Noncooperative Bargaining in Apex Games and the Kernel

Montero, M.P.

Publication date:
1999

Citation for published version (APA):
Montero, M. P. (1999). *Noncooperative Bargaining in Apex Games and the Kernel*. (CentER Discussion Paper; Vol. 1999-61). Microeconomics.

Noncooperative Bargaining in Apex Games and the Kernel

Maria Montero
CentER for Economic Research
P.O. Box 90153
5000 LE Tilburg
The Netherlands

July 6, 1999

Abstract

This paper studies non-cooperative bargaining with random proposers in apex games. Two di¤erent protocols are considered: the egalitarian propocol, which selects each player to be the proposer with the same probability, and the proportional protocol, which selects each player with a probability proportional to his number of votes. Expected equilibrium payo¤s coincide with the kernel for the grand coalition regardless of the protocol. Expected payo¤s conditional on a coalition may depend on the protocol: given a coalition of the apex player with a minor player, an egalitarian protocol yields a nearly equal split whereas a proportional protocol leads to a proportional split.

Keywords: noncooperative bargaining, apex games, kernel, random proposers.

JEL classi…cation numbers: C72, C78.

I'm indebted to Eric van Damme, Anne van den Nouweland, Alex Possajennikov and Marco Slikker for useful suggestions and discussion. Financial support from the NWO is gratefully acknowledged.

1 Introduction

An apex game is a simple n-player game with one major player (the apex player) and $n - 1 \geq 3$ minor players (also called base players). A winning coalition can be formed by the apex player together with at least one of the minor players or by all the minor players together. Apex games can be interpreted as weighted majority games in which the major player has $n - 2$ votes, each of the $n - 1$ minor players has one vote, and $n - 1$ votes are required for a majority.

Since the apex player only needs one of the minor players he can play them off against each other to obtain favorable terms. Each minor player has two options: either try to unite with the other minor players (and run the risk that one of the others yields to an advantageous offer of the apex player) or compete with them for the favor of the apex player. Apex games have received a lot of attention both in theory[1] and in experiments, and their importance may be compared to that of the prisoner's dilemma[2].

This paper addresses three questions concerning apex games:

1) What coalition(s) are likely to form?

2) How will the gains from cooperation be divided for each possible coalition?

3) What are the ex ante expected payoffs for the players?

There are very different answers in the literature to the first question. Some papers (Bennett, 1983; Morelli, 1998) predict that all minimal winning coalitions are possible, whereas others limit the possible outcomes to the coalition of all small players (Aumann and Myerson, 1988; Hart and Kurz, 1984) or to coalitions of the major player with a minor player (Chatterjee et al., 1993).

As for the second question, equal division of gains seems indicated if all minor players form a coalition. If the apex player forms a coalition with a

[1] Some of the theoretical literature will be reviewed in section 4.

[2] Rapoport et al. (1979) put it the following way:

> The centrality of the Apex player, which produces the conflict faced by each Base player of whether to cooperate and trust all the remaining n-2 Base players or to do his best, negotiating from weakness, against the Apex player, has aroused intense interest in apex games, such that they may become to n-person experimental games what the Prisoner's Dilemma has been to two-person noncooperative games.

minor player, the division of gains is not so clear-cut. The answers given in the literature point either to the "egalitarian" $\frac{1}{2} : \frac{1}{2}$ split corresponding to the kernel (Davis and Maschler, 1965) or to the "proportional" (to the number of votes) $\frac{n-2}{n-1} : \frac{1}{n-1}$ split that comes from observing that a small player can not expect more than $\frac{1}{n-1}$ if all the minor players form a coalition. The bargaining set (Aumann and Maschler, 1964) includes these two extremes and all outcomes in between.

Most of the literature has little to say about ex ante payo¤s. They are either very extreme (as the major player receives a payo¤ of zero) or undetermined (when several coalitions are possible, ex ante expected payo¤s depend on the likelihood of each coalition, and this is left undetermined). On the other hand, ex ante concepts like the Shapley value give no predictions about coalitions or division of gains. The current paper attempts to provide an answer to the three questions simultaneously.

In this paper a noncooperative procedure with random proposers (Baron and Ferejohn (1989), Okada (1996)) is used to model bargaining in apex games. Two types of protocol are examined: the "egalitarian" protocol in which each player is selected to be the proposer with equal probability, and the "proportional" protocol, in which each player is selected with a probability proportional to his number of votes[3]. The solution concept is stationary perfect equilibrium with symmetric strategies for the minor players.

Intuitively, the apex player should bene…t from a proportional protocol since he is chosen more often to be the proposer. However, we show that this is not the case: expected equilibrium payo¤s are proportional to the number of votes of the players for both protocols. The reason is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be the proposer with a higher probability, the other players make o¤ers to him with a lower probability so that his ex ante expected payo¤ remains unchanged.

We also show that all minimal winning coalitions may form, and the probability of a coalition being formed depends on the protocol (the coalition of all minor players being more frequent under a proportional rule). Expected payo¤s conditional on a coalition of the major player and a minor player

[3] This would be the case in a parliamentary system where the probability of a party being asked to form a government is proportional to the number of seats it holds. Baron and Ferejohn (1989) use a proportional protocol in one of their examples. Okada (1996) only considers the egalitarian protocol.

depend on the protocol: for an egalitarian protocol, the expected division is "close" to the egalitarian division (and converges to it when the number of players tends to infinity); for a proportional protocol, the expected division is proportional.

The rest of the paper is organized as follows: section 2 describes the model and the results, section 3 relates the resulting expected payoffs to the kernel for the grand coalition, section 4 reviews some of the literature, section 5 discusses possible extensions and section 6 concludes.

2 Bargaining with random proposers in apex games

2.1 The Model

Apex games consist of one major player (the apex player) and $n-1$ minor players. If $N = \{1, 2, \dots, n\}$ and 1 is the apex player, then $v(S) = 1$ if either $1 \in S$ and $S \backslash \{1\} \neq \emptyset$, or $S = N \backslash \{1\}$. Apex games can be interpreted as weighted majority games in which the major player has $n-2$ votes, each minor player has 1 vote, and $n-1$ votes are required to obtain a majority[4].

