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Publication date:
1998

Citation for published version (APA):
Koster, M. A. L. (1998). *Multi-Service Serial Cost Sharing: An Incompatibility with Smoothness*. (CentER Discussion Paper; Vol. 1998-122). Microeconomics.

Multi-service serial cost sharing: an incompatibility with smoothness

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Abstract: We focus on the radial serial rule as a natural extension of the Moulin-Shenker cost sharing rule. We show that it is the unique *regular* rule that is compatible with the *radial serial principle.* In particular, this shows the incompatibility of the *serial principle* with differentiability of a cost sharing rule as a function of the individual demands.

JEL-Classification: C69, D49.

Keywords: Cost sharing, serial rule, multi-service facilities.

1 Introduction

This paper centers on the problem of allocating the cost of usage of a production facility for a finite set of (infinitely) divisible goods M that is jointly owned by a fixed group of agents N. Each of the agents $i \in N$ has a particular interest for the goods in M, which is expressed by a vector $d_i \in \mathbb{R}_+^M$; d_{ij} is the demand of agent i for good j. Then the aggregate bundle $q = \sum_{i \in N} q_i$ is produced and we aim for distributing the corresponding total cost $c(q)$. We focus on cost sharing rule that incorporate the ideas behind the serial cost sharing rule (Moulin and Shenker (1992a,1992b)). Most importantly, we will require from such a cost sharing rule, that it is a *serial extension*, i.e. on the class of homogeneous problems it should determine the same cost shares as the serial rule for the naturally corresponding one good situation. In Kolpin (1994) and Koster *et al.* (1997) it is shown that in fact there are infinitely many serial extensions. However, though this class of all serial extensions can thus considered to be rich, it is poor in another respect. As Kolpin (1996) and Koster *et al.* (1997) learn, one easily runs into incompatibilities by combining the serial spirit with other worthwhile principles.

We will focus on the *serial principle*, which says the following. Given the fact that an agent i pays less than another agent j with respect to some demand profile q and cost structure c, then an increase of agent j's demand should have no effect on agent i's cost share. It is the natural extension of the serial principle in Sprumont (1997), that deals with situations where agents are confined to have demand for one good, and by which each can be identified. The latter model will be referred to below as the *personalized good model.* The serial principle generalizes the property *independence of size of larger demands* of the serial cost sharing rule (Moulin and Shenker (1992a)) which requires that the cost share of the agent with the smaller demand should not be affected by an increased demand of the agent with the larger demand. The principle can be seen as the essential characteristic of serial cost sharing, by which individual agents are protected against contingent excessive behavior of others. More or less, it is this aspect of the serial rule that explains the succes of the serial rule in environments exhibiting either negative or positive externalities where demands are chosen strategically (Moulin and Shenker (1992a), Moulin (1996)).

We will show that the serial principle essentially conflicts a *smoothness* condition, which requires the existence of the directional derivatives of the cost sharing rule (as a mapping of demands) such that for opposed directions the corresponding directional derivatives sum up to 0. So, this smoothness condition is weaker than the requirement of differentiability. We deduce that the *radial serial rule* is the unique smooth cost sharing rule that satisfies only a weakened version of the serial principle. The radial serial rule does not satisfy the serial principle, therefore we automatically get the incompatibility of the combination of smoothness together with the serial principle.

This result contrasts with Sprumont (1997), where it is shown that for the personalized good model there is essentially one smooth rule that is compatible with the serial principle, and that is the rule introduced as the *Moulin-Shenker rule*. Alternative characterizations of the Moulin-Shenker rule are found in Koster (1998).

2 The model

Throughout this paper we will restrict ourselves to a fixed and finite group of agents $N = \{1, 2, \ldots, n\}$. The group of agents jointly own some production facility for some finite set of divisible goods $M = \{1, 2, \ldots m\}$. Then any level of output is described by a vector $q \in \mathbb{R}_+^M$, where q_i is the output of good i. We assume that the costs associated with each level of output are summarized by a cost function $c : \mathbb{R}_+^M \to \mathbb{R}_+$. We will treat only those situations where there are no fixed costs, i.e. $c(0) = 0$. In addition we restrict ourselves to the domain $\mathcal{C}$ of all increasing and continuously differentiable cost functions with partial derivatives bounded away from 0 and ∞. So if $c \in \mathcal{C}$, then with its i-th partial derivative denoted by $D_i c$, there are positive real numbers $a(c)$ and $b(c)$ with $a(c) \leq D_i c(q) \leq b(c)$ for all $q \in \mathbb{R}_+^M$. We will denote the derivative of c in the direction $z \in \mathbb{R}_+^M \backslash \{0\}$ by $D_z c$, i.e. for all $q \in \mathbb{R}_+^M$ it holds $D_z c(q) = \lim_{t \downarrow 0} \frac{1}{t}(c(q + tz) - c(q))$.

