On the extreme points of the core of neighbour games and assignment games

Hamers, H.J.M.; Klijn, F.; Solymosi, T.; Tijs, S.H.; Pere Villar, J.

Publication date:
1999

On the Extreme Points of the Core of Neighbour Games and Assignment Games

HERBERT HAMERS*, FLIP KLIJN*,
TAMÁS SOLYMOSI†, STEF TIJS*, AND JOAN PERE VILLAR‡

Abstract

Neighbour games arise from certain matching or sequencing situations in which only some specific pairs of players can obtain a positive gain. As a consequence, neighbour games are as well assignment games as line graph restricted games. We will show that the intersection of the class of assignment games and the class of line graph restricted games yields the class of neighbour games.

Further, we give a necessary and sufficient condition for the convexity of neighbour games. In spite of the possible non-convexity of neighbour games, it turns out that for any neighbour game the extreme points of the core are marginal vectors. Moreover, we prove this for assignment games in general. Hence, for any assignment game the core is the convex hull of some marginal vectors.

Journal of Economic Literature Classification Numbers: C71, C78
Keyword: Assignment games

* CentER and Department of Econometrics, Tilburg University.

† Department of Operations Research, Budapest University of Economic Sciences. This author's work has been supported by CentER and Department of Econometrics, Tilburg University, and also the Foundation for the Hungarian Higher Education and Research (AMFK)

‡ Departament d'Economia I d'Historia Economica, Universitat Autonoma de Barcelona.

1 Introduction

In this paper we introduce a class of cooperative games, called neighbour games, and show that these games satisfy the CoMa-property, i.e. the core of these games is the convex hull of some marginal vectors. Moreover, we prove that all assignment games (cf. [10]) satisfy the CoMa-property.

The appealing feature of the CoMa-property is that the extreme points of the core are exactly those marginal vectors that are in the core. Hence, for these games it is rather easy to generate core elements. A well-known class of games that satisfies the CoMa-property is the class of convex games: the core of a convex game is the convex hull of all marginal vectors, (cf. [9], [5]). A non-convex class of games that satisfies the CoMa-property is the class of information games, (cf. [6]), which is a subclass of minimum cost spanning tree games (cf. [3]).

The following two examples describe situations that give rise to neighbour games. In the first example we consider a sequencing situation in which customers are lined in a queue and waiting for a taxi. The taxi company that provides the service has two types of cars: one that transports only one customer (type A) and one that can only transport two customers (type B). The first customer in the queue can decide to pick a taxi of type A or wait for the next customer in the queue. In the latter case they decide both to share a taxi of type B or the second customer will wait on the third customer. In the latter case the first customer has to pick a taxi of type A. This procedure is repeated until all customers are transported in a taxi. Since the costs of sharing a taxi of type B are lower than taking two taxis of type A, it is obvious that the customers can save costs by sharing a taxi of type B. However, each customer faces the problem that the cost of a taxi (of type B) is not fixed, because it depends on the trip to bring the customers to the right locations. Hence, we have that only customers that are neighbours in the queue can obtain cost savings, and customers that take a taxi of type A have cost savings equal to zero. All customers in the queue want to choose a combination of taxis of type A and B such that their cost savings are maximized. Moreover, they looking for an allocation of these cost savings that satisfies some specific properties.

The second example can be viewed as a restricted matching problem. Suppose a river runs through several countries. To be able to utilize this cheap transportation possibility, harbours have to be built. Each country is able, from financial viewpoint, to build at most one harbour. Neighbour countries might join to build a harbour at their border (which then can serve both countries) and save costs. The countries are interested in maximizing their cost savings and finding some proper allocation of these cost savings.

For analysing both examples we can use cooperative game theory, since one of the topics in cooperative game theory is the investigation of the stability of allocation rules, i.e. whether the allocation is contained in the core of the related cooperative game. For this purpose we introduce neighbour games.

