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The Myerson value for union stable systems

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Abstract

We study cooperation structures with the following property: Given any two feasible coalitions with non-empty intersection, its union is a feasible coalition again. These combinatorial structures have a direct relationship with conference structures à la Myerson. Characterizations of the *Myerson value* in this context are provided by means of the introduction of the concept of *basis* for *union stable systems*.

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1 Introduction

Several models of restricted cooperation have been proposed, among which are those derived from *communication situations* as introduced by Myerson [4] [5]. This line of research was continued by Owen [8], Borm, Owen and Tijs [3], van den Nouweland, Borm and Tijs [6], van den Nouweland [7], Potters and Reijnierse [9] and Algaba et al. [1]. In Myerson's model, the bilateral relations among the players are represented by means of an undirected graph and the feasible coalitions are those that induce connected subgraphs.

In our restricted cooperation model, if two feasible coalitions have common elements, these ones will act as intermediaries between the two coalitions in order to establish meaningful cooperation in the union of these coalitions. These feasible coalition systems will be called *union stable systems*. Section 2 formally introduces our model of restricted cooperation. A relation is established between Myerson conference structures and union stable systems by means of the *basis* of a union stable system. Section 3 introduces the *Myerson value* for games restricted by union stable systems and studies in detail some properties of this

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value. The concept of basis allows to extend the axiomatic characterizations given for the Myerson value given by Myerson [4] and van den Nouweland [7].

2 Union stable systems

Definition 2.1 *Let $N = \{1, 2, \ldots, n\}$ be a finite set of players and $\mathcal{F} \subseteq 2^N$ a system of feasible coalitions. The set system $\mathcal{F}$ is called union stable if for all $A, B \in \mathcal{F}$ with $A \cap B \neq \emptyset$ it is satisfied that $A \cup B \in \mathcal{F}$.*

A communication situation is a triple (N, v, E), where (N, v) is a game and (N, E) is a simple graph. It is easy to see that the set system $\mathcal{F}$, defined by

$$\mathcal{F} = \{S \subseteq N : (S, E(S)) \text{ is a connected subgraph of } (N, E)\},$$

is union stable. However, a union stable system can not always be modelled by a communication situation. Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$. We define inductively the families

$$\mathcal{G}^{(0)} = \mathcal{G}, \;\; \mathcal{G}^{(n)} = \left\{ S \cup T : S, T \in \mathcal{G}^{(n-1)}, \; S \cap T \neq \emptyset \right\} \;\; (n = 1, 2, \ldots)$$

Notice that $\mathcal{G}^{(0)} \subseteq \mathcal{G}^{(n-1)} \subseteq \mathcal{G}^{(n)} \subseteq \mathcal{F}$, since $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{F}$ is union stable.

Definition 2.2 *Let $\mathcal{F}$ be a union stable system and let $\mathcal{G} \subseteq \mathcal{F}$. We define $\overline{\mathcal{G}}$ by $\overline{\mathcal{G}} = \mathcal{G}^{(k)}$, where k is the smallest integer such that $\mathcal{G}^{(k+1)} = \mathcal{G}^{(k)}$.*

We are interested to obtain, for each union stable family, a minimal subset that by the above process generates the whole union stable family. Let $\mathcal{F}$ be a union stable system and $\mathcal{G} \subseteq \mathcal{F}$. If $\mathcal{G}$ is union stable, there can be feasible coalitions which can be written as the union of two feasible coalitions with non-empty intersection. So, we can consider the following set:

$$\mathcal{D}(\mathcal{G}) = \{G \in \mathcal{G} : G = A \cup B, \; A \neq G, \; B \neq G, \; A, B \in \mathcal{G}, \; A \cap B \neq \emptyset\}.$$

Note that $\mathcal{D}(\mathcal{G})$ is composed of those feasible coalitions which can be written as the union of two distinct feasible coalitions with non-empty intersection.