Agent i's demand for each of the goods is summarized by a vector $d_i \in \mathbb{R}_+^M$; d_{ij} stands for the requested number of units of good j. Then $d = (d_i)_{i \in N} \in (\mathbb{R}_+^M)^N$ is called the demand profile for N. Now the aggregate of the demands $\sum_{i \in N} d_i$ is produced and total costs $c(\sum_{i \in N} d_i)$ have to be shared by N. A *cost sharing problem* is an ordered pair $(q, c) \in (\mathbb{R}_+^M)^N \times \mathcal{C}$. Denote the space of all cost sharing problems by $\mathcal{G}$. A *cost sharing rule* is a mapping $x : \mathcal{G} \to \mathbb{R}_+^N$, associating each cost sharing problem with an efficient vector of cost shares, i.e. $\sum_{i \in N} x_i(q, c) = c(\sum_{i \in N} q_i)$ for all $(q, c) \in \mathcal{G}$.

For any finite set $S \subset \mathbb{N}$, the space $\mathbb{R}^S$ is partially ordered through the natural ordering $\leq$ on $\mathbb{R}$. For two vectors $y, y' \in \mathbb{R}^S$ we will write $y \leq y'$ if for all $i \in S$ it holds that $y_i \leq y'_i$. If at least one of these inequaltities is strict we write $y < y'$. We use the notation $y \ll y'$ whenever $y_i < y'_i$ for all $i \in S$. The natural inner product on $\mathbb{R}_+^M$ is defined by $\langle x, y \rangle := \sum_{i \in M} x_i y_i$. The corresponding Euclidean norm of a vector $x \in \mathbb{R}_+^M$ is denoted by $\|x\|$. The normalization of a vector $x \neq 0$ is defined as $\tilde{x} := x\|x\|^{-1}$; for $x = 0$ we take $\tilde{x}$ to be an arbitrary non-zero normalized vector. The profile out of $q \in \mathbb{R}_+$ where the demand of player i is interchanged with $t \in \mathbb{R}_+^M$ is denoted by (q^{-i}, t). For $S \subseteq N$ and $z \in (\mathbb{R}_+^M)^S$ we define $\mathcal{D}_S(z)$ as the set of all demand profiles $q \in (\mathbb{R}_+^M)^S$ such that $q_i = \lambda_i z_i$ for some $\lambda_i \in \mathbb{R}_+$ for all $i \in N$.

Next we will define the class of *path generated cost sharing rules.* For $S \subseteq N$ a *path* in $(\mathbb{R}_+^M)^S$ is a continuous mapping $\pi : \mathbb{R}_+ \to (\mathbb{R}_+^M)^S$ with $\pi(0) = 0$. The path π is increasing if $\pi_i(t) < \pi_i(t')$ for all $i \in S$ implies $t < t'$. In our setting, with the argument of π thought of as being time, an increasing path may be considered as a program for production. At time t a bundle of goods equal to $\pi_i(t)$ is produced for agent i.
Given $c \in \mathcal{C}$, demand profile z and $d \in \mathcal{D}_S(z)$ let $\pi^{d,c,S,z}$ be an increasing path for S such that $\pi^{d,c,S,z}(t) \in \mathcal{D}_S(z)$ for all $i \in \mathbb{R}_+$ and such that for each $q \in \mathcal{D}_S(z)$ there is $t \in \mathbb{R}_+$ with $\pi^{d,c,S,z}(t) > q$. Such a path will be considered

to describe a fictitious production plan for coalition S from level d, where agent i's demand is a multiple of z_i . The last variable of the quartet is used to define the ratio of the different goods that an agent receives through production device $\pi^{d,c,S,z}$ as part of his demand. In addition we will assume that whenever $z' \in D_S(z)$, such that for all $i \in S$, $z'_i = \lambda_i z_i$ and $\lambda_i > 0$, then $\pi^{d,c,S,z} = \pi^{d,c,S,z'}$. This is to make sure that the paths do only depend on the individual 'ratio profiles' z_i. Since we will allow for such a production plan to depend on the exogeneous information of costs that is summarized by $c \in \mathcal{C}$, it is included as a parameter as well.

Π is defined as the collection of all those paths, one for each quartet (d, c, S, z). We will refer to Π as a *path collection.* A path collection Π defines for each cost sharing problem $(q, c) \in \mathcal{G}$ a production plan in the following way.