A neighbour game can be viewed as an assignment game and as a σ-component additive game (cf. [2]). The latter one is a special type of Γ-component additive game (cf. [8]) where the restricted graph is a line graph.

More precisely, we show that the intersection of these two classes of games yield the class of neighbour games. As a consequence, neighbour games has many appealing properties, such as: the core is a non-empty set and coincides with the set of competitive equilibria ([10]), the core is equal to the bargaining set and the nucleolus coincides with the kernel ([8]), the existence of easy algorithms to calculate the nucleolus for neighbour games ([4]). Besides, neighbour games satisfy the already mentioned CoMa-property and some egaliterian solutions can be easily obtained ([4]).

This paper is organized as follows. Section 2 provides the necessary definitions of the relevant games and presents the intersection result. Convexity and the CoMa-property of neighbour games are discussed in Section 3. Finally, the proof of the CoMa-property of assignment games is provided in Section 4.

2 Neighbour games: assignment games and component additive games

In this section we introduce neighbour games. We show that the intersection of the class of assignment games and the class of σ-component additive games results in the class of neighbour games. Before we present this result we need the following notions from cooperative game theory.

A *transferable utility cooperative game* is an ordered pair (P, v) where $P = \{1, ..., p\}$ is a finite set of players and $v : 2^P \to I\!R$ is a map that assigns to each coalition $S \in 2^P$ a real number $v(S)$, such that $v(\emptyset) = 0$. Here 2^P is the collection of all subsets (coalitions) of P.

Assignment games, introduced by Shapley and Shubik ([10]), arise from bipartite matching situations. Let M and N be two disjoint sets. For each $i \in M$ and $j \in N$ the value of a matched pair (i, j) is $a_{ij} \geq 0$. From this situation an assignment game is defined in the following way. On the player set $M \cup N$, the worth of coalition $S \cup T, S \subseteq M, T \subseteq N$ (that will be denoted by (S, T) later on) is defined to be the maximum that (S, T) can achieve by making suitable pairs from its members. If $S = \emptyset$ or $T = \emptyset$ no suitable pairs can be made and therefore the worth in this situation is 0. Formally, an *assignment game* $((M, N), v)$ is defined for all $(S, T), S \subseteq M, T \subseteq N$ by

$$v(S, T) = \max\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S, T)\},$$

where $\mathcal{M}(S, T)$ denotes the set of matchings between S and T.

Component additive games, introduced by Curiel et al. ([2]), are a special class of Γ-component additive games, discussed in ([8]), which in turn are a special class of graph restricted games in the sense of Owen ([7]). Let (P, v) be a cooperative game and let $\Gamma = (P, E)$ be a undirected line graph. Then a *component additive game* (P, w_Γ) is defined for each $S \subseteq P$ by

$$w_\Gamma(S) = \sum_{T\in S\backslash\Gamma} v(T),$$

where $S\backslash\Gamma$ is the set of connected components of S with respect to Γ.

The situations discussed in the introduction that motivate the interests for neighbour games, give rise to a model in which players are lined up in a one-dimensional queue. In this queue, players can only directly cooperate with at most one of their neighbours in this queue. From this point of view neighbour games are defined as restricted assignment games: only pairs that are neighbours in the queue can be matched. Formally, let P be the player set and $\sigma : P \rightarrow \{1, ..., p\}$ be an order on P. Obviously, P can be partioned in M and N such that M contains the players in odd positions and N the players in even positions according to σ. Let $a_{ij} \geq 0$ if the players $i \in M, j \in N$ in the pair (i, j) are neighbours, i.e. either $\sigma(j) = \sigma(i) + 1$ or $\sigma(i) = \sigma(j)$, and $a_{ij} = 0$ otherwise. Then a *neighbour game* is defined for all $(S, T), S \subseteq M, T \subseteq N$ by

$$w(S, T) = \max\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{N}(S, T)\},$$

where $\mathcal{N}(S, T)$ is the set of matchings between S and T in which each matching only consists of pairs (i, j) that are neighbours.