Definition 2.3 *Let $\mathcal{F}$ be a union stable system. The set $\mathcal{B}(\mathcal{F}) = \mathcal{F} \setminus \mathcal{D}(\mathcal{F})$, is called the basis of $\mathcal{F}$, and the elements of $\mathcal{B}(\mathcal{F})$ are called supports of $\mathcal{F}$.*

We remark that the basis $B(\mathcal{F})$ is the minimal subset of the union stable system $\mathcal{F}$ such that $\overline{B(\mathcal{F})} = \mathcal{F}$ (see Algaba el al. [1]).

Definition 2.4 *Let $\mathcal{G} \subseteq 2^N$ be a set system and let $S \subseteq N$. A set $T \subseteq S$ is called a $\mathcal{G}$-component of S if it is satisfied that $T \in \mathcal{G}$ and there exists no $T' \in \mathcal{G}$ such that $T \subset T' \subseteq S$.*

Therefore, the $\mathcal{G}$-components of S are the maximal feasible coalitions that belong to $\mathcal{G}$ and are contained in S. We denote by $C_{\mathcal{G}}(S)$ the collection of the $\mathcal{G}$-components of S.

Proposition 2.1 *The set system $\mathcal{F} \subseteq 2^N$ is union stable if and only if for any $S \subseteq N$ with $C_{\mathcal{F}}(S) \neq \emptyset$, the $\mathcal{F}$-components of S are a partition of a subset of S.*

Proof. Let $\mathcal{F}$ be union stable. Let C^1, C^2, be two maximal feasible coalitions of S. If $C^1 \cap C^2 \neq \emptyset$, then $C^1 \cup C^2 \in \mathcal{F}$ since $\mathcal{F}$ is union stable and $C^1 \cup C^2 \subseteq S$. This contradicts the fact that C^1 and C^2 are $\mathcal{F}$-components of S.

Conversely, assume for any S such that $C_{\mathcal{F}}(S) \neq \emptyset$, that its $\mathcal{F}$-components form a partition of a subset of S. Suppose that $\mathcal{F}$ is not union stable, then there are $A, B \in \mathcal{F}$, with $A \cap B \neq \emptyset$ and $A \cup B \notin \mathcal{F}$. Hence, there must be an $\mathcal{F}$-component $C_1 \in C_{\mathcal{F}}(A \cup B)$, with $A \subseteq C_1$ and an $\mathcal{F}$-component $C_2 \in C_{\mathcal{F}}(A \cup B)$, with $B \subseteq C_2$ such that $C_1 \neq C_2$. This contradicts the fact that the $\mathcal{F}$-components of $A \cup B$ are disjoint. □

It is obvious that if $\mathcal{F}$ is a union stable system such that $\{i\} \in \mathcal{F}$, for all $i \in N$, then the $\mathcal{F}$-components of S form a partition of S. We have also the following consequence of the definitions.

Proposition 2.2 *Let $\mathcal{F}$ be a union stable system. Let $S \subseteq N$ and consider the collection $\mathcal{F}_S = \{F \in \mathcal{F} : F \subseteq S\}$. Then, the following conditions are satisfied:*

(a) *$\mathcal{F}_S$ is union stable.*

(b) *$C_{\mathcal{F}}(S) = C_{\mathcal{F}_S}(N)$.*

(c) *$\mathcal{B}(\mathcal{F}_S) = \{B \in \mathcal{B}(\mathcal{F}) : B \subseteq S\}$.*

In order to establish a relation between conference structures à la Myerson and union stable systems, we will give the following results. Moreover, the next theorem will be essential in order to prove the uniqueness in the axiomatization of Myerson value in union stable systems.

Definition 2.5 *Let $\mathcal{F}$ be a union stable system. The players $i, j \in N$, are called connected by $\mathcal{B}(\mathcal{F})$ if there exists a sequence of supports $(B_1, \ldots, B_k)$, such that $i \in B_1$, $j \in B_k$ and if $k \geq 2$, $B_p \cap B_{p+1} \neq \emptyset$, for all $p = 1, \ldots, k-1$.*

Theorem 2.3 *Let $\mathcal{F}$ be a union stable system. Let $S \in \mathcal{F}$ and $i, j \in N$, $i \neq j$. Then $\{i, j\} \subseteq S$ if and only if i and j are connected by supports in $\mathcal{C}(\mathcal{F})$ contained in S, where $\mathcal{C}(\mathcal{F}) = \{B \in \mathcal{B}(\mathcal{F}) : |B| \geq 2\}$.*