We start at production level 0. Initially, we take the path for N, $\pi^{0,c,N,q}$ as a production device, telling us for each moment in time what is produced for the individual agents. So follow $\pi^{0,c,N,q}$ up to the earliest moment t_1 that some agents $M_1 \subseteq N$ are satisfied, i.e.

$$\pi_i^{0,c,N,q}(t_1) = q_i \text{ for all } i \in M_1.$$

Define π on $[0, t_1]$ by $\pi(t) = \sum_{i \in N} \pi_i^{0,c,N,q}(t)$. Let d^1 denote the vector of demands that that is processed for each of the agents so far, $d^1 = \pi(t_1)$. Still, an agent $i \in N \backslash M_1$ needs the bundle $q_i - d_i^1$ in order to be satisfied. Next, we take $\pi^{d^1,c,N \backslash M_1,q}$ as the additional production plan for $N \backslash M_1$ until the first moment t_2 that some agents $M_2 \subseteq N \backslash M_1$ are satisfied, i.e.

$$\pi^{d^1,c,N \backslash M_1,q}(t_2) = q_i - d_i^1 \text{ for all } i \in M_2.$$

The definition of π is now completed up to moment $t_1 + t_2$ by

$$\pi(t + t_1) := d^1 + \sum_{i \in N \backslash M_1} \pi_i^{d^1,c,N \backslash M_1,q}(t) \text{ for all } t \in (0, t_2].$$

Let $d^2 = \pi(t_1 + t_2)$. Follow the production device $\pi^{d^2,c,N \backslash (M_1 \cup M_2),q}$ until moment t_3 where the first agents $M_3 \subseteq N \backslash (M_1 \cup M_2)$ are fulfilled with their remaining needs $q_{M_3} - d^2_{M_3}$. Then define

$$\pi(t + t_1 + t_2) = d^2 + \sum_{i \in N\backslash(M_1 \cup M_2)} \pi_i^{d^2,c,N\backslash(M_1\cup M_2),q}(t) \quad \text{for all } t \in (0, t_3].$$

In this way we can go on and complete the definition of π. We just proceed by determining time levels $t_4, t_5, \ldots$ and corresponding groups of agents $M_4, M_5, \ldots$ until the first moment $t_1 + \ldots + t_k$ such that there are no remaining demanders, i.e. $N\backslash(M_1 \cup \ldots \cup M_k) = \emptyset$. Set $\pi(t) = \sum_{i \in N} q_i$ for $t > t_1 + t_2 + \ldots + t_k$. We will say that π is *the path for* (q, c) *generated by* Π.

Definition 2.1 The solution for the cost sharing problem $(q, c) \in \mathcal{G}$ generated by a path collection Π is the vector $x^\Pi(q, c) \in \mathbb{R}^N_+$ defined as follows. Let π be the path for (q, c) generated by Π. Suppose that according to π agent i is satisfied at moment t_i. Without loss of generality, assume that $t_i \leq t_j$ whenever $i \leq j$ for all $i, j \in N$. We split the successive cost increments $c(\pi(t_{i+1})) - c(\pi(t_i))$ equally among the agents requiring service on the interval $(t_i, t_{i+1}]$. By assumption this is the set of agents $\{i+1, i+2, \ldots, n\}$. Then this boils down to the following formula. The cost share of agent 1 is given by

$$x_1^\Pi(q, c) := \frac{c(\pi(t_1))}{n}.$$

Then proceed inductively by defining for $i \geq 2$,

$$x_i^\Pi(q, c) = x_{i-1}^\Pi(q, c) + \frac{c(\pi(t_i)) - c(\pi(t_{i-1}))}{n - i + 1}.$$

By varying over all cost sharing problems in $\mathcal{G}$ this yields a cost sharing rule x^Π, generated by the path collection Π.

We will also say that in the above definition the cost shares for the problem (q, c) are generated by Π. Note that in essence for a path generated method only the images of the paths are of importance for determining the cost allocation; any other parametrization of the paths determines the same rule. The following lemmata are concerned with a common characteristic of path generated cost sharing rules, and that is the radial serial principle. But first we provide the formal definitions of both the serial as the radial serial principle.