Example 2.1 Let $P = \{1, 2, 3, 4\}$ be the player set and let σ describe the order $1 \prec 2 \prec 3 \prec 4$. The pairs that are neighbours with respect to σ are (1,2), (3,2) and (3,4). Hence, all other pairs have a worth equal to zero. Take, for instance, $a_{12} = 1, a_{32} = 2$, and $a_{34} = 3$. Then the corresponding neighbour game $(M \cup N, w)$, where $M = \{1, 3\}$ and $N = \{2, 4\}$, is depicted in Table 2.1.

S	{1,2}	{1,2,3}	{1,2,4}	{1,3,4}	{2,3}	{2,3,4}	{3,4}	{1,2,3,4}
w(S)	1	2	1	3	2	3	3	4

Table 2.1: a neighbour game.

□

Let (P, w) be a neighbour game that arises from an order σ and let (P, w_Γ) be the component additive game that arises from this neighbour game and the line graph Γ in which the vertices are ordered according to σ. It easy to verify that $w(S) = w_\Gamma(S)$ for all $S \subseteq P$. Hence, from the definition of neighbour games and this latter argument we conclude that any neighbour game is as well an assignment game as a component additive game. The next Proposition shows that also the reverse holds.

Proposition 2.2 *Let $\mathcal{NB}$, $\mathcal{A}$, and $\mathcal{CA}$ be the classes of neighbour games, assignment games, and component additive games, respectively, consisting of n players. Then*

$$\mathcal{NB} = \mathcal{A} \cap \mathcal{CA}.$$

PROOF: From the argument before Proposition 2.2 and the definition of neighbour games as restricted assignment games if follows that we only need to show that $\mathcal{A} \cap \mathcal{CA} \subset \mathcal{NB}$.

Let $(P, v) \in \mathcal{A} \cap \mathcal{CA}$. Since $(P, v) \in \mathcal{A}$ there exists a partition $P = (M, N)$ and a non-negative matrix $[a_{ij}]_{(i,j)\in(M,N)}$ that generates (P, v). Since $(P, v) \in \mathcal{CA}$, there exists a line graph Γ that orders the players according to some map σ. Because $v(\{i\}) = 0$ for all $i \in P$, we have that $v(\{i, j\}) = 0$ whenever i and j are not neighbours with respect to σ. Since $v(\{i, j\}) = a_{ij}$ we can conclude that (P, v) is a neighbour game. □

3 On the extreme points of the core of neighbour games

In this section we investigate the core of neighbour games. We will present the result that the core of a neighbour game is the convex hull of the marginal vectors that are in the core of the game. This property is henceforth called the CoMa-property. As a consequence, we have that each extreme point of the core coincides with at least one marginal vector. Moreover, we give a necessary and sufficient condition for the convexity of neighbour games. Before we state this result we will recall the notions of the core, convexity, and introduce the CoMa-property.

The core of a game (P, v) consists of all vectors that distribute the gains $v(P)$ obtained by P among the players in such a way that no subset of players can be better off by seceding from the rest of the players and act on their own behalf. Formally, the *core* of a game (P, v) is

$$Core(v) = \{x \in I\!R^P \mid x(S) \geq v(S) \text{ for all } S \subset P \text{ and } x(P) = v(P)\},$$

where $x(S) = \sum_{i \in S} x_i$. In general, the core may be an empty set. A game is called balanced whenever its core is non-empty.

Assignment games and component additive games are both balanced games. As a consequence of Proposition 2.2, neighbour games are also balanced. Moreover, Potters and Reijnierse ([8]) showed that for Γ-component additive games, in which Γ is a tree, the bargaining set coincides with the core and the kernel coincides with the nucleolus. Hence, these two features also hold for neighbour games. Moreover, Raghavan and Solymosi ([11]) provided an algorithm to calculate the nucleolus of assignment games. This algorithm has been simplified by Hamers et al. ([4]) to calculate the nucleolus of neighbour games.