Proof. Let $\{i, j\} \subseteq S$. If $S \in \mathcal{C}(\mathcal{F})$, it suffices to take $k = 1$ and $B_1 = S$. If $S \notin \mathcal{C}(\mathcal{F})$, then $S = A \cup B$, with $A, B \in \mathcal{F}$, and $A \cap B \neq \emptyset$. If $A, B \in \mathcal{C}(\mathcal{F})$ then we obtain the result. Otherwise, we repeat this decomposition and proceeding in this manner, we obtain the sequence of supports. The converse is obvious. □

Corollary 2.4 *Let $\mathcal{F}$ be a union stable system. Let $i, j \in N$, $i \neq j$. Then i and j are in the same $\mathcal{F}$-component of N if and only if i and j are connected by $\mathcal{C}(\mathcal{F})$.*

Example. Myerson [4] introduced the term *conference*, to refer to any set of two or more players who might meet together to discuss their cooperative plans. A *conference structure* CS is any collection $\mathcal{Q} \subseteq \{S \subseteq N : |S| \geq 2\}$. Given a conference structure $\mathcal{Q} \in CS$, two players i and j are connected by $\mathcal{Q}$ if $i = j$ or there exists some sequence of conferences $(S_1, \ldots, S_k)$ such that $i \in S_1$, $j \in S_k$, $\{S_1, \ldots, S_k\} \subseteq \mathcal{Q}$, and $S_p \cap S_{p+1} \neq \emptyset$ for all $p = 1, \ldots, k-1$.

If $\mathcal{F}$ is a union stable system then the set formed by the non-unitary supports is a Myerson's conference structure. Conversely, given a Myerson's conference structure, the set system

$$\mathcal{F} = \{S \subseteq N : \text{each pair of players } i, j \in S \text{ are connected by conferences in } S\}$$

is union stable.

3 The Myerson value: properties and axiomatizations

This section deals with a solution concept for games restricted by union stable structures: the *Myerson value.* We recall that this value is the *Shapley value* Φ of the *$\mathcal{F}$-restricted game.*

Definition 3.1 *Let (N, v) be a cooperative n-person game in coalitional form and $\mathcal{F} \subseteq 2^N$ a union stable system. The $\mathcal{F}$-restricted game $v^{\mathcal{F}} : 2^N \longrightarrow \mathbb{R}$, is defined by*

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T).$$

A *union stable structure* is a triple $(N, v, \mathcal{F})$ where $N = \{1, \ldots, n\}$ is the set of players, (N, v) is a game $v : 2^N \longrightarrow \mathbb{R}$ with $v(\emptyset) = 0$, and $\mathcal{F}$ is a union stable system.

Definition 3.2 *The Myerson value of a union stable structure $(N, v, \mathcal{F})$ is given by the vector $\mu(N, v, \mathcal{F}) = \Phi\left(N, v^{\mathcal{F}}\right)$.*

The following example illustrates the concepts introduced above.

Example. Consider the player set $N = \{1, 2, 3, 4\}$ and the union stable system given by $\mathcal{F} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, N\}$. Let $v : 2^N \longrightarrow \mathbb{R}$ be the game defined by $v(S) = |S| - 1$, $S \neq \emptyset$, and $v(\emptyset) = 0$. Then, $\mathcal{B}(\mathcal{F}) = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}\}$ and $\mathcal{C}(\mathcal{F}) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. In this case, it is clear that

$$v^{\mathcal{F}}(S) = \begin{cases} |S| - 1 & \text{if } S \in \mathcal{F} \\ 0 & \text{otherwise,} \end{cases}$$

and the Myerson value is $\mu(N, v, \mathcal{F}) = \frac{1}{12}(5, 13, 13, 5)$.

We now consider some properties that would be desirable for *an allocation rule*, and we focus on the study of these properties for the Myerson value. The set of all union stable structures with player set N will be denoted by US^N.