Definition 2.2 A cost sharing rule x satisfies the *serial principle* if for all cost sharing problems $(q, c) \in \mathcal{G}$ it holds that for all $i \in N$ and $j \in N \backslash \{i\}$ with $x_i(q, c) \leq x_j(q, c)$ it holds that for $t \geq q_j$,

$$x_i((q^{-j}, t), c) = x_i(q, c).$$

The *radial serial principle* requires robustness of the cost share rule with respect to the smaller agent for only those increases in the demand of the larger agent that correspond to the same direction of the original demand. More formally this reads as,

Definition 2.3 A cost sharing rule x satisfies the *radial serial principle* if for all cost sharing problems $(q, c) \in \mathcal{G}$ it holds that for all $i \in N$ and $j \in N \backslash \{i\}$ with $x_i(q, c) \leq x_j(q, c)$ it holds that for all $\alpha \geq 1$,

$$x_i((q^{-j}, \alpha q_j), c) = x_i(q, c).$$

The above properties are generalizations of the *independence of size of larger demands* property in Moulin and Shenker (1992a). However, the generalization is not straightforward. *Independence of size of larger demands* is makes use of direct comparison of the individual demands: "if an agent with a larger demand raises his demand, then the cost shares of the agents with the smaller demands will not be affected". But clearly this makes little or no sense in the multi-good case since, lacking a natural complete order on the demand space, comparison of demands becomes ambiguous. But cost shares can be compared still. Implicitly, the cost sharing mechanism makes a statement about how different demand profiles relate with respect to the size.

Lemma 2.4 *A cost sharing rule x that is generated by a collection of paths, satisfies the radial serial principle.*

Proof Let $(q, c) \in \mathcal{G}$ and take $i, j \in N$ such that $x_i(q, c) \leq x_j(q, c)$. Suppose that agent j raises his demand from q_j to $q'_j = \alpha q_j$ for some $\alpha \geq 1$. Then since the ratio between the demands for the different goods is kept the same, this will actually not affect the paths that are used in order to determine all the intermediate production levels at which cost increments are shared by sets of agents including i and j. So, agent i's cost share does not change. $\Box$

Lemma 2.5 *If a cost sharing rule is continuous on $\mathcal{D}_N(z)$ for all $z \in (\mathbb{R}^M_+)^N$, and satisfies the radial serial principle then it is generated by a collection of paths, and vice versa.*

Proof Suppose that x is a cost sharing rule satisfying the radial serial principle, such that $q \mapsto x(q, c)$ is continuous on each set $\mathcal{D}_N(z)$ for $z \in (\mathbb{R}^M_+)^N$. We will use arguments similar to those in Sprumont (1997) in order to conclude that for each $z \in (\mathbb{R}^M_+)^N$ and $c \in \mathcal{C}$ there is a collection of paths $\Pi(z, c)$ by which the cost shares for the subclass of cost sharing problems $\{(q, c) \,|\, q \in \mathcal{D}_N(z)\}$ are generated. Then, by varying over $c \in \mathcal{C}$ and $z \in (\mathbb{R}^M_+)^N$ we conclude that x is generated by $\cup_{z,c}\Pi(z, c)$.

Consider the subclass of cost sharing problems where the agents are restricted in their demand possibilities, i.e. each agent $i \in N$ may demand a bundle of goods that is a multiple of a certain bundle $z_i \in \mathbb{R}^M_+$. We will assume that $z_i \neq 0$ for all $i \in N$. The demand space of an agent i is then described by the radial through the vector z_i. In this framework, the demand of an agent i can also be expressed by the real number q_i if his demand is $q_i z_i$. In this way we create a *personalized good* model as in Sprumont (1997). Hence the arguments as in Sprumont (1997) can be repeated. Fix $z = (z_i)_{i \in N}$ and $c \in \mathcal{C}$. Define the set $p(0, c, N, z)$ by

$$p(0, c, N, z) := \{y \in \mathcal{D}_N(z) \,|\, x_i(y, c) = |N|^{-1} c(\sum_{j \in N} y_j) \text{ for all } i \in N\}.$$

Like in Sprumont (1997) it can be proved that the following two statements must hold:

i) $y, y' \in p(0, c, N, z), y \neq y' \Longrightarrow y' \ll y$ or $y \ll y$

ii) for each $t \in \mathbb{R}_+$, there is a $y \in p(0, c, N, z)$ such that $c(\sum_{j \in N} y_j) = t$.

