i\}}) &= \max\{\mu(\{i\}, v_{\{i\}}),\ \mu(\{i, 3\}, v_{\{i,3\}}) - \frac{1}{2}\} \\ &= \max\{0, 1 - \frac{1}{2}\} = \frac{1}{2},\ i = 1, 2\end{aligned}$$

and $\mu(T, v^{\mu,T,x}) = \mu(N, v)(x_1 + x_2) = 1 \times \frac{1}{2} = \frac{1}{2}$.
Clearly, all games $(T, v)$ with $v(\{i\}) = \frac{1}{2}$, $i = 1, 2$ and $v(T) \leq \frac{1}{2}$ satisfy these conditions and hence the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ is not uniquely determined.

Thus, in general DM-$(\mu, x)$-reduced games need not exist nor be unique. However, it turns out that they are uniquely determined on $\mathcal{G}$ if $\mu$ is linear on $\mathcal{G}$ and positive for all unanimity games.

**Theorem 5.3** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear on $\mathcal{G}$ and positive for all unanimity games. Let $(N, v) \in \mathcal{G}$ be an $n$-player game and $x \in \mathcal{S}^n$. Then for every $T \subset N$, the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ exists and is uniquely determined.*

Proof: For given $(N, v) \in \mathcal{G}$, $T \subset N$, a share vector $x \in \mathcal{S}^n$ and a linear function $\mu$ on $\mathcal{G}$ being positive for all unanimity games, let $(T, v^{T,\mu,x}) \in \mathcal{G}$ be a DM-$(\mu, x)$-reduced game for the coalition $T$. To show the existence and uniqueness of $(T, v^{T,\mu,x})$, we prove that there exist unique dividends $\Delta_{v^{\mu,T,x}}(E)$, $E \subset T$, by induction on $|E|$.

First, suppose that $|E| = 1$. Then the linearity of $\mu$ implies that $\mu(E, v^{T,\mu,x}_E) = \Delta_{v^{T,\mu,x}_E}(E)\mu(E, u^E)$. Since $\mu(E, u^E) > 0$ by assumption, it holds that the dividend $\Delta_{v^{T,\mu,x}}(E) = \Delta_{v^{T,\mu,x}_E}(E) = \frac{\mu(E, v^{T,\mu,x}_E)}{\mu(E, u^E)}$ is uniquely determined. Proceeding by induction assume that for some given integer $k \geq 1$ and for any $H \subset T$ with $|H| \leq k$ we have determined the dividends $\Delta_{v^{T,\mu,x}}(H)$, and let $E \subset T$ be such that $|E| = k+1$. First, consider the case that $|E| < |T|$. Then by definition, the DM-$(\mu, x)$-reduced game $(T, v^{T,\mu,x})$ must satisfy

$$\mu(E, v_E^{T,\mu,x}) = \max_{F \subset N \setminus T} \left\{ \mu(E \cup F, v_{(E \cup F)}) - \mu(N, v) \sum_{i \in F} x_i \right\}.$$

Let the maximum be attained at $F^* \subset N \setminus T$. Then

$$\mu(E, v_E^{T,\mu,x}) = \mu(E \cup F^*, v_{(E \cup F^*)}) - \mu(N, v) \sum_{i \in F^*} x_i,$$

which is uniquely determined. Also, in case $|E| = |T|$ (and hence $E = T$), we have by definition that

$$\mu(E, v_E^{T,\mu,x}) = \mu(N, v) \sum_{i \in T} x_i$$

is uniquely determined. Linearity of $\mu$ implies that

$$\mu(E, v_E^{T,\mu,x}) = \sum_{H \subset E} \Delta_{v_E^{T,\mu,\rho}}(H) \mu(E, u^H),$$

and by $\mu(E, u^E) > 0$ it then holds that

$$\Delta_{v_E^{T,\mu,x}}(E) = \frac{\mu(E, v_E^{T,\mu,x}) - \sum_{\substack{H \subset E \\ H \neq E}} \Delta_{v_E^{T,\mu,x}}(H) \mu(E, u^H)}{\mu(E, u^E)}.$$

Since $\Delta_{v_E^{T,\mu,x}}(H) = \Delta_{v_H^{T,\mu,x}}(H)$ for all $H \subset E$, the induction hypothesis implies that $\Delta_{v_E^{T,\mu,x}}(E)$ is uniquely determined. Hence all the dividends $\Delta_{v^{T,\mu,x}}(E) = \Delta_{v_E^{T,\mu,x}}(E)$ exist and are uniquely determined and therefore it holds that all the values $v^{T,\mu,x}(E) = \sum_{H \subset E} \Delta_{v^{T,\mu,x}}(E) u^H(E)$ exist and are uniquely determined for all $E \subset T$, and so are $(T, v^{T,\mu,x})$ for all $T \subset N$. □

In the following we will give an axiomatic characterization of the $\mu$-share-core. To do so, we first introduce a modified version of the standard converse reduced game property in case $\mu$ is given by $\mu^S(N, v) = v(N)$ and prove that the $\mu$-share-core satisfies this so-called *converse DM-$\mu$-reduced game property* for share mappings. To state this property, for given vector $x \in \mathcal{S}^n$ and subset $H \subset N$, let the $|H|$-dimensional vector $x^H \in \mathcal{S}^{|H|}$ for every $j \in H$ be given by

$$x_j^H = \begin{cases} \frac{x_j}{\sum_{i \in H} x_i} & \text{if } \sum_{i \in H} x_i \neq 0, \\ \frac{1}{|H|} & \text{if } \sum_{i \in H} x_i = 0. \end{cases} \tag{2}$$

**Axiom 5.4 (Converse DM-$\mu$-reduced game property)** *For given $\mu: \mathcal{G} \to \mathbb{R}$, a share mapping $M$ on $\mathcal{G}$ satisfies the converse DM-$\mu$-reduced game property on a subset $\mathcal{C}$ of $\mathcal{G}$ if for every $(N, v) \in \mathcal{C}$ with $|N| \geq 2$ and for every nonnegative $x \in \mathcal{S}^n$ it holds that $x \in M(N, v)$ if $x^H \in M(H, v^{H,\mu,x})$ for every $H \subset N$ with $|H| = 2$.*

Although this property will not appear in the first characterization that we present in this section, the proof of that characterization makes use of the following lemma.