Definition 3.3 *An allocation rule on US^N is a map $\gamma : US^N \longrightarrow \mathbb{R}^N$, such that it is component-efficient and component-dummy, that is,*

(1) $\sum_{i\in M} \gamma_i (N, v, \mathcal{F}) = v(M)$, *for all* $(N, v, \mathcal{F}) \in US^N$ *and* $M \in C_{\mathcal{F}}(N)$.

(2) $\gamma_i (N, v, \mathcal{F}) = 0$, *for all* $i \notin \bigcup_{M\in C_{\mathcal{F}}(N)} M$.

Lemma 3.1 *The Myerson value $\mu : US^N \longrightarrow \mathbb{R}^N$ is an allocation rule.*

Proof. Let $(N, v, \mathcal{F}) \in US^N$. If $N \in \mathcal{F}$ then N is its unique $\mathcal{F}$-component, and hence $\sum_{i\in N} \mu (N, v, \mathcal{F}) = \sum_{i\in N} \Phi_i \left(N, v^{\mathcal{F}}\right) = v^{\mathcal{F}}(N) = v(N)$. Suppose, that $N \notin \mathcal{F}$ and, therefore, consider the set $C_{\mathcal{F}}(N)$. To each $\mathcal{F}$-component M of N is associated the game u^M, which is defined in the following way , with M fixed,

$$u^M : 2^N \longrightarrow \mathbb{R}, \quad u^M(T) = v^{\mathcal{F}} (T \cap M) = \sum_{H\in C_{\mathcal{F}}(T\cap M)} v(H), \text{ for all } T \subseteq N.$$

Moreover, for any coalition $T \subseteq N$, $C_{\mathcal{F}}(T) = \bigcup_{R\in C_{\mathcal{F}}(N)} C_{\mathcal{F}}(T \cap R)$, and hence, it is immediate that $v^{\mathcal{F}} = \sum_{R\in C_{\mathcal{F}}(N)} u^R$. Taking into account the above considerations for the game $\left(N, v^{\mathcal{F}}\right)$, we find

$$\sum_{i\in M} \mu (N, v, \mathcal{F}) = \sum_{i\in M} \Phi_i \left(N, u^M\right) + \sum_{\{R\in C_{\mathcal{F}}(N) : R\neq M\}} \left[\sum_{i\in M} \Phi_i \left(N, u^R\right)\right].$$

Since $\sum_{i\in M} \Phi_i \left(N, u^M\right) = v^{\mathcal{F}}(M)$, and $\Phi_i(N, u^R) = 0$, $R \neq M$, $i \in M$, the above expression implies that $\sum_{i\in M} \mu (N, v, \mathcal{F}) = v^{\mathcal{F}}(M) = v(M)$.

Component-dummy is immediate since if $i \notin \bigcup_{M\in C_{\mathcal{F}}(N)} M$ then we have $C_{\mathcal{F}}(S) = C_{\mathcal{F}}(S \setminus \{i\})$, for all $S \in \mathcal{F}$. Hence, the marginal contributions are $v^{\mathcal{F}}(S) - v^{\mathcal{F}} (S \setminus \{i\}) = 0$, and $\mu_i (N, v, \mathcal{F}) = 0$. □

Definition 3.4 *An allocation rule γ is fair if for all $(N, v, \mathcal{F})$, $B \in \mathcal{B}(\mathcal{F})$, there exists $c \in \mathbb{R}$ such that $\gamma_j (N, v, \mathcal{F}) - \gamma_j (N, v, \mathcal{F}') = c$, for all $j \in B$, where $\mathcal{F}' = \overline{\mathcal{B}(\mathcal{F}) \setminus \{B\}}$.*

So, according to a fair allocation rule all players in a support B lose or gain the same amount if the support B is deleted. We now extend the axiomatization of the Myerson value to union stable structures.

Theorem 3.2 *The Myerson value is the unique fair allocation rule on US^N.*

Proof. *(a)* Uniqueness: Let $(N, v, \mathcal{F}) \in US^N$. Suppose γ^1 and γ^2 are two fair allocation rules on US^N. We will prove by induction to the number $|\mathcal{C}(\mathcal{F})|$ of non-unitary supports in the basis of $\mathcal{F}$, that $\gamma^1 (N, v, \mathcal{F}) = \gamma^2 (N, v, \mathcal{F})$.