For the proof of $i)$ we use the radial serial principle instead of the serial principle. Then the combination of (i) and (ii) implies that $p(0, c, N, z)$ is the image of some increasing path in $(\mathbb{R}_+^M)^N$, say $p^{0,c,N,z}$. One can extend the above reasoning to coalitions of decreasing size, as follows. Take $u \in p^{0,c,N,z}(\mathbb{R}_+)$ and let $S \subset N$. Denote by z_S the restriction of z to the agents in S. Let $p(u, c, S, z)$ be the set

$$\Big\{y \in \mathcal{D}(z_S) \,|\, \forall i \in S, x_i(u + (0_{N \setminus S}, y), c) - x_i(u, c) = \\ |S|^{-1} \left(c(\textstyle\sum_{j \in N}(u + (0_{N \setminus S}, y)_j) - c(\sum_{j \in N} u_j)\right)\Big\}.$$

Then this is again the image of some increasing path for S, say $p^{u,c,S,z}$. Repeat the above procedure for all possible combinations of $u \in p^{0,c,N,z}$ and $S \subset N$. The next step consists of choosing $S \subset N$, $u \in p^{0,c,N,z}$ and $v \in p^{u,c,S,z}$. Let $u' = u + (0_{N \setminus S}, v)$. Interchange in the above definition of $p(u, c, S, z)$ the symbol S by T for $T \subset S$, and u by $u' = u + (0_{N \setminus S}, v)$. Then by repeating the same procedure shows that $p(u', c, T, z)$ is the image of an increasing path $p^{u',c,T,z}$ for T. Now proceed by taking subsets of T, etcetera.

Then after this completes the third stage of defining our path collection. It should be clear now how to continue with the other stages $4, 5, \ldots, |N|$. Finally, for all combinations $u \in (\mathbb{R}_+^M)^N$ and $S \subseteq N$ for which $p^{u,c,S,z}$ is still not defined after completion of stage $|N|$, we take it to be an arbitrary path in $(\mathbb{R}_+^M)^S$. Define $\Pi(z, c) := \{p^{u,c,S,z} \,|\, u \in (\mathbb{R}_+^M)^N, S \subseteq N\}$. Then, by construction it holds that $\Pi(z, c)$ generates the cost shares of all problems $\{(q, c) \,|\, q \in \mathcal{D}_N(z)\}$.
This is proved as follows. Take $(q, c) \in \mathcal{G}$ such that $q \in \mathcal{D}_N(z)$. Let $\pi : \mathbb{R}_+ \to [0, q]$ be the production plan for q as is specified through $\Pi(z, c)$. Then let $t_1, t_2, \ldots, t_k \in \mathbb{R}_+$ be the very moments that new agents become satisfied with production using π. Without loss of generality assume that $t_1 \leq t_2 \leq \ldots \leq t_k$. For $i = 1, 2, \ldots, k$ let $u_i = \pi(t_i)$ and let M_i be the set of agents that are precisely satisfied with production at t_i. In addition put $u_0 = 0$. Then by

construction of π it holds for $i = 1, 2, \ldots, k$ and $j \in M_i \cup M_{i+1} \cup \ldots \cup M_k$ that

$$x_j(u_i, c) - x_j(u_{i-1}, c) = \frac{c(u_i) - c(u_{i-1}}{|M_i \cup M_{i+1} \cup \ldots \cup M_k|}.$$

So in combination with the radial principle this gives for $j \in M_1$

$$x_j(q, c) = x_j(u_1, c) = x_j^\pi(u_1, c) = x_j^\pi(q, c).$$

Then the proof is completed using an induction argument. Suppose that for all $j \in M_1 \cup M_2 \cup \ldots \cup M_r$ it holds that $x_j(q, c) = x_j^\pi(q, c)$. Then for $j \in M_{r+1}$,

$$\begin{aligned} x_j(q, c) &= x_j(u_{r+1}, c) = x_j(u_r, c) + \frac{c(u_{r+1}) - c(u_r)}{|M_{r+1} \cup M_{r+2} \cup \ldots \cup M_k|} \\ &= x_j^\pi(u_r, c) + \frac{c(u_{r+1}) - c(u_r)}{|M_{r+1} \cup M_{r+2} \cup \ldots \cup M_k|} \\ &= x^\pi(u_{r+1}, c) \;=\; x^\pi(q, c). \end{aligned}$$

The fourth equality follows by the construction of u_{r+1} together with the radial serial principle. So whenever $(q, c) \in \mathcal{G}$ is such that $q \in \mathcal{D}_N(z)$ we have $x(q, c) = x^\Pi(q, c)$. By varying over all possible profiles z we get that x is generated by $\cup_{z,c}\Pi(z, c)$. □