Bargaining in apex games is modeled following Okada (1996), which in turn extends the model of Baron and Ferejohn (1989). Given the underlying cooperative (apex) game $(N; v)$, bargaining proceeds as follows: At every round $t = 1, 2, \dots$ Nature selects a player randomly to be the proposer. This player proposes a coalition $S \subseteq N$ to which he belongs and a division of $v(S)$, denoted by $x^S = (x_i^S)_{i \in S}$. The ith component x_i^S represents a payoff for player i in S: Given a proposal, the rest of players in S (called responders) accept or reject sequentially (the order does not affect the results). If all players in S accept, the proposal is implemented and the game ends[5]. If at least one player rejects, the game proceeds to the next period in which nature selects a new proposer (always with the same probability distribution). Players are

[4] This is of course only one of the many possible vectors of weights we can assign to the players. We have chosen a so-called homogeneous representation, in which all minimal winning colaitions have the same number of votes.

[5] Okada (1996) allows bargaining to continue among the remaining players until no coalition that may be formed has a positive value. Ending the game after one coalition has been formed is a simplifying assumption that does not affect the results for apex games.

risk-neutral and share a discount factor $\delta < 1$[6]. Thus, if a proposal x^S is accepted by all players in S at time t, each player in S receives a payoff $\delta^{t-1} x_i^S$. A player not in S remains a singleton and receives zero.

A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

(i) A proposal (S, x^S).

(ii) A response function assigning "yes" or "no" to all possible proposals of the other players.

The solution concept is symmetric stationary subgame perfect equilibrium. A stationary perfect equilibrium is a subgame perfect equilibrium with the property that players follow the same strategy at every round t. Equilibrium strategies must be symmetric in a weak sense: each minor player proposes coalition $N \setminus \{1\}$ with the same probability. Notice that this does not impose any restriction on the payoffs offered or on the strategy of the major player.

Concerning the probability of players being selected to be proposers, we will call the probability vector used by Nature a protocol, and we will denote it by $\theta := (\theta_i)_{i \in N}$, where $\theta_i > 0 \; \forall i \in N$ and $\sum_{i \in N} \theta_i = 1$.

Two natural protocols suggest themselves: the egalitarian protocol $\theta^E := (\frac{1}{n}, ..., \frac{1}{n})$, which selects each player with the same probability, and the proportional protocol $\theta^P := (\frac{n-2}{2n-3}, \frac{1}{2n-3}, ..., \frac{1}{2n-3})$, which selects each player with a probability proportional to his number of votes.

We will denote the noncooperative game described above $G(N, v, \theta, \delta)$. We will think of v as the characteristic function of an apex game, unless otherwise specified.

2.2 The equilibrium

The following lemma corresponds to theorem 1 in Okada (1996). Even though the original theorem assumes the egalitarian protocol θ^E, it can be applied to any protocol θ. The proof is included for completeness.

Lemma 1 (Okada, 1996) Consider a zero-normalized, essential and superadditive[7] game (N, v). In any stationary subgame perfect equilibrium of the game $G(N, v, \theta, \delta)$, every player i in N proposes a solution (S_i, y^{S_i}) of the maximization problem

---

[6] Alternatively, after a proposal is rejected the game ends with probability $1 - \delta$.

[7] A cooperative game (N, v) is zero-normalized and essential if $v(i) = 0 \; \forall i \in N$ and $v(N) > 0$; it is superadditive if $v(S \cup T) \geq v(S) + v(T) \; \forall S, T \subseteq N$, $S \cap T = \emptyset$.

$$\max_{S,y} \left(v(S) - \sum_{j\in S, j\neq i} y_j\right) \tag{1}$$

s.t.:

(i) $i \in S \subseteq N$

(ii) $y_j \geq \delta w_j \ \forall j \in S\setminus\{i\}$

where w_j is the equilibrium expected payoff of player j. Moreover, the proposal (S_i, y^{S_i}) is accepted.

Proof. For every $i = 1,\ldots,n$, let w_i^j be player i's equilibrium expected payoff conditional on player j becoming the proposer at time 1, and let m_i be the maximum value of (1). We first show that $w_i^i = m_i$.

Subgame perfection implies $w_i^i \geq m_i$. In a subgame perfect equilibrium any player j must accept any proposal that gives him at least δw_j, thus player i can get at least m_i.

Can player i get more than m_i? If player i proposes (S, y^S) at round 1 with $y_i > m_i$, the proposal will be rejected (otherwise at least one responder j is getting less than δw_j and could do better by rejecting the proposal) and i will get δw_i. Since the characteristic function is superadditive we have $\sum_{j\in N} w_j \leq v(N)$. This implies that the pair (N, w), $w = (w_j)_{j\in N}$ is a feasible proposal and thus we must have $\delta w_i \leq w_i \leq m_i$, therefore $w_i^i \leq m_i$.

Since $w_i^i \geq m_i$ and $w_i^i \leq m_i$, it follows that $w_i^i = m_i$.

To prove that player i makes an acceptable proposal, we must prove $\delta w_i < m_i$. We know that $\delta w_i \leq m_i$. If $\delta w_i = m_i$, then $w_i = m_i = 0$ since $w_i \leq m_i$ and $\delta < 1$. Since $(N, \delta w)$ is a feasible proposal, $m_i \geq (1-\delta)v(N) > 0$. ■

An immediate and useful corollary of Lemma 1 is the following:

Corollary 2 Consider a zero-normalized, essential and superadditive game (N, v). In any stationary subgame perfect equilibrium of the game $G(N, v, \mu, \delta)$, every player in N has a strictly positive expected payoff ($w_i > 0 \ \forall i \in N$).