**Lemma 5.5** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games. Then the $\mu$-share-core mapping $C_\mu$ on $\mathcal{G}$ satisfies the converse DM-$\mu$-reduced game property on $\mathcal{G}$.*

PROOF: Since $\mu$ is linear and positive for all unanimity games, we have that all reduced games exist and are uniquely determined according to Theorem 5.3. Let $(N, v) \in \mathcal{G}$ with $|N| \geq 2$ and suppose that the nonnegative vector $x \in \mathcal{S}^n$ satisfies $x^H \in C_\mu(H, v^{H,\mu,x})$ for all $H \subset N$ with $|H| = 2$.

Since $x$ is nonnegative and $\sum_{i \in N} x_i = 1$, for every nonempty $T \subset N$, $T \neq N$, there exists some $i \in T$ and $h \in N \setminus T$ such that $x_i + x_h > 0$. Take such an $i$ and $h$ and set $H = \{i, h\}$. By assumption it then holds that $x^H \in C_\mu(H, v^{H,\mu,x})$ and thus $\mu(\{i\}, v_{\{i\}}^{H,\mu,x}) \leq \mu(H, v^{H,\mu,x}) x_i^H$. Since

$$\begin{aligned} \mu(\{i\}, v_{\{i\}}^{H,\mu,x}) &= \max_{F \subset N \setminus H} \left( \mu(\{i\} \cup F, v_{(\{i\} \cup F)}) - \mu(N, v) \sum_{k \in F} x_k \right) \\ &\geq \mu(T, v_T) - \mu(N, v) \sum_{k \in T \setminus \{i\}} x_k \end{aligned}$$

and $x \in \mathcal{S}^n$ it then holds that

$$\begin{aligned}
\mu(T, v_T) - \mu(N, v) \sum_{k \in T} x_k &= \mu(T, v_T) - \mu(N, v) \sum_{k \in T \setminus \{i\}} x_k - \mu(N, v) x_i \\
&\leq \mu(\{i\}, v_i^{H,\mu,x}) - \mu(N, v) x_i \\
&\leq \mu(H, v^{H,\mu,x}) x_i^H - \mu(N, v) x_i \\
&= \mu(H, v^{H,\mu,x}) \frac{x_i}{x_i + x_h} - \mu(N, v) x_i \\
&= \mu(N, v) x_i - \mu(N, v) x_i = 0.
\end{aligned}$$

Thus, $\mu(N, v) \sum_{j \in T} x_j \geq \mu(T, v_T)$ for all $T \subset N$, $T \neq N$. But then $x \in C_\mu(N, v)$. Thus, the $\mu$-share-core mapping $C_\mu$ satisfies the converse DM-$\mu$-reduced game property. $\square$

We now state axioms that characterize the $\mu$-share-core mapping. The first one is well-known.

**Axiom 5.6 (Non-emptyness)** *A share mapping $M$ on $\mathcal{G}$ satisfies non-emptyness on $\mathcal{C} \subset \mathcal{G}$ if for every $(N, v) \in \mathcal{C}$ it holds that $M(N, v) \neq \emptyset$.*

The next two axioms generalize familiar properties in case $\mu$ is given by $\mu^S(N, v) = v(N)$ for all $(N, v) \in \mathcal{G}$.

**Axiom 5.7 ($\mu$-Individual rationality)** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be given. A share mapping $M$ on $\mathcal{G}$ satisfies $\mu$-individual rationality on $\mathcal{C} \subset \mathcal{G}$ if for every $(N, v) \in \mathcal{C}$ with $\mu(N, v) \neq 0$ and for every $x \in M(N, v)$, it holds that $x_i \geq \frac{\mu(\{i\}, v_{\{i\}})}{\mu(N, v)}$ for all $i \in N$.*

To state the next axiom, for two sets $X, Y \subset \mathbb{R}^n$ and positive real numbers $a, b > 0$ we define the set $aX + bY \subset \mathbb{R}^n$ by $aX + bY = \{ax + by \mid x \in X,\ y \in Y\}$.

**Axiom 5.8 ($\mu$-Superadditivity)** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be given. A share mapping $M$ satisfies $\mu$-superadditivity on $\mathcal{C} \subset \mathcal{G}$ if for all pairs of games $(N, v), (N, w) \in \mathcal{C}$ such that $(N, v + w) \in \mathcal{C}$ it holds that $\mu(N, v) M(N, v) + \mu(N, w) M(N, w) \subset \mu(N, v + w) M(N, v + w)$.*

The next axiom states that if $x$ belongs to the solution set $M(N,v) \in \mathcal{S}^n$ of a game, then for any $H \subset N$, the $|H|$ dimensional vector $x^H$ as defined in equation (2) belongs to the solution set $M(H, v^{H,\mu,x})$ of the reduced game.