If $|\mathcal{C}(\mathcal{F})| = 0$, then $C_{\mathcal{F}}(N) = \{\{i\} : \{i\} \in \mathcal{F}\}$. Applying component-efficiency and component-dummy we obtain that $\gamma^1 (N, v, \mathcal{F}) = \gamma^2 (N, v, \mathcal{F})$.

Now, assume that $\gamma^1(N,v,\mathcal{G}) = \gamma^2(N,v,\mathcal{G})$ for all $\mathcal{G}$ with $|\mathcal{C}(\mathcal{G})| \leq k-1$, and let $|\mathcal{C}(\mathcal{F})| = k$. Consider $C \in \mathcal{C}(\mathcal{F})$. Fairness implies that there exist numbers $c \in \mathbb{R}$ and $d \in \mathbb{R}$ such that

$$\gamma_j^1(N,v,\mathcal{F}) - \gamma_j^1\left(N,v,\overline{\mathcal{B}(\mathcal{F}) \setminus \{C\}}\right) = c,$$

$$\gamma_j^2(N,v,\mathcal{F}) - \gamma_j^2\left(N,v,\overline{\mathcal{B}(\mathcal{F}) \setminus \{C\}}\right) = d,$$

for all $j \in C$. Note that by the induction hypothesis

$$\gamma_j^1\left(N,v,\overline{\mathcal{B}(\mathcal{F}) \setminus \{C\}}\right) = \gamma_j^2\left(N,v,\overline{\mathcal{B}(\mathcal{F}) \setminus \{C\}}\right).$$

So there is a constant $\alpha = c - d$ such that

$$\gamma_j^1(N,v,\mathcal{F}) - \gamma_j^2(N,v,\mathcal{F}) = \alpha, \text{ for all } j \in C. \tag{1}$$

Given $M \in C_{\mathcal{F}}(N)$, by component-efficiency for γ^1 and γ^2, we obtain

$$\sum_{i \in M}\left[\gamma_i^1(N,v,\mathcal{F}) - \gamma_i^2(N,v,\mathcal{F})\right] = 0.$$

Applying theorem 2.3 and equality (1) recursively, we get

$$\gamma_i^1(N,v,\mathcal{F}) - \gamma_i^2(N,v,\mathcal{F}) = \alpha,$$

for all $i \in M$, with $M \in C_{\mathcal{F}}(N)$, and this implies

$$\sum_{i \in M}\left[\gamma_i^1(N,v,\mathcal{F}) - \gamma_i^2(N,v,\mathcal{F})\right] = |M|\,\alpha.$$

Therefore $|M|\,\alpha = 0$, and hence $\gamma^1(N,v,\mathcal{F}) = \gamma^2(N,v,\mathcal{F})$.

(b) Next, we show that the Myerson value is fair. Consider the game (N,w) given by $w(S) = v^{\mathcal{F}}(S) - v^{\mathcal{F}'}(S)$, for all $S \subseteq N$, where $\mathcal{F}' = \overline{\mathcal{B}(\mathcal{F}) \setminus \{B\}}$. Let $k \in B$. We may deduce that $w(S) = 0$, for all $S \subseteq N$, $B \nsubseteq S$, and, since for all coalitions $S \subseteq N$ with $B \subseteq S$ we have $B \nsubseteq S \setminus \{k\}$ and so $w(S \setminus \{k\}) = 0$. Thus, we can write for $k \in B$

$$\Phi_k(N,w) = \sum_{\{S\,:\,B \subseteq S\}} \frac{(s-1)!\,(n-s)!}{n!} w(S), \quad \text{where } s = |S|,\ n = |N|.$$

It follows that $\Phi_k(N,w) = \Phi_p(N,w)$, for all $p \in B$, and we obtain that the Myerson value is fair. □

Definition 3.5 *An allocation rule γ is called basis monotonic if for all $(N,v,\mathcal{F})$, for all $B \in \mathcal{B}(\mathcal{F})$, and for all $j \in B$ it holds $\gamma_j(N,v,\mathcal{F}) \geq \gamma_j(N,v,\mathcal{F}')$, where $\mathcal{F}' = \overline{\mathcal{B}(\mathcal{F}) \setminus \{B\}}$.*

This condition asserts that all the players always benefit from reaching an agreement and cooperate.