Proof. From the proof of Lemma 1 we know that each player gets a strictly positive expected payoff as a proposer ($m_i \geq (1-\delta)v(N) > 0$ for all i). As a responder, he can guarantee himself a payoff of zero by rejecting all proposals that are made to him. Since each player has a strictly positive probability of being selected to be the proposer ($μ_i > 0$ for all i), this implies $w_i > 0$ 8i 2 N: ■

Corollary 2 has the following implications for zero-normalized, superadditive and essential simple games[8]

Corollary 3 Consider a zero-normalized, superadditive and essential simple game (N; v): In any stationary subgame perfect equilibrium of the game G(N; v; μ; ±); all players propose winning coalitions in which each responder is pivotal.

Proof. Proposing a losing coalition cannot be a solution of (1), since it would yield at most zero for the proposer and he can get at least 1 ¡ ± > 0 by proposing the grand coalition and o¤ering $±w_j$ to each responder.

On the other hand, since by corollary 2 $w_i > 0$ 8i 2 N; a proposal including a responder who is not pivotal cannot be a solution of (1). ■

It does not follow from Corollary 3 that only minimal winning coalitions are proposed in equilibrium. In fact, a stationary perfect equilibrium may include proposals of coalitions in which the proposer is not pivotal (see Section 5.1). For apex games however only minimal winning coalitions are proposed, as Corollary 4 shows.

Corollary 4 If (N; v) is an apex game, in any stationary subgame perfect equilibrium of the game G(N; v; μ; ±) all players propose minimal winning coalitions.

Proof. A winning but not minimal winning coalition in an apex game must contain the apex player and at least two minor players, thus at least two players in the coalition are not pivotal (the two minor players). If such a coalition would be proposed in equilibrium, at least one responder would not be pivotal, contradicting Corollary 3. ■

The following lemma describes some characteristics of the equilibrium that are common to the protocols that treat all the minor players equally.

Lemma 5 Consider an apex game (N; v) and a protocol μ such that $μ_i = μ_j$ for all i; j 2 Nnf1g: Let ¾¤ be a symmetric stationary subgame perfect equilibrium of the game G(N; v; μ; ±): Then the following holds:

[8] In a simple game, v(S) = 0 or 1 for all S µ N: A coalition S in a simple game is called winning if v(S) = 1 and losing if v(S) = 0. A player i 2 S is called pivotal if v(S) = 1 and v(Snfig) = 0: If all players in S are pivotal S is called a minimal winning coalition.

a) The apex player proposes each coalition $\{1, i\}$ $(i \neq 1)$ with probability $\frac{1}{n-1}$.

b) $w_i = w_j$ for all $i, j \in N \setminus \{1\}$.

In other words, if the protocol treats all minor players symmetrically and each minor player proposes coalition $N \setminus \{1\}$ with the same probability, then in equilibrium the apex player proposes to each minor player with the same probability and all the minor players have the same expected equilibrium payoff[9].

Proof. See Appendix. ■

Proposition 6 describes the symmetric stationary perfect equilibrium of the game for the egalitarian protocol; proposition 7 describes it for the proportional protocol. Both protocols satisfy the assumptions of Lemma 5, thus equilibrium expected payoff must be the same for all minor players. In the sequel we will denote equilibrium expected payoffs by w_m for a minor player and w_a for the apex player.

Proposition 6 The unique symmetric stationary subgame perfect equilibrium of $G(N; v; \mu^E; \delta)$ is as follows

a) When selected as a proposer, the apex player proposes to form a coalition with each of the minor players with equal probability[10].

b) When a minor player is selected as a proposer, he randomizes between proposing a coalition with the apex player (with probability $\frac{\delta + (n-1)(n-3)}{\delta (n-1)(n-2)}$) and with all other minor players.

These proposals are accepted and expected payoffs are $\frac{n-2}{2n-3}$ for the apex player and $\frac{1}{2n-3}$ for each minor player.

---

[9] Remember that symmetry of the equilibrium imposes no requirements on the equilibrium strategy of the major player. As for the minor players, it constraints the probabilities of proposing coalitions, not the payoffs offered. Thus, if i and j are minor players, symmetry of the equilibrium together with the fact that only minimal winning coalitions are proposed implies that i includes j in a proposal with the same probability that j inlcudes i, but it does not require that i offers to j the same payoff j offers to i: Because of subgame perfection, we know that j would offer δw_j to j and j would offer δw_i to i, but symmetry of the equilibrium does not require $w_i = w_j$.

[10] It suffices to describe the equilibrium strategies of the players by a probability distribution over the coalitions they propose. Lemma 1 implies that each responder j will accept any offer that gives him at least δw_j and will be offered exactly δw_j. This fact together with the probability distribution used by the proposers determines $(w_j)_{j \in N}$, therefore we need to specify neither the payoffs offered to the responders nor the set of proposals players accept.

Proof. The equilibrium strategy of the apex player follows from Lemma 5. Corollary 4 leaves three possibilities for the minor players: they may propose a coalition with the apex player, a coalition including all the minor players, or they may randomize.

Suppose they propose a coalition to the apex player. The continuation payoffs are then found from the following system of equations, where w_a denotes the continuation payoff for the apex player and w_m denotes the continuation payoff for a minor player:

$$\begin{aligned} w_a &= \frac{1}{n}[1 - \delta w_m] + \frac{n-1}{n}\delta w_a \\ w_m &= \frac{1}{n}[1 - \delta w_a] + \frac{1}{n(n-1)}\delta w_m \end{aligned}$$

The solution to this system of equations is $w_a = \frac{(n-\delta-1)}{n(n-1)-\delta(n^2-2n+2)}$ and $w_m = \frac{(1-\delta)(n-1)}{n(n-1)-\delta(n^2-2n+2)}$. When δ is close to 1, w_a is close to 1 and w_m is close to 0, thus this strategy combination cannot be an equilibrium (a minor player would prefer to form a coalition with the other minor players and get a payoff close to 1, instead of following his prescribed strategy and get a payoff close to zero).

Suppose each minor player proposes to the rest of the minor players. Then the continuation payoffs are found from the following system of equations:

$$\begin{aligned} w_a &= \frac{1}{n}[1 - \delta w_m] \\ w_m &= \frac{1}{n}[1 - (n-2)\delta w_m] + \frac{1}{n(n-1)}\delta w_m + \frac{n-2}{n}\delta w_m \end{aligned}$$

The solution to this system of equations is $w_a = \frac{(n-1-\delta)}{n(n-1)-\delta}$; $w_m = \frac{(n-1)}{n(n-1)-\delta}$. Clearly, $w_a < w_m$, thus these strategies can not constitute an equilibrium (a minor player would prefer to propose to the apex player).