In this section we concentrate on the extreme points of the core of neighbour games. We need the notion of a *marginal vector* of a game (P, v). Let $\Pi(P)$ be the set of all permutations of $P = \{1, 2, ..., p\}$. Then the i-th coordinate of the *marginal vector* $m^\pi(v)$ is defined by

$$m_i^\pi(v) = v(\{j \in P \mid \pi(j) \leq \pi(i)\}) - v(\{j \in P \mid \pi(j) < \pi(i)\}).$$

Now, we are able to define the CoMa-property for a cooperative game. A game (P, v) satisfies the *Core is convex hull of Marginals (CoMa-) property* if

$$Core(v) = conv\{m^\pi(v) \mid m^\pi(v) \in Core(v)\}. \tag{1}$$

Hence, the CoMa-property yields that the core is the convex hull of some marginal vectors.

A well-known class of games that satisfy the CoMa-property is the class of convex games. A game (P, v) is called *convex* if for all $i \in P$ and all coalitions S and T with $S \subset T \subseteq P \backslash \{i\}$ it holds that

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Shapley ([9]) and Ichiishi ([5]) showed that a game is convex if and only if each marginal vector is an extreme point of the core. From this result the CoMa-property follows immediately, since the core is a convex set.

The next example shows that neighbour games need not be convex.

Example 3.1 Consider the player set $M = \{1, 3\}$, $N = \{2\}$ and let the values of the neighbour pairs be $a_{12} = 2$ and $a_{32} = 1$. Then the worth of the coalitions of the corresponding neighbour game $((M, N), w)$ is given in Table 3.1.

S	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
w(S)	0	0	0	2	0	1	2

Table 3.1: a non-convex neighbour game.

Take π such that $\pi(1) = 3, \pi(2) = 1$, and $\pi(3) = 2$.
Then $m^\pi(w) = (1, 0, 1) \notin Core(w)$. Hence $((M, N), w)$ is not convex. □

The following Proposition provides a necessary and sufficient condition for the convexity of neighbour games.

Proposition 3.2 *Let P be a player set that is partitioned into M and N according to the order $\sigma : 1 \prec 2 \prec ... \prec n$. Let $((M, N), w)$ be the corresponding neighbour game. Then $((M, N), w)$ is convex if and only if for any triple $j-1, j, j+1 \in P$ of consecutive players according to σ it holds that $w(\{j-1, j\}) = 0$ or $w(\{j, j+1\}) = 0$.*

PROOF: We first prove the 'only if' part. Suppose that $w(\{j-1,j\}) > 0$ and $w(\{j,j+1\}) > 0$ for some $j \in P$. Then

$$\begin{aligned} & w(\{j-1,j,j+1\}) - w(\{j-1,j\}) \\ = \; & \max\{w(\{j-1,j\}), w(\{j,j+1\})\} - w(\{j-1,j\}) \\ = \; & \max\{0, w(\{j,j+1\}) - w(\{j-1,j\})\} \\ < \; & w(\{j,j+1\}) - w(\{j\}). \end{aligned}$$

Hence, (P,w) is not convex.
Second, we prove the 'if' part. For any $S \subset T \subset P$ and $k \in P \backslash T$ we have

$$\begin{aligned} w(T \cup \{k\}) - w(T) &= \sum_{i \in A \cap T} w(\{i,k\}) \\ &\geq \sum_{i \in A \cap S} w(\{i,k\}) \\ &= w(S \cup \{k\}) - w(S), \end{aligned}$$

where A is the set defined by

$$A = \begin{cases} \{k-1,k+1\} & \text{if } k \neq 1,n \\ \{2\} & \text{if } k = 1 \\ \{n-1\} & \text{if } k = n \end{cases}$$

□

Although neighbour games need not be convex, they satisfy the CoMa-property.

Theorem 3.3 *Neighbour games satisfy the CoMa-property.*

The proof is omitted since Theorem 3.3 is an immediate consequence of Proposition 2.2 and Theorem 4.5 of the next section.