**Axiom 5.9 (DM-$\mu$-reduced game property)** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be given. A share mapping $M$ satisfies the DM-$\mu$-reduced game property on $\mathcal{C} \subset \mathcal{G}$ if for every $(N,v) \in \mathcal{C}$, $x \in M(N,v)$, and every $H \subset N$, $H \neq \emptyset$, it holds that $x^H \in M(H, v^{H,\mu,x})$.*

Next, following Peleg (1986) for $\mu^S(N,v) = v(N)$, we will characterize the $\mu$-share-core mapping on subgame closed subsets of the class of *$\mu$-balanced games*. For a function $\mu$ on $\mathcal{G}$, a game $(N,v) \in \mathcal{G}$ is called $\mu$-balanced if $C_\mu(N,v) \neq \emptyset$. Let $\mathcal{G}_\mu^B$ denote the collection of all $\mu$-balanced games $(N,v)$ in $\mathcal{G}_\mu$, i.e.

$$\mathcal{G}_\mu^B = \{(N,v) \in \mathcal{G} \mid \mu(N,v) \geq 0 \text{ and } C_\mu(N,v) \neq \emptyset\}.$$

The characterization follows from the next two lemma's.

**Lemma 5.10** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games. Then the $\mu$-share-core mapping $C_\mu$ satisfies non-emptyness on any subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu^B$, and $\mu$-individual rationality, the DM-$\mu$-reduced game property and $\mu$-superadditivity on any subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$.*

PROOF: Clearly, by definition $C_\mu$ satisfies non-emptyness since $\mathcal{C} \subset \mathcal{G}_\mu^B$. Moreover, by definition $C_\mu$ satisfies $\mu$-individual rationality on $\mathcal{C} \subset \mathcal{G}_\mu$. To show the DM-$\mu$-reduced game property on $\mathcal{C} \subset \mathcal{G}_\mu$, let $(N,v) \in \mathcal{G}_\mu^B$, $x \in C_\mu(N,v)$, and $H \subset N$, $H \neq \emptyset$. Since $\mu$ is linear and positive for all unanimity games, it follows from Theorem 5.3 that all reduced games $(H, v^{H,\mu,x})$, $H \subset N$, exist and are uniquely determined. We have to prove that $x^H \in C_\mu(H, v^{H,\mu,x})$ for all $H \subset N$. By definition of $x^H$ it holds that $\sum_{h \in H} x_h^H = 1$.

First, suppose that $\mu(N,v) = 0$. If there is an $E \subset N$ with $v(E) > 0$ then $C_\mu(N,v) = \emptyset$, and so the DM-$\mu$-reduced game property is satisfied in this case. Otherwise, $v(E) \leq 0$ for all $E \subset N$ and $C_\mu(N,v) = \mathcal{S}^n$. In that case it follows from Definition 5.1 that $\mu(H, v^{H,\mu,x}) = 0$ and $\mu(E, v^{H,\mu,x}) \leq 0$ for all $E \subset H$, $E \neq H$. But then $C_\mu(H, v^{H,\mu,x}) = \mathcal{S}^n = C_\mu(N, v)$. So, the DM-$\mu$-reduced game property is satisfied if $\mu(N, v) = 0$.

Now, suppose that $\mu(N, v) > 0$. Since $\mathcal{C}$ is subgame closed, according to Lemma 3.3 we have that $x_i \geq 0$, for all $i \in N$. Hence $x_h = 0$ for all $h \in H$ if $\sum_{h \in H} x_h = 0$. Since according to equation (1), $\mu(H, v^{H,\mu,x}) = \mu(N, v) \sum_{h \in H} x_h$ and $x \in \mathcal{S}^n$, for every $E \subset H,\ E \neq H$, it holds that

$$\begin{aligned}
\sum_{i \in E} \left(\mu(N, v)x_i - \mu(H, v^{H,\mu,x})x_i^H\right) &= \sum_{i \in E} \left(\mu(N, v)x_i - x_i^H \mu(N, v) \sum_{h \in H} x_h\right) \\
&= \mu(N, v) \left(\sum_{i \in E} \left(x_i - x_i^H \sum_{h \in H} x_h\right)\right).
\end{aligned}$$

When $\sum_{h \in H} x_h = 0$, then $x_h = 0$ for all $h \in H$ and hence $x_i = 0$ for all $i \in E$, so that $\mu(N, v) \left(\sum_{i \in E} \left(x_i - x_i^H \sum_{h \in H} x_h\right)\right) = 0$. When $\sum_{h \in H} x_h > 0$, then by definition $x_i^H \sum_{h \in H} x_h = x_i$ for all $i \in E \subset H$ and also in this case it follows that $\mu(N, v) \left(\sum_{i \in E} \left(x_i - x_i^H \sum_{h \in H} x_h\right)\right) = 0$. Hence, for every $E \subset H,\ E \neq H$, it holds that

$$\mu(N, v) \sum_{i \in E} x_i = \mu(H, v^{H,\mu,x}) \sum_{i \in E} x_i^H. \tag{3}$$

Thus, for every $E \subset H,\ E \neq H$, it holds that

$$\begin{aligned}
\mu(E, v_E^{H,\mu,x}) - \mu(H, v^{H,\mu,x}) \sum_{i \in E} x_i^H &= \mu(E, v_E^{H,\mu,x}) - \mu(N, v) \sum_{i \in E} x_i \\
&= \max_{F \subset N \setminus T} \left(\mu(E \cup F, v_{E \cup F}) - \mu(N, v) \sum_{i \in F} x_i\right) - \mu(N, v) \sum_{i \in E} x_i \\
&= \max_{F \subset N \setminus T} \left(\mu(E \cup F, v_{E \cup F}) - \mu(N, v) \sum_{i \in E \cup F} x_i\right) \leq 0,
\end{aligned}$$

since by assumption $x \in C_\mu(N, v)$ and hence $\mu(N, v) \sum_{i \in E \cup F} x_i \geq \mu(E \cup F, v_{E \cup F})$ for all $E \cup F \subset N$. Thus, $x^H \in C_\mu(H, v^{H,\mu,x})$.