Suppose a minor player proposes to the apex player with probability λ and to the other minor players with probability $1-\lambda$: The continuation payoffs and λ are found from the following system of equations (the third one being an indifference condition for the minor players):

$$w_a = \frac{1}{n}[1 - \delta w_m] + \frac{n-1}{n}\lambda \delta w_a$$
$$w_m = \frac{1}{n}[1 - \lambda \delta w_a - (1-\lambda)(n-2)\delta w_m] + \frac{1}{n(n-1)}\delta w_m + \frac{n-2}{n}(1-\lambda)\delta w_m$$
$$w_a = (n-2) w_m$$

The solution to this system of equations is $w_a = \frac{n-2}{2n-3}$; $w_m = \frac{1}{2n-3}$; $\lambda = \frac{\delta + (n-1)(n-3)}{\delta(n-1)(n-2)}$: When δ tends to 1, λ tends to $\frac{n-2}{n-1}$: ■

Proposition 7 The unique symmetric stationary subgame perfect equilibrium of $G(N; v; \mu^P; \delta)$ is as follows

a) When selected as a proposer, the apex player proposes to form a coalition with each of the minor players with equal probability.

b) When a minor player is selected as a proposer, he randomizes between proposing a coalition with the apex player (with probability $\frac{1}{n-1}$) and with all other minor players.

These proposals are accepted and expected payoffs are $\frac{n-2}{2n-3}$ for the apex player and $\frac{1}{2n-3}$ for a minor player.

Proof. It is easy to check that there is no equilibrium in which the minor players play pure strategies (see the proof of proposition 6).

Suppose a minor player proposes to the apex player with probability λ and to the other minor players with probability $1 - \lambda$: The continuation payoffs and λ are found from the following system of equations:

$$\begin{aligned} w_a &= \frac{n-2}{2n-3}[1 - \delta w_m] + \frac{n-1}{2n-3}\lambda \delta w_a \\ w_m &= \frac{1}{2n-3}[1 - \lambda \delta w_a - (1-\lambda)(n-2)\delta w_m] + \frac{n-2}{(2n-3)(n-1)}\delta w_m \\ &\quad + \frac{n-2}{2n-3}(1-\lambda)\delta w_m \\ w_a &= (n-2) w_m \end{aligned}$$

The solution to this system of equations is $w_a = \frac{n-2}{2n-3}$; $w_m = \frac{1}{2n-3}$; $\lambda = \frac{1}{n-1}$: ■

Remark 8 Note that expected payoffs are the same for both protocols.

The reason why payo¤s are the same for both protocols is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be a proposer more often, equilibrium strategies adjust so that he becomes a responder less often and his expected payo¤ remains unchanged.

Since in equilibrium a winning coalition always forms without delay, we have $w_a + (n - 1)w_m = 1$: If the minor players follow a mixed strategy in equilibrium, this condition together with the indi¤erence condition $w_a = (n - 2)w_m$ determines expected payo¤s regardless of the protocol. These payo¤s are such that players are indi¤erent between all the minimal winning coalitions they can propose, thus there is always such an equilibrium provided that the corresponding λ is indeed a probability, that is, a number between 0 and 1. If we restrict ourselves to protocols that give the same probability of being the proposer to all minor players, all values of μ_a in the (open) interval $]0; \frac{1}{2}[$ yield the same ex ante payo¤s: Outside this interval, the minor players no longer randomize, there is no room for strategies to compensate changes in the protocol (one would need $\lambda < 0$ or $\lambda > 1$) and the intuitive result that a player gets a higher expected payo¤ if he is more often selected to be the proposer is obtained.

Remark 9 Expected payo¤s coincide with the kernel for the grand coalition.

We elaborate on this in the next section.

Remark 10 Consider any two protocols that yield the same expected payo¤s (for example, the egalitarian protocol and the proportional protocol). The payo¤ a player gets conditional on being the proposer is the same for both protocols; the same holds for the payo¤ a player gets as a responder.

As a responder, a player gets his expected payo¤ times the discount factor δ: As a proposer he must o¤er to the responders their expected payo¤ times δ: For any two protocols that yield the same expected payo¤s a player receives the same payo¤ as a responder and solves the same maximization problem as a proposer.

Remark 11 The proposer has an advantage that does not completely disappear in the limit when δ tends to 1.

The proposer is said to have an advantage if the payoff a player gets as proposer is higher than the payoff he gets as a responder. This advantage does not vanish in the limit when δ tends to 1, in contrast with the Rubinstein (1982) game. The reason is that in this game the impatience of the players is not the only source of advantage for the proposer; a second sort of advantage is the majority rule (as opposed to the unanimity rule). The proposer will offer to each responder δ times his continuation payoff. Since the proposer only needs to form a minimal winning coalition, he "saves" the expected payoff of players outside the minimal winning coalition. Because expected payoffs for the players are proportional to their number of votes, the proposer gets a payoff of at least one half.

Remark 12 The coalition of all minor players forms more often under a proportional protocol.

As we have argued above, the payoff a player gets as a proposer and the payoff he gets as a responder are the same for both protocols. If the protocol changes so that a player is selected more often to be a proposer, strategies change so that he is selected less often to be a responder to keep expected ex ante payoffs unchanged. Moreover, since a player gets a higher payoff as a proposer rather than as a responder (see remark 11), his total probability of being in a coalition (that is, the sum of his probability of being a proposer and his probability of being a responder) must decrease as well.

The reasoning above implies that the apex player must be less often in a coalition under a proportional protocol, thus the coalition of all minor players must form more often.

We can calculate the concrete probabilities using propositions 6 and 7.