Proposition 3.3 *Let $(N,v,\mathcal{F}) \in US^N$. If v is superadditive and zero-normalized, then $\mu(N,v,\mathcal{F})$ is basis monotonic.*

Proof. It suffices to prove that $w(S) \geq 0$ for any $S \subseteq N$ such that $B \subseteq S$, where for all $S \subseteq N$, $w(S) = v^{\mathcal{F}}(S) - v^{\mathcal{F}'}(S)$, with $\mathcal{F}' = \overline{\mathcal{B}(\mathcal{F}) \setminus \{B\}}$. Any maximal feasible coalition of S in $\mathcal{F}'$ is either a maximal feasible coalition of S in $\mathcal{F}$ or it is contained in an $\mathcal{F}$-component of S. Then, taking the $\mathcal{F}'$-components of S and taking into consideration that the game (N,v) is superadditive and zero-normalized, we obtain

$$v^{\mathcal{F}'}(S) = \sum_{T' \in C_{\mathcal{F}'}(S)} v\left(T'\right) \leq \sum_{T \in C_{\mathcal{F}}(S)} \left[v\left(\bigcup_{\{T' \in C_{\mathcal{F}'}(S) : T' \subseteq T\}} T' \right) \right] \leq v^{\mathcal{F}}(S).$$

□

To provide other axiomatic characterizations for the Myerson value, the next definitions are introduced (see van den Nouweland [7]). We use $\mathcal{C}_i(\mathcal{F})$ to denote the collection $\{C \in \mathcal{C}(\mathcal{F}) : i \in C\}$.

Definition 3.6 *A union stable structure $(N,v,\mathcal{F})$ is called point anonymous if there exists a function $f : \{0,1,\ldots,|D|\} \longrightarrow \mathbb{R}$ such that $v^{\mathcal{F}}(S) = f(|S \cap D|)$ for all $S \subseteq N$, where $D = \{i \in N : \mathcal{C}_i(\mathcal{F}) \neq \emptyset\}$.*

Definition 3.7 *An allocation rule γ satisfies point anonymity if for all point anonymous $(N,v,\mathcal{F})$, there exists $\alpha \in \mathbb{R}$ such that*

$$\gamma_i(N,v,\mathcal{F}) = \begin{cases} \alpha & \text{for all } i \in D, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.4 *The Myerson value satisfies point anonymity.*

Proof. Let $(N,v,\mathcal{F}) \in US^N$ be point anonymous. If $D = \emptyset$, then the restricted game $v^{\mathcal{F}}(S) = f(|S \cap \emptyset|) = f(0) = 0$, for all $S \subseteq N$. Hence, $\mu_i(N,v,\mathcal{F}) = 0$ for all $i \in N$. Let $D \neq \emptyset$. If $i \notin D$, obviously $S \cap D = (S \setminus \{i\}) \cap D$ and $\mu_i(N,v,\mathcal{F}) = 0$. On the other hand, if $i,j \in D$ applying the symmetry property of the Shapley value we have $\mu_i(N,v,\mathcal{F}) = \mu_j(N,v,\mathcal{F})$, and hence $f(|D|) = \sum_{i \in D} \mu_i(N,v,\mathcal{F}) = |D| \mu_i(N,v,\mathcal{F})$. Therefore, $\mu_i(N,v,\mathcal{F}) = f(|D|)/|D| = \alpha$, for all $i \in D$ and $\mu_i(N,v,\mathcal{F}) = 0$, otherwise. □