To show $\mu$-superadditivity of $C_\mu$, let $(N, v), (N, w) \in \mathcal{C}$ be such that $(N, v + w) \in \mathcal{C} \subset \mathcal{G}_\mu$, and let $x \in C_\mu(N, v)$ and $x' \in C_\mu(N, w)$. For every $E \subset N$ it then holds that $\mu(N,v)\sum_{i\in E} x_i \geq \mu(E,v_E)$, $\mu(N,w)\sum_{i\in E} x'_i \geq \mu(E,w_E)$, and $\sum_{i\in N} x_i = \sum_{i\in N} x'_i = 1$. But then additivity of $\mu$ and the fact that $v_E + w_E = (v+w)_E$ yield $\mu(N,v)\sum_{i\in N} x_i + \mu(N,w)\sum_{i\in N} x'_i = \mu(N,v) + \mu(N,w) = \mu(N,v+w)$, and for every $E \subset N$, $\mu(N,v)\sum_{i\in E} x_i + \mu(N,w)\sum_{i\in E} x'_i \geq \mu(E,v_E) + \mu(E,w_E) = \mu(E,(v+w)_E)$, and thus $\mu(N,v)x + \mu(N,w)x' \in \mu(N,v+w)C_\mu(N,v+w)$. Thus, $C_\mu$ satisfies $\mu$-superadditivity on subgame closed subsets $\mathcal{C}$ of $\mathcal{G}_\mu$. $\square$

The second lemma states the reverse.

**Lemma 5.11** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games. Let $M$ be a share mapping satisfying non-emptyness on $\mathcal{G}^B_\mu$, and $\mu$-individual rationality, the DM-$\mu$-reduced game property and $\mu$-superadditivity on $\mathcal{G}$. Then it holds that $M(N,v) = C_\mu(N,v)$ for every $(N,v) \in \mathcal{G}^B_\mu$.*

PROOF: Since $\mu$ is linear and positive for all unanimity games, according to Theorem 5.3 all reduced games $(H, v^{H,\mu,x})$, $H \subset N$, exist and are uniquely determined. To prove that the mapping $M$ must be equal to $C_\mu$ on $\mathcal{G}^B_\mu$, we first consider the case $n = 1$. Then non-emptyness and $\mu$-individual rationality of $M$ yields that $M(N,v) = \{x \in \mathbb{R} \mid x = 1\} = C_\mu(N,v)$. Next, for $n \geq 2$ we prove that for all $(N,v) \in \mathcal{G}^B_\mu$ it holds that both $M(N,v) \subset C_\mu(N,v)$ and $C_\mu(N,v) \subset M(N,v)$, so that $M(N,v) = C_\mu(N,v)$.

First, we show that $M(N,v) \subset C_\mu(N,v)$. Let $(N,v) \in \mathcal{G}^B_\mu$ and $x \in M(N,v)$. If $n = 2$, then $\mu$-individual rationality implies that $\mu(N,v)x_i \geq \mu(\{i\}, v_{\{i\}})$, for $i \in N$. Since also $\sum_{i\in N} x_i = 1$ it follows that $x \in C_\mu(N,v)$.
If $n \geq 3$, then the DM-$\mu$-reduced game property of $M$ implies that $x^H \in M(H, v^{H,\mu,x})$ for all $H \subset N$. In particular this is true for $|H| = 2$. But then, as shown above for $n = 2$, it follows from the property of $\mu$-individual rationality that $x^H \in C_\mu(H, v^{H,\mu,x})$, for all $H \subset N$, $|H| = 2$ and thus $x \in C_\mu(N,v)$ according to Lemma 5.5. So, $M(N,v) \subset C_\mu(N,v)$ for all $(N,v) \in \mathcal{G}^B_\mu$.

Second, to prove that $C_\mu(N,v) \subset M(N,v)$, let $(N,v) \in \mathcal{G}^B_\mu$ and $x \in C_\mu(N,v)$. We first consider the case that $n \geq 3$. Therefore, construct the game $(N,w)$ by setting $\mu(\{i\}, w_{\{i\}}) = \mu(\{i\}, v_{\{i\}})$ for all $i \in N$, and $\mu(E,w_E) = \mu(N,v)\sum_{i\in E} x_i$ for every

$E \subset N$ with $|E| \geq 2$. Since $x \in \mathcal{S}^n$ it holds that $\mu(N,w) = \mu(N,v)$, and thus

$$C_\mu(N,w) = \{x' \in \mathcal{S}^n \mid \mu(N,w) \sum_{i\in E} x'_i \; geq \mu(E, w_E),\ E \subset N\}$$

$$= \{x' \in \mathcal{S}^n \mid \sum_{i\in E} x'_i \geq \sum_{i\in E} x_i \text{ if } |E| \geq 2 \text{ and } \mu(N,v)x'_i \geq \mu(\{i\}, v_{\{i\}}),\ \forall i \in N\}$$