Under an egalitarian protocol, each minor player proposes the coalition of all minor players with probability $\frac{1}{n-1}$ (in the limit when δ tends to 1). Since one of the minor players is selected to be the proposer with probability $\frac{n-1}{n}$, this implies that the coalition of all minor players forms with probability $\frac{1}{n-1}\frac{n-1}{n} = \frac{1}{n}$: A coalition of the apex player with a given minor player forms with probability $\frac{1}{n}\frac{1}{n-1}+\frac{1}{n}\frac{n-2}{n-1} = \frac{1}{n}$ (the apex player is selected with probability $\frac{1}{n}$ and proposes to a given minor player with probability $\frac{1}{n-1}$; a given minor player is selected with probability $\frac{1}{n}$ and proposes to the apex player with probability $\frac{n-2}{n-1}$). Thus, each minimal winning coalition is equally likely.

Under a proportional protocol, each minor player proposes the coalition of all minor players with probability $\frac{n-2}{n-1}$: Since a minor player is selected to be the proposer with probability $\frac{n-1}{2n-3}$, this implies that the coalition of all minor players forms with probability $\frac{n-2}{2n-3}$ (approximately $\frac{1}{2}$ for large n). Analogous computations show that each coalition of the apex player with a minor player forms with probability $\frac{1}{2n-3}$.

Remark 13 Expected payoffs conditional on a concrete coalition being formed are $[\frac{1}{n-1}, \ldots, \frac{1}{n-1}]$ for a coalition containing all minor players and depend on the protocol for a coalition containing the apex player and a minor player.

The first part of the remark follows from the fact that expected payoffs are the same for all minor players and the equilibrium is symmetric.

As for the second part, we have argued above that the apex player will be in a coalition less often under the proportional protocol. Since his ex ante expected payoff remains unchanged, this implies that conditional on being in a coalition his payoff must be higher. We can calculate the exact (limit) expected payoff division using the equilibrium values of λ found in propositions 6 and 7 and the coalition probabilities computed above.

For the egalitarian protocol, a coalition of the apex player with a given minor player forms with probability $\frac{1}{n}$. It is proposed by the apex player with probability $\frac{1}{n}\frac{1}{n-1}$, and by the minor player with probability $\frac{1}{n}\frac{n-2}{n-1}$. As a proposer, the apex player gets $1-\frac{1}{2n-3}$; as a responder he gets $\frac{n-2}{2n-3}$. His expected payoff conditional on being in the coalition is then

$$\frac{\frac{1}{n}\frac{1}{n-1}\left[1-\frac{1}{2n-3}\right]+\frac{1}{n}\frac{n-2}{n-1}\frac{n-2}{2n-3}}{\frac{1}{n}} = \frac{n(n-2)}{(n-1)(2n-3)}$$

This value is close to $\frac{1}{2}$.

For the proportional protocol, a coalition of the apex player and a given minor player forms with probability $\frac{1}{2n-3}$. It is proposed by the apex player with probability $\frac{n-2}{2n-3}\frac{1}{n-1}$ and by the minor player with probability $\frac{1}{2n-3}\frac{1}{n-1}$. As for the egalitarian protocol, the apex player gets $1-\frac{1}{2n-3}$ as a proposer and $\frac{n-2}{2n-3}$ as a responder. His expected payoff conditional on being in the coalition is then

$$\frac{\frac{n-2}{2n-3}\frac{1}{n-1}\left[1-\frac{1}{2n-3}\right]+\frac{1}{2n-3}\frac{1}{n-1}\frac{n-2}{2n-3}}{\frac{1}{2n-3}} = \frac{n-2}{n-1}$$

That is, the division of payoffs conditional on a coalition of the apex player and a minor player is proportional to the number of votes each of the two players has.

3 Apex games and the kernel

We saw in the previous section that expected equilibrium payoffs of the game $G(N;v;\theta;\delta)$ coincide with the kernel for the grand coalition provided that $(N;v)$ is an apex game, θ is a protocol that treats all minor players equally and with $0 < \theta_a < \frac{1}{2}$, and δ is close to 1 (cf. remarks 7 and 8). We show now that this is not coincidental.

3.1 Definition of the kernel[11]

Consider a cooperative game $(N;v)$: Assume $v(S) \geq 0 \;\; \forall S \subseteq N$ and $v(i) = 0 \;\; \forall i \in N$: An outcome of the game is denoted by $(x;B)$ where x_i denotes the payoff to the ith player and $B \equiv \{B_1;...;B_n\}$ the coalition structure (partition of N) that was formed. The payoff vector is assumed to satisfy

$x_i \geq 0$, $i = 1;2;...;n$

$\sum_{i \in B_j} x_i = v(B_j)$; $j = 1;2;...;m$

A payoff vector satisfying these two conditions is called an imputation. The space of all imputations for the coalition structure B is denoted by $X(B)$.

Definition 14 Let x be an imputation in a game $(N;v)$ for an arbitrary coalition structure. The excess of a coalition S at x is $e(S;x) := v(S) - \sum_{i \in S} x_i$:

Definition 15 Let $(x;B)$ be an outcome for a cooperative game, and let k and l be two distinct players in a coalition B_j of B. The surplus of k against l at x is

$$s_{k;l}(x) := \max_{\substack{k \in S;\\ l \notin S}} e(S;x)$$

Definition 16 Let $(N;v)$ be a cooperative game and let B be a coalition structure. The kernel $K(B)$ for B is

$K(B) := \{x \in X(B) : s_{k;l}(x) > s_{l;k}(x) \Rightarrow x_l = 0;$ for all $k;l \in B \in B; k \neq l\}$

[11] The kernel was introduced by Davis and Maschler (1965). The definition of the kernel included here is taken from Maschler (1992). The two papers differ slightly in the terminology.

Suppose $(N; v)$ is an apex game and consider the coalition structure $\{\{1; 2\}; \{3\}; \{4\}; \ldots; \{n\}\}$ (1 is the apex player): The kernel for this coalition structure predicts that 1 and 2 receive $\frac{1}{2}$ each. Since the payoff for a coalition must equal its value, all singletons must receive zero. The surplus of 1 against 2 then equals $1 - x_1$, whereas the surplus of 2 against 1 equals $1 - x_2$: Thus $x_1 = x_2$. Player 1 needs only one of the other players, whereas player 2 needs all of them; however, since all those players receive zero, this makes no difference for payoffs. This is somehow disappointing, because intuitively the apex player is stronger and it seems that he should get more than half.