Typically, many cost sharing rules in the cost sharing literature determine cost shares through measuring the marginal cost along some curve in the demand space towards the aggregate demand. For instance, the Aumann-Shapley rule determines a unit price for each of the goods through measurement of the total marginal costs along the straight line towards the aggregate demand q. For the personalized model among the *curve-based* solutions one finds the Friedman-Moulin (1995) rule and the Moulin-Shenker rule (Sprumont (1997), Koster (1998)). For a given demand vector $q \in \mathbb{R}_+^N$ with $q_1 \leq q_2 \leq \ldots \leq q_n$, the Friedman-Moulin rule calculates marginal costs along the curve that interconnects the vectors $(q_1, \ldots, q_1), (q_1, q_2, \ldots, q_2), \ldots (q_1, q_2, \ldots, q_n)$, and is linear inbetween. In other cases, the cost shares are determined by performing an ordering of the demand profile first. The Moulin-Shenker rule can be seen as the non-additive counterpart of the Friedman-Moulin rule. Typically, for every cost sharing problem (q,c) it uses a different path $\gamma^{q,c}$ that is the unique solution of a particular system of differential equations that uses the parameters q and c as inputs. Then in the cost sharing problem (q,c) an agent i is charged $\int_{\gamma^{q,c}} D_i c$. We will generalize the approach that leads to the Moulin-Shenker rule, which will result in the *radial serial rule.* Given a cost sharing problem (q,c) we will define a production path such that the total of marginal costs corresponding to the individual agents is kept constant and equal to one for those agents requiring additional amounts of the goods. By the marginal costs it is meant the derivative with respect to the direction that is specified by the demand bundle of the agent. Loosely spoken this means for the case of 2 agents and 2 goods, that from production level 0 we will produce three times as much of the normalized bundle of agent 2 as that of agent 1 if the latter is three times as expenxive. Especially, in this way a vector field is specified. At each starting point in the production space the marginal costs of the normalized demands determine the direction specifying the ratio of marginal production for the different goods. multiples of the individual normalized bundles of the different agents that have to be produced. Once an agent is satisfied with production, the same principle is applied for the remaining agents that require some production still.

The above ideas lead to considering the following system of differential equations for $(q,c) \in \mathcal{G}$ with $q_i \neq 0$ for all $i \in N$. Let $\gamma : \mathbb{R}_+ \to (\mathbb{R}_+^M)^N$ be a mapping such that for all $i \in N, j \in M$, and $t \geq 0$,

$$D_j \gamma_i(t) = \begin{cases} \dfrac{\tilde{q}_{ij}}{D_{\tilde{q}_i} c(\sum_{k \in N} \gamma_k(t))} & \text{if } \gamma_{ij}(t) < q_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Recall that $\tilde{q}_i$ stands for the normalized demand of agent i and $D_{\tilde{q}_i} c$ for the corresponding directional derivative of c. It can be shown that there is a unique mapping $\gamma^{q,c} : \mathbb{R}_+ \to (\mathbb{R}_+^M)^N$ that satisfies the above equalities with initial value $\gamma(0) = 0$ (see Coddington and Levinson (1955)). Essentially, this is due to the regularity assumptions for cost functions, by which it is ascertained that for each $i \in N$ the mapping $z \mapsto (D_{\tilde{q}_i} c(z))^{-1}$ verifies a Lipschitz condition. Note the dependency of the curve on both the demand profile q and the cost function c. The curve $\gamma^{q,c}$ can also be interpreted as a production device. Suppose that the intensity at which an agent i is served at moment t is measured by the corresponding marginal cost in the direction of his demand, i.e. $D_{\tilde{q}_i}c(\sum_{k\in N}\gamma_k^{q,c}(t))\|D\gamma_i^{q,c}(t)\|$. Then $\gamma^{q,c}$ is just a device by which the unfullfilled agents are served with equal intensity 1, since

$$\begin{aligned}
\|D\gamma_i^{q,c}(t)\|^2 &= \langle D\gamma_i^{q,c}(t), D\gamma_i^{q,c}(t)\rangle = \sum_{j\in M}\left(\frac{1}{D_{\tilde{q}_i}c(\sum\limits_{k\in N}\gamma_k^{q,c}(t))}\cdot\tilde{q}_{ij}\right)^2 \\
&= \left(\frac{1}{D_{\tilde{q}_i}c(\sum\limits_{k\in N}\gamma_k^{q,c}(t))}\right)^2\cdot\sum_{j\in M}\tilde{q}_{ij}^2 \\
&= \left(\frac{1}{D_{\tilde{q}_i}c(\sum\limits_{k\in N}\gamma_k^{q,c}(t))}\right)^2\cdot\|\tilde{q}_{ij}\|^2 = \left(\frac{1}{D_{\tilde{q}_i}c(\sum\limits_{k\in N}\gamma_k^{q,c}(t))}\right)^2.
\end{aligned}$$

Furthermore, observe that whenever the agents do not ask for mixed bundles of goods but have a demand for one specific good instead, and such that there is just one demander per good, then this system of differential equations is just the one that is used to define the production curve for the Moulin-Shenker rule. We will now define the cost share of an agent by the total of marginal cost in his direction if the production plan $\gamma^{q,c}$ is carried out.