Definition 3.8 *Let $(N,v,\mathcal{F}) \in US^N$. Then player $i \in N$ is called superfluous for $(N,v,\mathcal{F})$ if $v^{\mathcal{F}}(S) = v^{\mathcal{F}}(S \setminus \{i\})$, for all $S \subseteq N$. An allocation rule γ satisfies the superfluous player property if for all $(N,v,\mathcal{F})$ and every player $i \in N$ that is superfluous for $(N,v,\mathcal{F})$ it holds $\gamma(N,v,\mathcal{F}) = \gamma\left(N,v,\mathcal{F}_{N\setminus\{i\}}\right)$, where $\mathcal{F}_{N\setminus\{i\}} = \{F \in \mathcal{F} : F \subseteq N \setminus \{i\}\}$.*

Proposition 3.5 *The Myerson value satisfies the superfluous player property.*

Proof. Let $i \in N$ be a superfluous player for $(N, v, \mathcal{F}) \in US^N$. We have to prove $\mu(N, v, \mathcal{F}) = \mu(N, v, \mathcal{F}_{N \setminus \{i\}})$. We observe that i is a zero player in $v^{\mathcal{F}}$ and this implies that $\mu_i(N, v, \mathcal{F}) = 0$. Further, $\mu_i(N, v, \mathcal{F}_{N \setminus \{i\}}) = 0$, because $i \notin \bigcup_{M \in C_{\mathcal{F}_{N \setminus \{i\}}}(N)} M$ and μ satisfies component-dummy (lemma 3.1).

For the other players, it suffices to show that $v^{\mathcal{F}}(S) = v^{\mathcal{F}_{N \setminus \{i\}}}(S)$, or equivalently, as i is a superfluous player for $(N, v, \mathcal{F})$, that $v^{\mathcal{F}}(S \setminus \{i\}) = v^{\mathcal{F}_{N \setminus \{i\}}}(S)$, for all $S \subseteq N$. The components satisfy $C_{\mathcal{F}}(S \setminus \{i\}) = C_{\mathcal{F}_{N \setminus \{i\}}}(S)$, and therefore

$$v^{\mathcal{F}}(S \setminus \{i\}) = \sum_{T \in C_{\mathcal{F}}(S \setminus \{i\})} v(T) = \sum_{T \in C_{\mathcal{F}_{N \setminus \{i\}}}(S)} v(T) = v^{\mathcal{F}_{N \setminus \{i\}}}(S),$$

for all $S \subseteq N$. □

Definition 3.9 *An allocation rule* γ *is called additive if for all* $(N, v, \mathcal{F})$ *and* $(N, w, \mathcal{F})$ *then* $\gamma(N, v + w, \mathcal{F}) = \gamma(N, v, \mathcal{F}) + \gamma(N, w, \mathcal{F})$.

We obtain immediately that the Myerson value is additive.

Lemma 3.6 *If* γ *is an additive allocation rule that satisfies the superfluous player property, then* $\gamma(N, v, \mathcal{F}) = \gamma(N, v^{\mathcal{F}}, \mathcal{F})$*, for all* $(N, v, \mathcal{F}) \in US^N$.

Proof. By additivity of γ, it suffices to show that $\gamma(N, v - v^{\mathcal{F}}, \mathcal{F}) = 0$, for all $(N, v, \mathcal{F}) \in US^N$. Indeed, for any $S \subseteq N$,

$$\left(v - v^{\mathcal{F}}\right)^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} \left(v - v^{\mathcal{F}}\right)(T) = \sum_{T \in C_{\mathcal{F}}(S)} \left[v(T) - v^{\mathcal{F}}(T)\right] = 0.$$

Therefore, all players are superfluous for any $(N, v - v^{\mathcal{F}}, \mathcal{F}) \in US^N$. Hence, taking recursively all players in the same maximal component $M \in C_{\mathcal{F}}(N)$

$$\gamma\left(N, v - v^{\mathcal{F}}, \mathcal{F}\right) = \gamma\left(N, v - v^{\mathcal{F}}, \mathcal{F}_{N \setminus M}\right).$$

For all $i \in M$, we obtain

$$\gamma_i\left(N, v - v^{\mathcal{F}}, \mathcal{F}\right) = \gamma_i\left(N, v - v^{\mathcal{F}}, \mathcal{F}_{N \setminus M}\right) = 0,$$

since $i \notin \bigcup_{H \in C_{\mathcal{F}_{N \setminus M}}(N)} H$. It follows that $\gamma_i\left(N, v - v^{\mathcal{F}}, \mathcal{F}\right) = 0$, for all $i \in N$. □