For the grand coalition, the kernel predicts $(\frac{n-2}{2n-3}; \frac{1}{2n-3}; \ldots; \frac{1}{2n-3})$: To see this, notice that all the minor players must get the same payoff; call this payoff x_2. The surplus of the apex player against a minor player then equals $1 - x_2 - x_1$, whereas the surplus of a minor player against the apex player equals $1 - (n-1)x_2$: The equality $1 - x_2 - x_1 = 1 - (n-1)x_2$ together with $x_1 + (n-1)x_2 = 1$ (that is, x must be an imputation) yields the result. If we interpret apex games as weighted majority games, the kernel predicts payoffs that are proportional to the number of votes of the players[12].

3.2 Why expected equilibrium payoffs coincide with the kernel for the grand coalition

We now come back to the equilibrium of the noncooperative game described in section 2. Expected equilibrium payoffs follow from the indifference condition for a minor player together with the fact that players propose winning coalitions and there is no delay in equilibrium (see remark 7).

The indifference condition of the minor player, $w_a = (n-2)w_m$, implies $1 - w_a = 1 - (n-2)w_m$: Substracting w_m from both sides we get

$$1 - w_a - w_m = 1 - (n-1)w_m$$

That is, in the language of the kernel, the surplus of the apex player against a minor player equals the surplus of a minor player against the apex player.

Because there is no delay in equilibrium (and players always propose winning coalitions), the sum of all expected payoffs equals 1, that is

[12] While proportional payoffs may seem only too obvious, one must take into account that neither the Shapley value nor the Banzhaf value assign proportional payoffs in an apex game.

$w_a + (n - 1)w_m = 1$

In the language of the kernel, $(w_i)_{i \in N}$ is an imputation.

4 Related literature

This section reviews some of the literature on apex games. This literature is divided in three groups: the stable demands literature, the two-stage literature and the imperfect competition literature.

4.1 Stable aspirations

The stable demands literature predicts that any minimal winning coalition may form. Given that a coalition forms, payoff division will be proportional[13]. Since probabilities are not assigned to each minimal winning coalition, there is no prediction of ex ante expected payoffs. If the situation is modeled as an extensive form game (Bennett and van Damme (1991), Bennett (1997), Morelli (1998)) the equilibrium strategies are not unique: the apex player can propose to any minor player, and each minor player can propose the minor players coalition or a coalition with the apex player. "Natural" assumptions on the selection of the first proposer (egalitarian or proportional protocols) and on the mixed strategies (each player plays all strategies that yield the same payoff with equal probability) do not lead to expected ex ante payoffs proportional to the number of votes[14]. Ex post payoffs are proportional to the number of votes; ex ante payoffs may be proportional to the number of votes (if initial probabilities and mixed strategies are chosen in an appropriate way) but need not be so. On the other hand, given that a coalition forms the payoff division is always proportional and does not depend on the protocol or on who was the proposer. Thus, the stable demands approach makes robust predictions ex post, whereas the current approach makes robust predictions ex ante.

[13] The rationale for this division varies across the stable demands literature. A simple justification is the partnership condition: given two players, either each of them needs the other to get his payoff demands, or neither needs the other.

[14] Neither do they lead to the Shapley or Banzhaf values.

4.2 Two-stage models

The two-stage approach provides possible justifications for the $\frac{1}{2}:\frac{1}{2}$ split in the two-person minimal winning coalition. This type of coalition, however, never arises in equilibrium and the "major" player always gets a zero payoff.

A reason why the kernel assigns the unintuitive $\frac{1}{2}:\frac{1}{2}$ split to a coalition of the apex player and a minor player is the assumption that, when players consider alternative coalitions, they assign to the other players the payoff they get in the current coalition structure. Hart and Kurz (1983) instead assume that payoffs are given by the Owen value[15], and that if a group of players deviates the new payoffs are given by the Owen value of the new coalition structure. A coalition structure is then considered to be stable if no group of players can reorganize themselves in such a way that all its members are strictly better-off.

The payoff division corresponding to a coalition of the apex player with a minor player is $\left(\frac{1}{2},\frac{1}{2}\right)$ if the other minor players are together in a coalition and $\left(\frac{n-2}{n-1},\frac{1}{n-1}\right)$ if they remain singletons (the kernel makes the same prediction for both structures). This coalition structure is not stable because the apex player can form a coalition (larger than minimal winning) with other minor players. The coalition of all small players is stable if $n \geq 5$ and coalitions break up when one member leaves.

Aumann and Myerson (1988) consider a link formation game in which players are offered the opportunity to form links and payoffs are determined by the Myerson[16] value of the resulting graph. The $\frac{1}{2}:\frac{1}{2}$ split in a two-player coalition is then justified since communication is not possible among players in different coalitions, so that the apex and the minor player are in a symmetric situation. Because of this, the big player prefers to form a coalition larger than the "minimal winning".

In equilibrium, all minor players form a coalition. The reason is that, if any of them links with the apex player, the apex player will then link with his "optimal" number of minor players, and each minor player would get less than what he would get if he linked with all the other minor players.

[15] The Owen value assigns to a player his average contribution, where the average is computed over the orderings that are "consistent" with the coalition structure, i.e., players arrive randomly but players in the same coalition arrive successively.

[16] The Myerson value is the Shapley value of the graph-restricted game.

4.3 Imperfect competition

Chatterjee et al. (1993) consider a proposal-making model in which a rule of order selects the first proposer and the order in which players respond to a proposal, and the first player to reject becomes the next proposer. They predict that a coalition of the apex player and a minor player will form and split the payoff equally regardless of the number of players. Expected payoffs depend on the rule of order and the strategy of the apex player. If the first proposer is selected randomly and the apex player randomizes among all minor players expected payoffs are $(\frac{1}{2}; \frac{1}{n}; ...; \frac{1}{n})$.

The reason for the equal-split prediction is that the game fails to reflect competition between the minor players.