Definition 2.6 The *radial serial rule* x^r computes agent *i's* cost share for $(q,c)\in\mathcal{G}$ by the integral over the marginal costs in direction $\tilde{q}_i$ along the path $\gamma^{q,c}$,

$$x_i^r(q,c) := \int_0^\infty \left\langle Dc(\sum_{k\in N}\gamma_k^{q,c}(s)), D\gamma_i^{q,c}(s)\right\rangle ds.$$

Each personalized good cost sharing problem fits in our setting as a special case, for which x^r calculates the same cost shares as the Moulin-Shenker rule. Therefore, x^r can be considered as its extension.

Fix a cost function $c \in \mathcal{C}$. For each $d \in (\mathbb{R}_+^M)^N$, let $c^d \in \mathcal{C}$ be the cost function that relates each increase of demand $y \in \mathbb{R}_+^M$ after the aggregate of d to the corresponding incremental cost, i.e. $c^d(y) := c(y+\sum_{j\in N} d_j)-c(\sum_{j\in N} d_j)$ for all $y \in \mathbb{R}_+^M$. Then, an ordered pair $(d,c) \in (\mathbb{R}_+^M)^N \times \mathcal{C}$ gives rise to a system of differential equations in the following way. For $q \in (\mathbb{R}_+^M)^N$ with $q_k \neq 0$ for all $k \in N$, let $\gamma : \mathbb{R}_+ \to (\mathbb{R}_+^M)^S$ be such that for all $t \in \mathbb{R}_+$ and all $i \in S$, $j \in M$,

$$D_j\gamma_i(t) = \frac{1}{D_{\tilde{q}_i}c^d(\sum_{k\in N}\gamma_k(t))}\tilde{q}_{ij}.$$

By the regularity assumptions on c this system has a unique solution, which we will denote by $\gamma^{d,c,S,q}$. By varying over all possible quartets (d,c,S,q) this leads to a collection of paths Γ, which in turn generates the radial serial rule.

As we will show now, x^r is the unique cost sharing rule that satisfies the radial serial principle together with some smoothness property.

Theorem 2.7 *Consider a cost sharing rule x that satisfies the radial serial principle. If for all cost sharing problems $(q,c) \in \mathcal{G}$ it holds that for all $i \in N$, the derivatives of the mapping $t \mapsto x((q^{-i},t),c)$ in both the directions q_i and $-q_i$ exist and sum up to 0, then $x = x^r$.*

Proof First of all, notice that x^r has all the enlisted properties. As we explained above, x^r is generated by a collection of paths Γ and consequently, as a result of Lemma 2.4, we conclude that x^r satisfies the radial serial principle. The regularity conditions are easily checked by restriction of x^r to subclasses of cost sharing problems of type $\{(q,c) \,|\, q \in \mathcal{D}_N(z)\}$ for fixed $z \in (\mathbb{R}_+^M)^N$. Furthermore, as we explained earlier, x^r mimicks the Moulin-Shenker rule on each of the subclasses of cost sharing problems $\{(q,c) \,|\, q \in \mathcal{D}_N(z)\}$ with fixed $z \in (\mathbb{R}_+^M)^N$. The existence of the directional derivatives for the mapping $y \mapsto x^r((q^{-i},y),c)$ at q_i with respect to $\tilde{q}_i$ and $-\tilde{q}_i$ follows from the fact that the Moulin-Shenker rule is continuously differentiable (Sprumont (1997)), and for the same reason the total sum of both must equal 0. Especially, continuity of $q \mapsto x^r(q,c)$ on $\mathcal{D}_N(z)$ for each $z \in (\mathbb{R}^M_+)^N$ is implied for all $c \in \mathcal{C}$.

Suppose that there is another cost sharing rule ξ that satisfies the above properties. Then, first of all, by Lemma 2.5 it holds that ξ is generated by a collection of paths. Now fix $(q,c) \in \mathcal{G}$. Just for notational convenience we will assume that there are only two agents, such that $N = \{1,2\}$, though such a restriction does not change the basics of the below arguments. Suppose that the corresponding path for (q,c), say π, does not equal $\gamma^{q,c}$. Without loss of generality assume that π is differentiable and that there is a moment $t \in \mathbb{R}_+$ at which the agents are served at different intensities, say

$$D_{\tilde{q}_1} c(\pi_1(t) + \pi_2(t)) \|D\pi_1(t)\| < D_{\tilde{q}_2} c(\pi_1(t) + \pi_2(t)) \|D\pi_2(t)\|. \tag{1}$$