4 On the extreme points of the core of assignment games

In this section we show that assignment games satisfy the CoMa-property. We first show that we can restrict attention to assignment games in which the cardinality of the disjoint sets that have to be matched are equal. After that we provide a relation between the extreme points in the core of assignment games and the components of the corresponding tight graph. Finally, we provide the proof of the CoMa-property. However, before we can provide the proof of this result, we need some preparations.

Let $((M,N),w)$ be an assignment game. Then an allocation of the grand coalition, $w(M,N)$, will sometimes, for convenience, be denoted by $(u,v) \in I\!R^M \times I\!R^N$, where u and v are the vectors that correspond to the payoffs of the players in M and N, respectively.

The following Lemma, due to Shapley and Shubik ([9]), shows that each pair (i,j) that is in an optimal matching between M and N shares in any core allocation the reward a_{ij}.

Lemma 4.1 *Let $((M,N),w)$ be an assignment game and let μ be an optimal matching between M and N. Then for any $(i,j) \in \mu$ and $(u,v) \in Core(w)$ it holds that $u_i + v_j = w(\{i,j\})$.*

As a consequence of Lemma 4.1, each player that is not matched in an optimal matching between M and N obtains in each core allocation a payoff equal to zero.

Let $((M,N),w)$ be an assignment game in which $\mid M \mid<\mid N \mid$. Let μ be an optimal matching between M and N. Then μ induces a set $N_M \subset N$ such that M and N_M are completely matched, which implies that $\mid M \mid=\mid N_M \mid$. If $((M,N_M),\overline{w})$ is the restricted assignment game of $((M,N),w))$, then it is straightforward to verify that there is a one-to-one correspondence between the extreme points of their cores, i.e. $x \in ext\{Core(\overline{w})\}$ if and only if $y \in ext\{Core(w)\}$, where $y_i = x_i$ if $i \in M \cup N_M$ and $y_i = 0$ otherwise. The following Lemma shows that it is sufficient to prove the CoMa-property for assignment games that arise from situations in which $\mid M \mid=\mid N \mid$.

Lemma 4.2 *Let $((M,N),w)$ be an assignment game in which $\mid M \mid<\mid N \mid$ and let $((M,N_M),\overline{w})$ be the restricted assignment game. If $((M,N_M),\overline{w})$ satisfies the CoMa-property, then $((M,N),w)$ satisfies the CoMa-property.*

PROOF: Let y be an extreme point of the $Core(w)$ and let x be the corresponding extreme point in $Core(\overline{w})$. Since $((M, N_M), \overline{w})$ satisfies the CoMa-property, there exists an order σ on (M, N_M) such that $x = m^{\sigma}(\overline{w})$. Then it is straightforward to verify that any order σ' on (M, N), defined by $\sigma'(j) = \sigma(j)$ if $j \in M \cup N_M$ and $\sigma(j) \geq \mid M \cup N_M \mid$ otherwise, leads to the marginal vector $m^{\sigma'}(w)$ that is equal to y. □

As a consequence of Lemma 4.2 we can restrict our discussion in the remaining part of this section to assignment games that arise from situations in which $\mid M \mid = \mid N \mid$. Since then we can regard N as a disjoint copy of M, without loss of generality we may also assume that one optimal matching between M and N is the one that matches all identical pairs (i, i). This optimal matching will be denoted by μ^*.

Given an assignment game $((M, N), w)$ and a core allocation $(u, v) \in Core(w)$, in the *tight graph* $G^w(u, v) = (V, E)$, the set of vertices V equals the player set (M, N) and the edge set is defined by $E = \{(i, j) \mid i \in M, j \in N, u_i + v_j = w(\{i, j\})\}$. In a tight graph we distinguish between two types of edges with respect to μ^*. All edges corresponding to μ^* are referred to as thick edges and all other edges are referred to as thin edges. Note that according to the assumption on μ^*, we have that the thick edges are the pairs (i, i). A *tight tree*, which is a subgraph of a component of a tight graph, is a tree that covers all vertices of the component and contains all thick edges. Notice that a tight tree need not be uniquely determined by the tight graph. The following Lemma establishes a relation between the extreme points of an assignment game and the components of the corresponding tight graph.