Suppose the minor players propose to the apex player. A minor player who rejects an offer will propose to the apex player in the next period and get a continuation payoff $z_m = \delta(1 - \delta z_a)$; this payoff is the same for all minor players and does not depend on the proposing strategy of the apex player: If the apex player rejects a proposal, he gets $z_a = \delta(1 - \delta z_m)$ no matter to which minor player he proposes. These two equations determine z_a and z_m independently of the number of minor players. In the present paper, the payoff of a player who rejects a proposal depends on how often other players propose to him, so that competitive pressures are reflected in the expected payoffs of the players[17].

5 Possible extensions?

The main result of this paper, namely the fact that expected payoffs are proportional to the number of votes, easily extends to all simple games with one large player and $n - 1$ identical small players. It certainly does not extend to all weighted majority games. This section includes two counterexamples, one for the egalitarian protocol and one for the proportional protocol.

[17] Note that in the game considered by Chatterjee et al. (1993) expected payoffs and continuation payoffs may be very different. Consider a protocol that always appoints a given minor player i to be the proposer. This implies that the expected payoff for a minor player $j \neq i$ (w_j in the notation of this paper) equals 0, whereas his continuation payoff z_i is approximately $\frac{1}{2}$. In the game we consider there is a close relation between expected payoffs and continuation payoffs. Since nature selects a new proposer once a proposal is rejected, we have $z_i = \delta w_i$.

5.1 Counterexample 1: egalitarian protocol

Consider a weighted majority game in which two large players have $k \geq 2$ votes each, $2k - 1$ players have one vote each and $2k$ votes are needed to obtain a majority. The number of players is then $n = 2k + 1$, and the total number of votes is $4k - 1$. There are two types of minimal winning coalitions: the two large players, and a large player together with k small players. Suppose expected ex ante payoffs are proportional to the number of votes of the players; this implies that players will propose only minimal winning coalitions. A small player will then include one and only one of the large players in the coalition he proposes; a large player is indifferent between proposing to the other large player or to k minor players.

One may suspect that a large player cannot get a payoff proportional to his number of votes for n large enough. As n grows, a large player becomes a proposer less often, thus he must be a responder more often to keep his expected payoff equal to his proportion of votes $\frac{k}{4k-1}$. Since the proportion of the total votes a large player has is roughly constant regardless of the number of players, his probability of being a responder must be close to 1 when the number of players is large. However, there is an upper bound to his probability of being the responder since a small player proposes to only one of the large players (together with $k - 1$ small players).

The expected payoff for a large player w_l must satisfy the following equation, where λ is the probability with which a large player proposes to the other large player[18]

$$w_l = \frac{1}{n}[1 - \delta w_l] + \frac{n-2}{n}\frac{1}{2}\delta w_l + \frac{1}{n}\lambda \delta w_l.$$

The maximum possible value of w_l (corresponding to $\lambda = 1$) is smaller than $\frac{k}{4k-1}$ for any $k \geq 2$!

One can prove that in equilibrium a small player will propose a coalition of himself and the two large players with positive probability and a large player receives $k - 1$ times what a small player receives instead of k times. Thus, large players are underpaid and coalitions larger than minimal form.

[18] This formula takes into account that the small players must propose a coalition to each of the large players with probability $\frac{1}{2}$; in order for the two large players to have the same expected payoff.

5.2 Counterexample 2: proportional protocol

Consider a weighted majority game with four players, one of them with 3 votes, two of them with 2 votes and one of them with 1 vote. 5 votes are needed to obtain a majority, and thus there are two types of minimal winning coalitions: a player with 3 votes together with one of the players with 2 votes, or the two players with 2 votes together with the player with 1 vote. Thus, the largest and the smallest player are never together in a minimal winning coalition. If expected ex ante payoffs are proportional, only minimal winning coalitions form in equilibrium, thus the player with 3 votes must propose to one of the players with 2 votes, and the player with 1 vote to both of them. A player with 2 votes can propose two minimal winning coalitions, one including the largest player and the other including the smallest player.

Suppose that expected equilibrium payoffs are proportional to the number of votes under a proportional protocol. If μ is the probability that a player with 2 votes proposes to the largest player, the following equations must be satisfied:

$\frac{3}{8} = \frac{3}{8}(1 - \delta\frac{2}{8}) + \frac{4}{8}\mu\delta\frac{3}{8}$; describing the expected payoff for the largest player.

$\frac{1}{8} = \frac{1}{8}(1 - 2\delta\frac{2}{8}) + \frac{4}{8}(1 - \mu)\delta\frac{1}{8}$; describing the expected payoff for the smallest player.

For δ close to 1, μ needs to be close to $\frac{1}{2}$ for the largest player to receive a proportional payoff, and close to 0 for the smallest player to receive a proportional payoff, a contradiction.

6 Concluding remarks

This paper considers an application of the Baron and Ferejohn (1989) bargaining model to apex games. Expected ex ante payoffs are found to be proportional to the number of votes of the players, and this result is robust to (not too extreme) changes in the protocol. The probability of a coalition being formed as well as the expected division of payoffs given that a coalition is formed depends on the protocol.

The proposer has two sources of advantage in this game: the impatience of the players and the majority (rather than unanimity) rule. In fact, a proposer always gets more than half of the total payoff regardless of the number of players. The reason is that each responder receives a payoff proportional to his share of the total votes, and, since only minimal winning coalitions form, the sum of these shares is always smaller than $\frac{1}{2}$. This is striking if we think of large apex games with a minor player selected as a proposer.

Expected payoffs proportional to the number of votes may be desirable for fairness reasons. However, the fairness argument only applies for homogeneous majority games (games in which each minimal winning coalition has the same number of votes). This paper concerns homogeneous apex games only. If, the game is not homogeneous, the result that all minor players have the same expected payoff is not obviously attractive. Moreover, the assumption that the protocol treats all the minor players equally and the requirement that the minor players follow similar strategies are less reasonable if the game is not homogeneous.