$$= \{x\}.$$

Because $(N,v) \in \mathcal{G}^B_\mu$, we have that $\mu(N,w) = \mu(N,v) \geq 0$. Moreover $C_\mu(N,w) = \{x\}$ is not empty. Hence $(N,w) \in \mathcal{G}^B_\mu$ and thus by assumption $M(N,w)$ is not empty. However above we have shown that $M(N,w) \subset C_\mu(N,w) = \{x\}$. Hence $M(N,w) = C_\mu(N,w) = \{x\}$. Now, define the game $(N,z)$ by $z(E) = v(E) - w(E)$ for all $E \subset N$. Linearity of $\mu$ implies that $\mu(E, z_E) = \mu(E, v_E) - \mu(E, w_E)$ for all $E \subset N$. Hence, by construction of $(N,w)$ it follows that $\mu(\{i\}, z_{\{i\}}) = 0$ for all $i \in N$ and that $\mu(N,z) = \mu(N,v) - \mu(N,w) = 0$. Furthermore, since $x \in C_\mu(N,v)$, it holds that $\mu(E, z_E) = \mu(E, v_E) - \mu(N,v)\sum_{i\in E} x_i \leq 0$ for every $E \subset N$, $2 \leq |E| \leq n$. Hence $C_\mu(N,z) = \mathcal{S}^n$ is not empty. Since $\mu(N,z) = 0$, it follows that $(N,z) \in \mathcal{G}^B_\mu$ and hence $M(N,z) \subset \mathcal{S}^n$ is not empty. Since, $C_\mu(N,z) = \mathcal{S}^n$, it follows that $M(N,z) \subset C_\mu(N,z)$. Applying the $\mu$-superadditivity of $M$ on $v = z + w$ it follows with $\mu(N,z) = 0$ and $(M(N,w) = \{x\}$ that $\mu(N,w)\{x\} = \mu(N,z)M(N,z) + \mu(N,w)M(N,w) \subset \mu(N,v)M(N,v)$, and thus $x \in M(N,v)$. Hence $C_\mu(N,v) \subset M(N,v)$ for every $(N,v) \in \mathcal{G}^B_\mu$ with $n \geq 3$.

It remains to consider the case $n = 2$. Therefore, for $h \notin N$, let the three player game $(N', w)$ be given by $N' = N \cup \{h\}$, $\mu(N', w) = \mu(N,v)$, and $\mu(E, w_E) = \sum_{i\in E\cap N} \mu(\{i\}, v_{\{i\}})$ for every $E \subset N'$. Suppose $x \in C_\mu(N,v)$. Then, for $x' \in \mathcal{S}^3$ given by $x'_i = x_i$ for $i \in N$, and $x'_h = 0$ it holds that $x' \in C_\mu(N', w)$. Hence $C_\mu(N', w)$ is not empty and $\mu(N', w) = \mu(N,v) \geq 0$, thus $(N', w) \in \mathcal{G}^B_\mu$. Since $|N'| = 3$ the case $n \geq 3$ applies and hence it holds that $x' \in M(N', w)$ as shown above. Further, from the construction of $(N', w)$ and $x'$, it follows that for $N \subset N'$ the reduced game $(N, w^{N,\mu,x'}) = (N,v)$. The DM-$\mu$-reduced game property then implies that $(x')^N = x \in M(N, w^{N,\mu,x'}) = M(N,v)$ and thus also in case $n = 2$ it holds that $C_\mu(N,v) \subset M(N,v)$. □

The proof of the Main Theorem follows immediately from the two lemma's stated above.

**Theorem 5.12** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games and let $M$ be a mapping on a subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu^B$. Then $M$ satisfies non-emptyness, $\mu$-individual rationality, the DM-$\mu$-reduced game property and $\mu$-superadditivity on $\mathcal{C}$ if and only if $M$ is the $\mu$-share-core mapping on $\mathcal{C}$.*

PROOF: The proof follows immediately from Lemma 5.10 and Lemma 5.11. $\square$

The Main Theorem characterizes the $\mu$-share-core mapping as the unique non-empty mapping satisfying $\mu$-individual rationality, the DM-$\mu$-reduced game property and $\mu$-superadditivity on a given subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu^B$.

We conclude this section by generalizing another result from Peleg (1986). In that result we use *$\mu$-individual rationality for two person games* which requires the $\mu$-individual rationality property to hold only for games with two players.

**Theorem 5.13** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be linear and positive for all unanimity games and let $\mathcal{C}$ be a subgame closed subset of $\mathcal{G}_\mu$. Then the share mapping $M$ satisfies the DM-$\mu$-reduced game property, the converse DM-$\mu$-reduced game property, and $\mu$-individual rationality for two person games on $\mathcal{C}$ if and only if it is equal to the $\mu$-share-core mapping.*

PROOF: First, observe that $C_\mu$ satisfies the three properties. According to Lemma 5.5 the converse DM-$\mu$-reduced game property is satisfied. The DM-$\mu$-reduced game property and $\mu$-individual rationality for two person games is satisfied according to Lemma 5.10.

Second, suppose that $M$ satisfies the three properties on $\mathcal{G}$, and let $(N,v) \in \mathcal{G}$. If $n = 1$ then by definition $M(N,v) = \{x \in \mathbb{R} \mid x = 1\} = C_\mu(N,v)$. If $n = 2$ then $\mu$-individual rationality for two person games and the fact that $M(N,v) \subset \mathcal{S}^2$ imply that $M(N,v) = C_\mu(N,v)$.

It remains to consider $n \geq 3$. Observe that all reduced games exist and are uniquely determined because $\mu$ is linear and positive for all unanimity games. Now, let $x \in M(N,v)$. The DM-$\mu$-reduced game property of $M$ then implies that $x^H \in M(H, v^{\mu,H,x})$ for every $H \subset N$ with $|H| = 2$. But then $x^H \in C_\mu(H, v^{H,\mu,x})$ for every $H \subset N$ with $|H| = 2$, as shown above. The converse DM-$\mu$-reduced game property of $C_\mu$ then implies that $x \in C_\mu(N,v)$. Thus, $M(N,v) \subset C_\mu(N,v)$. Along the same lines it follows that $C_\mu(N,v) \subset M(N,v)$. Hence $M(N,v) = C_\mu(N,v)$. $\square$

Note that Theorem 5.13 characterizes the $\mu$-share-core mapping on subgame closed subsets of $\mathcal{G}_\mu$, whereas Theorem 5.12 characterizes it on subgame closed subsets of $\mathcal{G}_\mu^B$ for which it is not empty by definition.