Theorem 3.7 *The Myerson value is the unique allocation rule on* US^N *that satisfies additivity, the superfluous player property and point anonymity.*

Proof. Let γ be an allocation rule on US^N that also satisfies additivity, the superfluous player property and point anonymity. From lemma 3.6 we deduce $\gamma(N,v,\mathcal{F}) = \gamma\left(N,v^{\mathcal{F}},\mathcal{F}\right)$. The unanimity games $\{u_S : S \in \mathcal{F},\, S \neq \emptyset\}$ form a basis for the vector space of the $\mathcal{F}$-restricted games (see Bilbao [2]), that is,

$$v^{\mathcal{F}} = \sum_{\{S\in\mathcal{F}:S\neq\emptyset\}} \alpha_S u_S$$

for some coefficients α_S. Applying additivity, it suffices to show that $\gamma(N,\alpha u_S,\mathcal{F})$, is uniquely determined for all $S \in \mathcal{F},\ S \neq \emptyset$ and $\alpha \in \mathbb{R}$. Fix S and α. If $i \in N\setminus S$ then for all coalitions $T \subseteq N$

$$\alpha u_S(T) = \alpha \Longleftrightarrow S \subseteq T \Longleftrightarrow S \subseteq T\setminus\{i\} \Longleftrightarrow \alpha u_S(T\setminus\{i\}) = \alpha.$$

We deduce that any player that is not in S is superfluous and hence by the superfluous player property:

$$\gamma(N,\alpha u_S,\mathcal{F}) = \gamma\left(N,\alpha u_S,\mathcal{F}_{N\setminus(N\setminus S)}\right) = \gamma(N,\alpha u_S,\mathcal{F}_S).$$

Since $C_{\mathcal{F}_S}(N) = C_{\mathcal{F}}(S) = \{S\}$, component-dummy implies that

$$\gamma_i(N,\alpha u_S,\mathcal{F}_S) = 0,$$

for all $i \in N\setminus S$. It remains only to compute $\gamma_i(N,\alpha u_S,\mathcal{F}_S)$ for all $i \in S$. First, for all $T \subseteq N$, we have

$$(\alpha u_S)^{\mathcal{F}_S}(T) = \sum_{H\in C_{\mathcal{F}_S}(T)} \alpha u_S(H) = \alpha \Longleftrightarrow \exists H \in \mathcal{F}_S,\, S \subseteq H \subseteq T.$$

If $H \in \mathcal{F}_S$ then $H \subseteq S$, and hence $(\alpha u_S)^{\mathcal{F}_S}(T) = \alpha$ if and only if $S \subseteq T$. Therefore, $(\alpha u_S)^{\mathcal{F}_S} = \alpha u_S$ implies

$$(\alpha u_S)^{\mathcal{F}_S}(T) = \alpha u_S(T) = \alpha \Longleftrightarrow S \subseteq T \Longleftrightarrow S\cap T = S.$$

It follows that there exists a function $f : \{0,1,\ldots,|S|\} \longrightarrow \mathbb{R}$, such that $(\alpha u_S)^{\mathcal{F}_S}(T) = f(|S\cap T|)$, for all $T \subseteq N$, where $f(0) = \cdots = f(|S|-1) = 0$, and $f(|S|) = \alpha$. Hence $(N,\alpha u_S,\mathcal{F}_S)$ is point anonymous and applying point anonymity to the rule γ, there exists $\beta \in \mathbb{R}$ such that

$$\gamma_i(N,\alpha u_S,\mathcal{F}_S) = \begin{cases} \beta & \text{if } i \in S, \\ 0 & \text{otherwise.}\end{cases}$$

Further, $C_{\mathcal{F}_S}(N) = \{S\}$, and using component-efficiency we get

$$\sum_{i\in S} \gamma_i(N,\alpha u_S,\mathcal{F}_S) = \alpha = |S|\,\beta.$$

Then $\beta = \alpha/|S|$ and we deduce that $\gamma(N,v,\mathcal{F})$ is the Myerson value. □

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