7 Appendix

Proof of Lemma 5. Corollary 4 implies that the apex player will propose a coalition of the form $\{1, i\}$ $(i \neq 1)$ whereas a minor player i will propose $\{1, i\}$ or $N \backslash \{1\}$. Symmetry requires each minor player to propose coalition $N \backslash \{1\}$ (and thus coalition $\{1, i\}$) with the same probability.

The equilibrium expected payoff for a minor player is then given by the following expression:

$$w_i = \theta_m w_i^{\omega} + (n-2)\theta_m(1-\lambda)\delta w_i + \theta_a \mu_i \delta w_i \qquad (2)$$

where θ_m denotes the probability each minor player has to be the proposer, θ_a denotes the probability for the apex player to be the proposer ($\theta_a = 1-(n-1)\theta_m$), λ denotes the probability that a minor player proposes to the apex player, and μ_i denotes the probability that the apex player proposes to player i ($\sum_{j \in N \backslash \{1\}} \mu_j = 1$).

Solving for w_i in (2), we get

$$w_i = \frac{\theta_m w_i^{\omega}}{1-\delta((n-2)\theta_m(1-\lambda)+\theta_a \mu_i)} \qquad (3)$$

We can distinguish two cases:

a) $\lambda > 0$ (the minor players propose to the apex player with positive probability).

b) $\lambda = 0$ (the minor players never propose to the apex player).

Case a): $\lambda > 0$ implies $w_i^i = 1 - \delta w_1 = w_j^j \; \forall i, j \in N \setminus \{1\}$. Thus, given two minor players i and j, w_i can only be different from w_j if $\mu_i \neq \mu_j$. We now prove that $\mu_i = \mu_j \; \forall i, j \in N \setminus \{1\}$.

Suppose not, say, $\mu_i > \mu_j$. From (3), this implies $w_i > w_j$. But this in turn implies $\mu_i = 0$ (the apex player will never propose to player i, since he can do better by proposing to player j), a contradiction. Thus $\mu_i = \mu_j = \frac{1}{n-1}$ (since by Corollary 4 the apex player always proposes to a minor player) and $w_i = w_j$ for all $i, j \in N \setminus \{1\}$.

Case b): Take a player j such that $\mu_j > 0$. The apex player will only propose to a minor player if he is (one of) the cheapest, thus it must be the case that $w_j = \min_{k \in N \setminus \{1\}} w_k$.

Suppose not all minor players have the same equilibrium expected payoff. Then $\exists i \in N \setminus \{1\}$ s.t. $w_i > w_j$. By assumption $\lambda = 0$, so each minor player proposes the coalition of all minor players in equilibrium. Moreover, optimization by the apex player implies $\mu_i = 0$.

Equilibrium expected payoffs for i and j are then given by the following expressions:

$$w_i = \theta_m \Big(1 - \sum_{k \in N \setminus \{1,i,j\}} \delta w_k - \delta w_j\Big) + (n-2)\theta_m \delta w_i \tag{4}$$

$$w_j = \theta_m \Big(1 - \sum_{k \in N \setminus \{1,i,j\}} \delta w_k - \delta w_i\Big) + (n-2)\theta_m \delta w_j + \theta_a \mu_j \delta w_j \tag{5}$$

Substracting (5) from (4) and re-arranging terms, we get

$$\Delta = -\frac{\theta_a \mu_j \delta w_j}{1 - (n-1)\delta\theta_m} \tag{6}$$

where Δ denotes $w_i - w_j$.

Since $1 - (n-1)\delta\theta_m > \theta_a > 0$ and $\theta_a \mu_j \delta w_j > 0$, we get $\Delta < 0$, a contradiction. Therefore, all minor players must have the same equilibrium payoff. ■

References

Aumann, R. and Maschler, M. (1964): The Bargaining Set for Cooperative Games, in Advances in Game Theory (M. Dresher, L.S. Shapley and A.W. Tucker, eds), 443-476.

Aumann, R. and Myerson, R. (1988). Endogenous formation of links between players and of coalitions: an application of the Shapley value, in The Shapley value, Essays in Honor of Lloyd Shapley (A. Roth, ed). Cambridge: Cambridge University Press.

Baron, David P., and Ferejohn, John A. (1989). Bargaining in Legislatures. American Political Science Review, 83, 1181-1206.

Bennett, E. (1983). The Aspiration Approach to Predicting Coalition Formation and Payo¤ Distribution in Sidepayment Games. International Journal of Game Theory 12, 1-28.

Bennett, E. (1997). Multilateral Bargaining Problems. Games and Economic Behavior 19, 151-79.

Bennett, E. and van Damme, E. (1991). Demand Commitment Bargaining: The Case of Apex Games, in Game Equilibrium Models, Vol III, Strategic Bargaining (R.Selten, ed.) 118-140. Berlin: Springer-Verlag.

Chatterjee, K., Dutta, B., Ray, D. and Sengupta, K. (1993). A Noncooperative Theory of Coalitional Bargaining. Review of Economic Studies 60, 463-477.

Davis, M. and Maschler, M. (1965). The Kernel of a Cooperative Game. Naval Research Logistics Quarterly 12, 223-259.

Hart, S. and Kurz, M. (1983). Endogenous Formation of Coalitions, Econometrica 51, 1047-1064.

Hart, S. and Kurz, M. (1984). Stable Coalition Structures, in Coalitions and Collective Action, ed. by M.J. Holler. Physica-Verlag, Wuerzburg (Germany).

Maschler, M. (1992). The Bargaining Set, Kernel, and Nucleolus, in Handbook of game theory with economic applications, volume 1, ed. by R.J. Aumann and S. Hart. North-Holland, 1992, 591-667.

Morelli, M. (1998). The Stable Aspirations Set. Characterization and Implementation for Majority Games. Manuscript.

Okada, A. (1996). A Noncooperative Coalitional Bargaining Game with Random Proposers, Games and Economic Behavior 16, 97-108.

Rapoport, A., Kahan, J., Funk, S. and Horowitz, A. (1979). Coalition Formation by Sophisticated Players. Springer-Verlag.

Rubinstein, A. (1982). Perfect Equilibrium in a Bargaining Model. Econometrica 50 , 97-108.