## 6 Marginal shares and $\mu$-convex games

In this section we first give an alternative definition of the class of share functions given in Theorem 2.1.(ii) by using marginal contributions. Therefore, we first extend Theorem 2.1 from subclasses of the class $\mathcal{G}_\mu^+$ of $\mu$-positive games to subclasses of the set $\mathcal{G}_\mu$, so allowing for games to which the function $\mu$ assigns the value zero. The next corollary follows immediately from part (i) of Theorem 2.1 by requiring that $\rho^\mu$ satisfies the *equal share property* in case $(N,v)$ is a game with $\mu(N,v) = 0$, i.e. $\rho_i^\mu(N,v) = \frac{1}{n}$ for all $i \in N$ when $(N,v) \in \mathcal{C} \cap \mathcal{G}_0^\mu$.

**Corollary 6.1** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be additive and symmetric on $\mathcal{G}_\mu$, and let $\mathcal{C} \subset \mathcal{G}_\mu$ be a subgame closed set containing all positively scaled unanimity games. Then there exists a unique share function $\rho^\mu$ on $\mathcal{C}$ satisfying (i) symmetry and $\mu$-additivity on $\mathcal{C}$, (ii) the null player property on $\mathcal{C} \cap \mathcal{G}_+^\mu$, and (iii) the equal share property on $\mathcal{C} \cap \mathcal{G}_0^\mu$*

The *marginal value vector* of game $(N,v)$ and permutation $\pi: N \to N$ is the vector $m^\pi(N,v)$ given $m_i^\pi(N,v) = m^i_{P(\pi,i)\cup\{i\}}(N,v) = v(P(\pi,i) \cup \{i\}) - v(P(\pi,i))$ with $P(\pi,i) := \{j \in N \mid \pi(j) < \pi(i)\}$ for all $i \in N$. Thus, the marginal value vector corresponding to permutation $\pi$ assigns to player $i$ its marginal contribution to the worth of the coalition consisting of all its predecessors in $\pi$. Using these marginal value vectors the Shapley value can also be expressed as the average of these marginal value vectors over all permutations, i.e. $Sh(N,v) = \frac{1}{n!}\sum_{\pi\in\Pi(N)} m^\pi(N,v)$, where $\Pi(N)$ denotes the collection of all permutations on $N$. A similar expression can be given for share functions. So, let $\mu\colon\mathcal{G}\to\mathbb{R}$ be given and suppose that the entrance of the players in forming the 'grand coalition' is given by permutation $\pi\colon N\to N$ on the set $N$ of players. Then we define the share function $\rho^{\mu,\pi}$ in which the shares are determined by the marginal contributions of the players according to the permutation $\pi$ by

$$\rho_i^{\mu,\pi}(N,v) = \begin{cases} \frac{\mu(P(\pi,i)\cup\{i\},v_{P(\pi,i)\cup\{i\}})-\mu(P(\pi,i),v_{P(\pi,i)})}{\mu(N,v)} & \text{if } \mu(N,v)\neq 0 \\ \frac{1}{n} & \text{otherwise.}\end{cases} \tag{4}$$

The share vector $\rho^{\mu,\pi}(N,v)$ is called the *marginal share vector* of $(N,v)$ corresponding to $\mu$ and permutation $\pi$. If players enter the 'grand coalition' according to permutation $\pi$ then the first player $\pi(1)$ when entering gets full share

$$\rho_{\pi(1)}^{\mu,\pi}(\{\pi(1)\},v_{\{\pi(1)\}}) = 1 = \begin{cases} \frac{\mu(\{\pi(1)\},v_{\{\pi(1)\}})}{\mu(\{\pi(1)\},v_{\{\pi(1)\}})} & \text{if } \mu(\{\pi(1)\},v_{\pi(1)\}})\neq 0 \\ \frac{1}{|\{\pi(1)\}|} & \text{otherwise.}\end{cases}$$

When the second player, $\pi(2)$, enters the coalition then the share of the first player is adapted to $\rho_{\pi(1)}^{\mu,\pi}(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})$

$$= \begin{cases} \frac{\mu(\{\pi(1)\},v_{\{\pi(1)\}})}{\mu(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})} & \text{if } \mu(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})\neq 0 \\ \frac{1}{|\{\pi(1),\pi(2)\}|}=\frac{1}{2} & \text{otherwise.}\end{cases}$$

and the share of the new player $\pi(2)$ is the remainder $\rho_{\pi(2)}^{\mu,\pi}(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}}) = 1-\rho_{\pi(1)}^{\mu,\pi}(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})$. This equals $\frac{\mu(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})-\mu(\{\pi(1)\},v_{\{\pi(1)\}})}{\mu(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})}$ if $\mu(\{\pi(1),\pi(2)\},v_{\{\pi(1),\pi(2)\}})\neq 0$, and is $\frac{1}{2}$ otherwise. Proceeding in this way yields the shares expressed by (4).

Next we state another property for $\mu$-functions. The function $\mu\colon\mathcal{G}\to\mathbb{R}$ is *null player independent* on $\mathcal{C}\subset\mathcal{G}$ if for every $(N,v)\in\mathcal{C}$ and every null player $i$ in $(N,v)$ such that $(N\setminus\{i\},v_{N\setminus\{i\}})\in\mathcal{C}$ it holds that $\mu(N,v)=\mu(N\setminus\{i\},v_{N\setminus\{i\}})$. Examples of null player independent functions are $\mu^S$ and $\mu^B$. Note that $\mu^{\overline{B}}$ is not null player independent. Now, every share function of Corollary 6.1 with $\mu$ satisfying null player independence and such that $\mu$ assigns zero to null games is equal to the average of all share vectors $\rho^{\mu,\pi}(N,v)$ over all $\pi \in \Pi(N)$.