Lemma 4.3 *Let $((M, N), w)$ be an assignment game. Then $(u, v) \in ext\{Core(w)\}$ if and only if each component of the tight graph $G^w(u, v)$ contains at least one player with payoff equal to zero.*

PROOF: First, we show the 'only if' part. Let $(u, v) \in ext\{Core(w)\}$ and let C be a component of $G^w(u, v)$ in which the vertices are (S, T). Since μ^* is an optimal

matching and C is connected, we have that $\mid S \mid = \mid T \mid$. Suppose that the restriction of (u,v) to (S,T), denoted by $(u,v)_{|(S,T)}$, has only positive elements. Then by Lemma 4.1, for sufficiently small $\epsilon > 0$ we have that the vectors $x, y \in I\!R^S \times R^T$ defined by $x_i = u_i + \epsilon$, $y_i = u_i - \epsilon$ for all $i \in S$; $x_j = v_j - \epsilon, y_j = v_j + \epsilon$ for all $j \in T$, are both in $Core(w_{|(S,T)})$ such that $\frac{1}{2}x + \frac{1}{2}y = (u,v)_{|(S,T)}$. This implies that also (u,v) itself can be written as a convex combination of two different vectors in $Core(w)$, which contradicts the fact that $(u,v) \in ext\{Core(w)\}$. Hence, the 'only if' part of the Lemma follows.
To see the 'if' part, we have to show that the system

$$u(S) + v(T) \geq w((S,T)) \text{ for all } S \subseteq M, T \subseteq N, \quad (2)$$

contains $2 \mid M \mid$ tight equations that are linear independent. Assume that the tight graph $G^w(u,v)$ can be partitioned in k components, say $C_1, C_2, ..., C_k$. Each component C_i contains a tight tree. Then the system of equations, generated by the edges of such a tree, is a linear independent system (cf. [1]). Hence, we have $\sum_{i=1}^{k}(\mid C_i \mid -1)$ linear independent tight equations. Combining these equations with the tight equation in each component, that is generated by the player with zero payoff, we obtain a system of $\sum_{i=1}^{k}(\mid C_i \mid) = 2 \mid M \mid$ linear independent equations. Hence, we can conclude that $(u,v) \in ext\{Core(w)\}$. □

The following Lemma provides the worth of some specific $(r-s)$-path coalitions. Here, an $(r-s)$-*path coalition* consists of all players that are contained in the path between r and s in a tight graph, where r and s are both in the same component.

Lemma 4.4 *Let (u,v) be an extreme point of the core of an assignment game $((M,N),w)$ and let $(r-s)$ be a path in a tight tree of $G^w(u,v)$ such that vertex r corresponds to a player that has a payoff equal to zero in (u,v). If S is an $(r-s)$ path coalition, then $w(S) = \sum_{j \in S \cap M} u_j + \sum_{j \in S \cap N} v_j$.*

PROOF: Let μ be the complete matching that covers S, except r in case the cardinality of S is odd, and consists only of edges contained in the $(r-s)$ path. Without loss of generality we may assume that u_1 corresponds to the vertex r, i.e.

$u_1 = 0$. Then from the definition of μ, the definition of a tight graph, and the assumption that $u_1 = 0$ it follows that

$$\sum_{j \in S \cap M} u_j + \sum_{j \in S \cap N} v_j = \sum_{(i,j) \in \mu} a_{ij}. \tag{3}$$

From the definition of an assignment game and (3) we have that

$$w(S) \geq \sum_{j \in S \cap M} u_j + \sum_{j \in S \cap N} v_j. \tag{4}$$

Since (u, v) is a core-element we have that

$$\sum_{j \in S \cap M} u_j + \sum_{j \in S \cap N} v_j \geq w(S). \tag{5}$$

Combining (4) and (5) completes the proof. □

Now, we can present the main result of this section.

