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THE STRUCTURE OF THE SET OF EQUILIBRIA

FOR

TWO PERSON MULTICRITERIA GAMES

Peter Borm, Dries Vermeulen, and Mark Voorneveld

Abstract

In this paper the structure of the set of equilibria for two person multicriteria games is analysed. It turns out that the classical result for the set of equilibria for bimatrix games, that it is a finite union of polytopes, is only valid for multicriteria games if one of the players only has two pure strategies. A full polyhedral description of these polytopes can be derived when the player with an arbitrary number of pure strategies has one criterion.

JEL Code: C61, C72

1. Introduction

Multicriteria strategic form games were first introduced by Blackwell (1956). The difference between these games and ordinary strategic form games is that a player in a strategic form game only has one criterion (his payoff) to evaluate the outcome of the game (i.e. the profile of strategies chosen by the players of the game) while in a multicriteria game each player may have an arbitrary number of criteria (payoffs) that are intrinsically uncomparable with each other.

Nash introduced the notion of an equilibrium for non-cooperative games in strategic form in his papers in 1950 and 1951. Since then the equilibrium concept has been and still is being studied extensively. One of the topics in this investigation is the structure of the set of equilibria of a bimatrix game. (A bimatrix game is a non-cooperative game in strategic form with two players.) Over the last decades a fair number of papers has been published on this topic. It turned out that the set of equilibria of a bimatrix game is a finite union of polytopes. Proofs of this fact can for example be found in Winkels (1979), Jansen (1981) and Jansen and Jurg (1990).

From a computational point of view these results are quite important. The main reason for this is that the original proof of Nash of the existence of equilibria is not constructive. It shows that the assumption that a non-cooperative game does not have an equilibrium leads to a contradiction. It does therefore not tell you how to find an equilibrium for a given game. Also the basic inequalities in the definition of the equilibrium concept are not of much help. In general (without further assumptions on the structure of the game) these inequalities are polynomial and it is not clear how one can actually calculate one single solution given these inequalities, let alone how to find a parametric representation of the complete set of equilibria.

In the case of bimatrix games life is much simpler. For such a game it is possible to show that the set of equilibria is a finite union of polytopes and it is moreover possible to derive a polyhedral description of each of these polytopes. Hence, by using some theory of linear inequalities, it is possible to compute all extremal points of such a polytope and in this way find a parametric description of the set of equilibria. There are also a number of exact algorithms for the computation of one specific equilibrium, such as the algorithm of Lemke and Howson (1964), that are based on the special structure of the set of equilibria for bimatrix games.

In this paper we investigate to what extent the results on the structure of the set of equilibria of a bimatrix game can be carried over to the equilibrium concept introduced by Shapley (1959) for two person multicriteria games. Unfortunately our results are on the negative side of the spectrum. First of all we provide an example to show that the set of equilibria may have a quadratic component whenever both players have three or more pure strategies and one of the players has more than one criterion. Secondly we show that the set of equilibria is indeed a finite union of polytopes if one of the players has two pure strategies. The actual polyhedral description of these polytopes cannot be computed directly though, unless the player with an arbitrary number of pure strategies has exactly one criterion.

Notation For a finite set F, the number of elements of F is denoted by $|F|$ and $\Delta(F)$ denotes the set of probability vectors on F. For an element $x \in \Delta(F)$, the *carrier* of x is the set $\{i \in F \mid x_i > 0\}$. The element e_i of $\Delta(F)$ denotes the probability vector that puts all weight on $i \in F$. For two vectors $x, y \in \mathbb{R}^n$ $\langle x, y \rangle := \sum_{i=1}^{n} x_i \cdot y_i$ is the inner product of x and y, and if $x_i < y_i$ holds for all

$i = 1, \ldots, n$ we say that y dominates x. We write $x \leq y$ if $x_i \leq y_i$ for all $i = 1, \ldots, n$.

2. Preliminaries

In a (two-person multicriteria) game the first player has a finite set M of pure strategies and player two has a finite set N of pure strategies. The players are supposed to choose their strategies simultaneously. Given their choices $m \in M$ and $n \in N$, player one has a finite set S of criteria to evaluate the pure strategy pair (m, n). For each criterion $s \in S$ the evaluation is a real number $(A_s)_{mn} \in \mathbb{R}$. Of course we also have an evaluation $(B_t)_{mn} \in \mathbb{R}$ for each criterion $t \in T$ of player two. Thus the game is specified by the two sequences

$$A := (A_s)_{s \in S} \qquad \text{and} \qquad B := (B_t)_{t \in T}$$

of $M \times N$-matrices

$$A_s := [(A_s)_{mn}]_{(m,n) \in M \times N} \qquad \text{and} \qquad B_t := [(B_t)_{mn}]_{(m,n) \in M \times N}.$$

Despite the fact that the players may have more than one criterion, we will refer to A and B as payoff matrices. The game is denoted by (A, B). The players of the game are also allowed to use mixed strategies. Given such mixed strategies $p \in \Delta(M)$ and $q \in \Delta(N)$ for players one and two resp, the vectors

$$pAq := (pA_sq)_{s \in S} \qquad \text{and} \qquad pBq := (pB_tq)_{t \in T}$$

are called *payoff vectors* (for players one and two, resp.).

BEST REPLIES AND EQUILIBRIA

In the context of bimatrix games (games in which each of the two players has exactly one criterion) the equilibrium concept of Nash is one of the best known ways to solve these games. A very convenient way to define equilibria, certainly when one wants to analyze their structure, is by means of best replies.