Existence of such a t follows from the fact that π is not the Moulin-Shenker path corresponding to this cost sharing problem.
Now let $q = \pi(t)$. Define the mapping $h : \mathbb{R}^M_+ \to \mathbb{R}_+$ by $h(z) = \xi_2((q^{-2}, z), c)$ for all $z \in \mathbb{R}^M_+$. We are about to show that $D_{-\tilde{q}_1} h(q) \neq -D_{\tilde{q}_1} h(q)$ which contradicts the premises for ξ. Let q^* be the demand profile for N with $q_1^* = \tilde{q}_1$ and $q_2^* = 0$. Consider

$$D_{\tilde{q}_1} h(q) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left\{ \xi_2(q + \epsilon q^*, c) - \xi_2(q,c) \right\}.$$

The above limit exists and equals 0 since by the radial serial principle we have for all ϵ,

$$\xi_2(q + \epsilon q^*, c) = \xi_2(q,c) = \frac{1}{2} c(q_1 + q_2).$$

Next, we will show that $D_{-\tilde{q}_1} h(q)$ also exists but is not equal to 0. First define $p : \pi_1(\mathbb{R}_+) \to \mathbb{R}^M_+$ by $p(t_1) = t_2 \Leftrightarrow (t_1, t_2) \in \pi(\mathbb{R}_+)$. Then for each $\epsilon < 0$ we have

$$\xi_2(q_1 + \epsilon q^*, c) - \xi_2(q,c) = c(q_1 + \epsilon \tilde{q}_1 + q_2) - c(q_1 + q_2) +$$

$$+ \tfrac{1}{2}\{c(q_1 + q_2) - c(q_1 + \epsilon \tilde{q}_1 + p(q_1 + \epsilon \tilde{q}_1))\}.$$

So, in addition, by taking the limits for $\epsilon \uparrow 0$ on both sides of the equality sign this gives

$$D_{-\tilde{q}_1} h(q) = D_{-\tilde{q}_1} c(q_1 + q_2) - \frac{1}{2} \lim_{\epsilon \uparrow 0} \frac{1}{\epsilon} \{c(q_1 + \epsilon \tilde{q}_1 + p(q_1 + \epsilon \tilde{q}_1)) - c(q_1 + q_2)\}.$$

The limit on the right hand side exists and equals

$$\lim_{\epsilon \uparrow 0} \frac{1}{\epsilon} \left\{ c(q_1 + \epsilon \tilde{q}_1 + \epsilon \frac{\|D\pi_2(t)\|}{\|D\pi_1(t)\|} \tilde{q}_2) - c(q_1 + q_2) \right\}.$$

But this term equals

$$D_{-\tilde{q}_1 - \frac{\|D\pi_2(t)\|}{\|D\pi_1(t)\|} \tilde{q}_2} c(q_1 + q_2) = -D_{\tilde{q}_1} c(q_1 + q_2) - \frac{\|D\pi_2(t)\|}{\|D\pi_1(t)\|} D_{\tilde{q}_2} c(q_1 + q_2).$$

Thus we get

$$D_{-\tilde{q}_1} h(q) = -\frac{1}{2} D_{\tilde{q}_1} c(q_1 + q_2) + \frac{1}{2} \frac{\|\pi_2(t)\|}{\|\pi_1(t)\|} D_{\tilde{q}_2} c(q_1 + q_2).$$

Then, since $q = \pi(t)$ and by (1),

$$-D_{\tilde{q}_1} h(q) = D_{-\tilde{q}_1} h(q) > -\frac{1}{2} D_{\tilde{q}_1} c(q_1 + q_2) + \frac{1}{2} D_{\tilde{q}_1} c(q_1 + q_2) = 0.$$

□

Theorem 2.8 *There is no cost sharing rule x that satisfies the serial principle such that the mapping $q \to x(q, c)$ is differentiable for all $c \in \mathcal{C}$.*

Proof Suppose that x is a cost sharing rule such that for all $c \in \mathcal{C}$ it holds that $q \to x(q, c)$ is differentiable. Then especially x satisfies the regularity conditions as in Theorem 2.7. So if x satisfies the serial principle, and thus also the radial serial principle, then according to Theorem 2.7 it must hold that $x = x^r$. But as is easily seen, x^r does not satisfy the serial principle. Changing the ratio of the demand for the different goods of an agent j will cause the entire path $\gamma^{q,c}$ to change. In this way also the cost share of a 'smaller' agent i will be affected. So, since x^r does not satisfy the serial principle it can not be the case for x, which leads to a contradiction. □

Note that the above result can be strengthened by interchanging differentiability with the smoothness condition that was used in Theorem 2.7.

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