Theorem 4.5 *Assignment games satisfy the CoMa-property.*

PROOF: Let (u, v) be an extreme point of an assignment game $((M, N), w)$. We have to show that there exists some order π on the player set (M, N) such that the corresponding marginal vector $m^{\pi}(w)$ coincides with (u, v). First we prove the case when the tight graph $G^w(u, v)$ is connected, i.e. the tight graph consists of only one component. Let $T^w(u, v)$ be a tight tree of $G^w(u, v)$ and let $x = (u, v)$ be such that x_i corresponds to the payoff of player i. Then Lemma 4.3 implies that there exists a vertex r in the tight tree, in which player r has a payoff equal to zero. Next, we will label the vertices in the tight tree via a depth first search procedure. More specifically, initially all vertices in the tight tree are unlabeled. In the following procedure we label the vertices by the increasing sequencing of numbers $1, 2, ..., 2 \mid M \mid$.

Step 1: give vertex r label 1.

Step 2: let a be the vertex that is labeled last, say by k;

Procedure:

(i) if there exist a thin edge that connects a with an unlabeled vertex b, then give

vertex b label $k+1$, and repeat Step 2; otherwise go to (ii)
(ii) if there exists a thick edge that connects a with an unlabeled vertex b, then give vertex b label $k+1$, and repeat Step 2; otherwise go to (iii)
(iii) if there exists no edge that connects a with an unlabeled vertex then return to the lowest labeled vertex b that is connected with a, set a:=b and repeat the **Procedure**.

Let π be the order on the players that is generated by the labels assigned in the described depth first search procedure. We will show that $m^{\pi}(w) = x$. Let S_j be the set of the first j labeled players in the depth first search procedure. We prove that

$$w(S_j) = \sum_{i \in S_j} x_i. \tag{6}$$

Let player m be the player that is labeled last in S_j. Then coalition S_j can be partitioned in $S_j(1)$ and $S_j(2)$, where $S_j(1)$ are the players on the unique path from r to m and $S_j(2)$ are all other players of S_j. Then Lemma 4.4 implies that that

$$w(S_j(1)) = \sum_{i \in S_j(1)} x_i. \tag{7}$$

Obviously, the proof is completed if $S_j(2) = \emptyset$. Hence, we may assume that $S_j(2) \neq \emptyset$. We now show that there exists a matching on $S_j(2)$ that consists only of thick edges and covers $S_j(2)$. Let $a \in S_j(1)$ and let $b \in S_j(2)$ be such that (a,b) is an edge in the tight tree. Since there exists a path from a to m, there are at least three edges incident to a. Since vertex b is visited using edge (a,b) before m is visited, it follows from item (i) in the depth first search procedure that (a,b) has to be a thin edge. Obviously, all vertices in $S_j(2)$ are labeled before m is labelled the depth first search procedure. Since each vertex is incident to a thick edge, we can conclude that there exists indeed a matching of $S_j(2)$ that consists only of thick edges. This observation gives

$$w(S_j(2)) = \sum_{i \in S_j(2)} x_i \tag{8}$$

since the optimal matching is provided by the thick edges. Now, we have

$$\begin{aligned} w(S_j) &\geq w(S_j(1)) + w(S_j(2)) \\ &= \sum_{i \in S_j(1)} x_i + \sum_{i \in S_j(2)} x_i \\ &= \sum_{i \in S_j} x_i \\ &\geq w(S_j), \end{aligned}$$

where the first inequality holds since the merger of optimal matchings of $S_j(1)$ and $S_j(2)$ gives a matching for S_j, the first equality holds by (7) and (8), the second equality since $S_j(1)$ and $S_j(2)$ form a partition of S_j and the second inequality holds since x is in the core of the assignment game.

From (6) it follows immediately that

$$m^{\pi}_{j+1}(w) = w(S_{j+1}) - w(S_j) = x_{j+1},$$

which completes the proof in case the tight graph consists of one component.