**Theorem 6.2** *Let $\mu: \mathcal{G} \to \mathbb{R}$ be additive, symmetric and null player independent on $\mathcal{G}_\mu$ and satisfy $\mu(N,v^0) = 0$, and let $\rho^\mu$ be the share function of Corollary 6.1. For every subgame closed subset $\mathcal{C}$ of $\mathcal{G}_\mu$ containing all positively scaled unanimity games, and $(N,v) \in \mathcal{C}$ it then holds that*

$$\rho^\mu(N,v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} \rho^{\mu,\pi}(N,v) \text{ for all } (N,v) \in \mathcal{C}.$$

PROOF: In van den Brink and van der Laan (1999) (Theorem 3.5) it is shown that $(N,v^\mu) \in \mathcal{G}_{\mu^S}$ and that $\rho^\mu(N,v) = \rho^{\mu^S}(N,v^\mu)$, where $(N,v^\mu)$ is given by $v^\mu(E) = \mu(E,v_E)$ for all $E \subset N$. With the corresponding property of the Shapley value it is easy to verify that the theorem holds for the Shapley share function $\rho^{\mu^S}$. For the other share functions the theorem then follows from the fact that $\rho^{\mu,\pi}(N,v) = \rho^{\mu^S,\pi}(N,v^\mu)$. $\square$

Thus, the share function $\rho^\mu(N,v)$ satisfying the properties of Corollary 6.1 can be obtained as the average of the marginal share vectors in a similar way as the corresponding value function can be obtained as the average of the marginal value vectors.

We now turn to the class of convex games $(N,v)$. As stated before the (standard) Core of a convex game is not empty. Moreover, it is equal to the convex hull of all marginal vectors of $(N,v)$, i.e. $Core(N,v) = Conv\{m^\pi(N,v) \mid \pi \in \Pi(N)\}$ and the Shapley value $Sh(N,v)$ is the barycenter of $Core(N,v)$ if $(N,v)$ is a convex game (see Shapley (1971) and Ichiichi (1981)). From this it is easy to verify that for convex non-null games $(N,v)$, $v \neq v^0$, it holds that the Shapley share core $C(N,v) = C_{\mu^S}(N,v) = Conv\left\{\frac{m^\pi(N,v)}{v(N)} | \pi \in \Pi(N)\right\}$, and that the Shapley share vector $\rho^S(N,v)$ is the barycenter of $C_{\mu^S}(N,v)$.

To generalize these results for other $\mu$ functions we first generalize the concept of convexity. For given function $\mu$, we call a game *$\mu$-convex* if the game that assigns to every coalition the $\mu$-value assigned to the subgame restricted to that coalition is convex.

**Definition 6.3** *Let $\mathcal{C} \subset \mathcal{G}_\mu$ be subgame closed, and let $\mu : \mathcal{C} \to \mathbb{R}$ be given. The game $(N, v) \in \mathcal{G}_\mu$ is* $\mu$-**convex** *on $\mathcal{C}$ if for every $E, F \subset N$ it holds that*

$$\mu(E \cup F, v_{E\cup F}) + \mu(E \cap F, v_{E\cap F}) \geq \mu(E, v_E) + \mu(F, v_F).$$

For a subgame closed set $\mathcal{C} \subset \mathcal{G}_\mu$, $(N, v) \in \mathcal{C}$, and $\mu : \mathcal{C} \to \mathbb{R}$ define the characteristic function $w$ on $N$ by $w(E) = \frac{\mu(E, v_E)}{\mu(N, v)}$ for every $E \subset N$ if $\mu(N, v) \neq 0$, and $w = v^0$ otherwise. Note that we already used this transformed game in the proof of Theorem 4.6. As noticed in the proof of Theorem 4.6 it holds that $C_\mu(N, v) = C(N, w)$. The properties for convex games mentioned above then yield the following theorem.

**Theorem 6.4** *Let $\mu : \mathcal{G} \to \mathbb{R}$ be linear, symmetric and null player independent on the subgame closed subset $\mathcal{C} \subset \mathcal{G}_\mu$ containing all positively scaled unanimity games, and satisfy $\mu(N, v^0) = 0$. If $(N, v) \in \mathcal{C}$ is $\mu$-convex then*

(i) *$C_\mu(N, v) \neq \emptyset$;*

(ii) *$C_\mu(N, v) = Conv\{\rho^{\mu,\pi}(N, v) \mid \pi \in \Pi(N)\}$ if $\mu(N, v) \neq 0$, and $C_\mu(N, v) = \mathcal{S}^n$ if $\mu(N, v) = 0$;*

(iii) *$\rho^\mu(N, v)$ is the barycenter of $C_\mu(N, v)$.*

PROOF: Let $\mu : \mathcal{G} \to \mathbb{R}$ be symmetric and linear on the subgame closed subset $\mathcal{C} \subset \mathcal{G}_\mu$, and let $(N, v) \in \mathcal{C}$ be $\mu$-convex. Further, let the characteristic function $w$ on $N$ be given by $w(E) = \frac{\mu(E, v_E)}{\mu(N, v)}$ for all $E \subset N$ if $\mu(N, v) \neq 0$, and $w = v^0$ otherwise. Then $\mu$-convexity of $(N, v)$ implies convexity of $(N, w)$, and with $C_\mu(N, v) = C_{\mu^S}(N, w)$ we obtain the following.

Part (i) follows from the convexity of $(N, w)$ and non-emptyness of the Shapley share-core of convex games.

Part (ii) is true by definition when $\mu(N, v) = 0$. If $\mu(N, v) \neq 0$, then part (ii) follows from

$$\rho_i^{\mu,\pi}(N, v) = \frac{\mu(P(\pi, i) \cup \{i\}, v_{(P(\pi,i)\cup\{i\})}) - \mu(P(\pi, i), v_{P(\pi,i)})}{\mu(N, v)}$$

$$= w(P(\pi, i) \cup \{i\}) - w(P(\pi, i)) = m_i^\pi(N, w), \text{ for all } i \in N,$$

and the fact that $Core(N,w)$ of a convex game $(N,w)$ is equal to the convex hull of all marginal vectors $m^\pi(N,w)$.

To show Part (iii), first consider $\mu(N,v) \neq 0$. Then it follows from Theorem 6.2, the fact that $Sh(N,w)$ is the barycenter of $Core(N,w)$ and because it holds that $\frac{1}{n!}\sum_{\pi\in\Pi(N)} \rho_i^{\mu,\pi}(N,v)$

$$= \frac{1}{n!}\sum_{\pi\in\Pi(N)} \frac{\mu(P(\pi,i)\cup\{i\}, v_{(P(\pi,i)\cup\{i\})}) - \mu(P(\pi,i), v_{P(\pi,i)})}{\mu(N,v)}$$

$$= \frac{1}{n!}\sum_{\pi\in\Pi(N)} w(P(\pi,i)\cup\{i\}) - w(P(\pi,i)) = Sh_i(N,w),$$

for all $i \in N$. If $\mu(N,v) = 0$ then $\frac{1}{n!}\sum_{\pi\in\Pi(N)} \rho^{\mu,\pi}(N,v) = \frac{1}{n}$ is the barycenter of $\mathcal{S}^n = C_\mu(N,v)$. □

Note that $\mu^S$-convexity coincides with convexity of a game. Thus, trivially, all convex games are $\mu^S$-convex. Also, all convex games are $\overline{\mu}^B$-convex, but convex games need not be $\mu^B$-convex.

**Example 6.5** Consider the first game of Example 4.1. Then the extreme points of the share-cores $C_{\mu^S}(N,v)$ and $C_{\mu^B}(N,v)$, respectively, are given by the vectors $\rho^{\mu^S,\pi}(N,v)$ and $\rho^{\mu^B,\pi}(N,v)$, $\pi \in \Pi(N)$. The vectors $\rho^S(N,v) = (\frac{1}{2},\frac{1}{4},\frac{1}{4})^\top$ and $\rho^B(N,v) = (\frac{10}{24},\frac{7}{24},\frac{7}{24})^\top$ are the barycenters of the corresponding $\mu$-share-cores.

# References


BANZHAF, J.F. (1965), "Weighted Voting Doesn't Work: A Mathematical Analysis", *Rutgers Law Review*, 19, 317-343.

BRINK, R. VAN DEN, AND G. VAN DER LAAN (1998a), "Axiomatizations of the Normalized Banzhaf Value and the Shapley Value", *Social Choice and Welfare*, 15, 567-582.

Brink, R. van den and G. van der Laan (1998b), "The Normalized Banzhaf Value and the Banzhaf Share Function", *Game Theory and Applications Vol. 4* (eds L. Petrosjan and V. Mazalov), pp 11-31.

Brink, R. van den and G. van der Laan (1999), "Potential and Reduced Games for Share Functions", Tinbergen DP.

Deegan, J., and E.W. Packel (1979), "A New Index of Power for Simples $n$-Person Games", *International Journal of Game Theory*, 7, 113-123.

Dragan, I. (1996), "New Mathematical Properties of the Banzhaf Value", *European Journal of Operational Research*, 95, 451-463.

Funaki, Y, and T. Yamamoto (1997), "The Core and Conistency Properties", *Mimeo*, Toyo University.

Haller, H. (1994), "Collusion Properties of Values", *International Journal of Game Theory*, 23, 261-281.

Harsanyi J.C. (1959), "A Bargaining Model for Cooperative n-Person Games", in: *Contributions to the Theory of Games IV* (eds. A.W. Tucker and R.D. Luce), Princeton University Press, Princeton, pp. 325-355.

Hart, S., and A. Mas-Colell (1988), "The Potential of the Shapley Value", in: *The Shapley Value. Essays in Honor of L.S. Shapley* (ed. A.E. Roth), Cambridge University Press, pp. 127-137.

Hart, S., and A. Mas-Colell , (1989), "Potential, Value and Consistency", Econometrica, 57, 589-614.

Ichiishi, T. (1981), "Super-Modularity: Applications to Convex Games and the Greedy Algorithm for LP", *Journal of Economic Theory*, 25, 283-286.

Laan, G. van der, and Brink, R. van den (1998), "An Axiomated Class of Share Functions for N-Person Games", *Theory and Decision*, 44, 117-148.

Lehrer, E. (1988), "An Axiomatization of the Banzhaf Value", *International Journal of Game Theory*, 17, 89-99.

Peleg, B. (1986), "On the Reduced Game Property and Its Converse", *International Journal of Game Theory*, 15, 187-200.

SHAPLEY, L.S. (1953) "A Value for $n$-Person Games", *Annals of Mathematics Studies* 28 (Contributions to the Theory of Games Vol.2) (eds. H.W. Kuhn and A.W. Tucker), Princeton University Press, 307-317.

SHAPLEY, L.S. (1971) "Cores of Convex Games", *International Journal of Game Theory, 1,* 11-26.

TADENUMA, K. (1992), "Reduced Games, Consistency, and the Core", *International Journal of Game Theory*, 20, 325-334.

TIJS, S. (1981), "Bounds for the Core and the $\tau$-Value", in: *Game Theory and Mathematical Economics*, (eds. O. Moeschlin and D. Pallaschke), North-Holland Publishing Company, Amsterdam, pp.123-132.