Second, we prove the case in which the tight graph consists of more than one component. Suppose $G^w(u,v)$ consists of k components, say $C_1, ..., C_k$. Then from the first part it follows that there exists an order π_i on the player set S_i of C_i such that $m^{\pi_i}(w) = x_{|S_i}$ for all $1 \leq i \leq k$. Then it is straightforward to show that $m^{\pi}(w) = x$ where $\pi = (\pi_1, \pi_2, ..., \pi_k)$. □

The following example illustrates the outcome of the procedure used in the proof of Theorem 4.5 and shows that an extreme point can be generated by several marginal vectors.

Example 4.6 Let $x = (0,1,0,1,0,1,0,1)$ be an extreme point of the core of an assignment game (N, w). A tight tree that corresponds to x is depicted in figure 4.1. The weight of an edge is 1 if the edge is contained in the tight tree and 0 otherwise. The number in a vertex denotes the corresponding player.

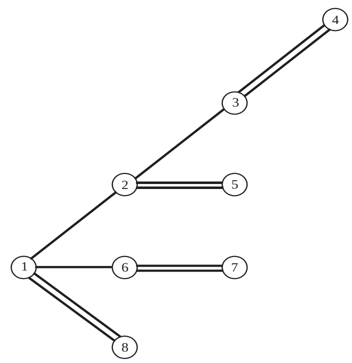

Figure 4.1: the tight graph $G^w(x)$.

Then the procedure, starting in the vertex corresponding to player 1, can give the order $\pi = (1, 2, 3, 4, 5, 6, 7, 8)$ and $\pi^* = (1, 6, 7, 2, 3, 4, 5, 8)$, respectively. Then it is easy to verify that $m^\pi(w) = m^{\pi^*}(w) = x$.

From the observations in Example 4.6 we propose the following allocation rule for a game (N, w) that satisfies the CoMa-property:

$$\gamma(w) = \frac{1}{\mid \{\pi : m^\pi(w) \in Core(w)\} \mid} \sum_{\pi : m^\pi(w) \in Core(w)} m^\pi(w).$$

Obviously, if (N, w) is convex, then γ equals the Shapley value. Otherwise γ can be considered as a generalized Shapley value with the property that its outcome is in the bary center of the core.

References

[1] CHVÁTAL V. (1983), Linear Programming, *W.H. Freeman and Company, USA.*

[2] CURIEL I., POTTERS J. RAJENDRA PRASAD V., TIJS S. AND VELTMAN B., Sequencing and cooperation, *Operations Research*, 42, 566-568.

[3] GRANOT D. AND HUBERMAN G. (1981), Minimum cost spanning tree games, *Mathematical Programming*, 21, 1-18.

[4] HAMERS H., KLIJN F., SOLYMOSI T., TIJS S. AND VERMEULEN D. (1999), On the nucleolus and egalitarian solutions for neighbour games, *working paper, Tilburg University*.

[5] ICHIISHI T. (1981), Super-modularity: applications to convex games and the greedy algorithm for LP, *Journal of Economic Theory*, 25, 283-286.

[6] KUIPERS J. (1994), Combinatorial methods in cooperative game theory. *PhD., University of Maastricht, The Netherlands.*

[7] OWEN G. (1986), Values of graph-restricted games, *SIAM Journal of Algebraic and Discrete Methods*, 7, 210-220.

[8] POTTERS J. AND REIJNIERSE H., Γ-component additive games, *International Journal of Game Theory*, 24, 49-56.

[9] SHAPLEY L. (1971), Cores of convex games, *International Journal of Game Theory*, 1, 11-26.

[10] SHAPLEY L. AND SHUBIK M. (1972), The assignment game I: The core, *International Journal of Game Theory*, 1, 111-130.

[11] SOLYMOSI T. AND RAGHAVAN T.E.S. (1994), An algorithm for finding the nucleolus of assignment games, *International Journal of Game Theory*, 23, 119-143.