An analogous approach can be used in the case of multicriteria games. Shapley (1959) first introduced the notion of equilibrium for this type of games. His definition is a direct generalization of the equilibrium concept for strategic form games with only one criterion. In order to describe this definition we need to generalize the notion of a best reply.

Definition 1. Let $q \in \Delta(N)$ be a strategy of player two. A strategy $p \in \Delta(M)$ of player one is called a *best reply* of player one against q if there is no other strategy $p' \in \Delta(M)$ such that the payoff vector $p'Aq$ dominates the payoff vector pAq. The set of best replies of player one against q is denoted by $BR_1(q)$.

It almost goes without saying that we also can define best replies against a strategy p and the set $BR_2(p)$ for player two. Now the definition of equilibrium runs as follows.

Definition 2. A strategy pair (p^*, q^*) is called an *equilibrium* if p^* is a best reply of player one against q^* and q^* is a best reply of player two against p^*.

Remark It is also possible to define a more restrictive notion of equilibrium based on the dominance relation on $\mathbb{R}^n$ defined by "x dominates y if $x_i \geq y_i$ for all i, and at least one of these inequalities is strict". Since this relation does not necessarily yield a closed set of equilibria (see e.g. Borm, Tijs, and van den Aarssen (1988)), we decided to use the weaker version. Nevertheless, proofs similar to the ones presented in this paper show that also in this case we can find a decomposition of the set of equilibria into a number of relative interiors of polytopes.

3. Stability regions and structure

In case of bimatrix games, the proof that the set of equilibria is a finite union of polytopes is based on the fact that this set of equilibria can be chopped up in a finite number of sets. Then each of these sets can easily be shown to be a polytope. It turns out to be worthwhile to execute this procedure for multicriteria games as well.

SHAPLEY'S RESULT

First of all we need the result of Shapley (1959). Essentially Shapley (1959) provides a link between best replies and linear programs. In order to describe this link we need to introduce some terminology.

Recall that for each criterion $t \in T$ the real number $e_i B_t e_j$ is the payoff of player two according to his criterion t and B_t is the matrix whose entry on place i, j is this number $e_i B_t e_j$. Now suppose that player two decides to assign a weight $\mu_t \geq 0$ to each criterion $t \in T$ available to him (we assume that $\sum_{t \in T} \mu_t$ equals one). The vector $\mu = (\mu_t)_{t \in T}$ is called a *weight vector.* According to the criterion associated with this weight vector the evaluation of the outcome (e_i, e_j) is the real number

$$\sum_{t \in T} \mu_t e_i B_t e_j = e_i \left(\sum_{t \in T} \mu_t B_t \right) e_j.$$

So, given the weight vector $(\mu_t)_{t \in T}$, player two in effect uses the matrix

$$B(\mu) := \sum_{t \in T} \mu_t B_t$$

to calculate his payoff. With this terminology, the result of Shapley (1959) can be rephrased as follows.

Lemma 1. *Let p be a strategy of player one and let q be a strategy of player two. Then the following two statements are equivalent.*

(i) *q is a best reply of player two against p*

(ii) *there exists a weight vector $\mu := (\mu_t)_{t \in T}$ such that q is a best reply of player two against p according to the criterion associated with $B(\mu)$.*

In words, the Lemma states that q is a best reply of player two against p if and only if player two can assign to each criterion $t \in T$ a non-negative weight μ_t such that the resulting weighted criterion is maximal in q, given that player one plays p.

THE STRUCTURE OF THE SET OF EQUILIBRIA

In this section we will construct a decomposition of the set of equilibria of the game (A, B) into a finite number of sets that are easier to handle. This decomposition is in fact the multicriteria equivalent of the technique that is used to prove that the set of equilibria of a bimatrix game is a finite union of polytopes. In order to give the reader some background concerning the line of reasoning employed here, we will first give an informal discussion of this technique.

Suppose that we have a bimatrix game and a subset I of the set of pure strategies of player one. Then we can associate two areas with this set, one in the set of mixed strategies of player one and one in the set of mixed strategies of player two. For player one, this is the set $\Delta(I)$ of mixed strategies that put all weight exclusively on the pure strategies in I, and for player two this is the set $U(I)$ of mixed strategies of player two against which (at least) all strategies in $\Delta(I)$ are best replies. (Such a set $U(I)$ is called a stability region.) Obviously we can do the same for a subset J of the set of pure strategies of player two.

Now the crucial point is that (for a bimatrix game) all these sets $\Delta(I)$, $\Delta(J)$, $U(I)$, and $U(J)$ are polytopes (and for each of these polytopes it is even possible to find a describing system of linear inequalities). So, also the set

$$(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I))$$

is a polytope. Moreover there is only a finite number of such sets *and* it can be shown that their union equals the set of equilibria of the given bimatrix game.

Although the sets $U(I)$ and $U(J)$ not necessarily need to be polytopes in the multicriteria case, we can still carry out this procedure for two person multicriteria games.

To this end, let v be an element of $\mathbb{R}^n$ and let P be a polytope in $\mathbb{R}^n$. The vector v is said to *attain its maximum over* P in the point $x \in P$ if

$$\langle v, x \rangle \geq \langle v, y \rangle \quad \text{for all } y \in P.$$

Then we have the following well-known Lemma.

Lemma 2. *Let v be a vector in $\mathbb{R}^n$. Further, let P be a polytope in $\mathbb{R}^n$ and let F be a face of P. If v attains its maximum over P in some relative interior point x of F, then it also attains its maximum over P in any other point of F.*

Now let I be a subset of M. Slightly abusing notation we write $\Delta(I)$ for the set of strategies $p \in \Delta(M)$ whose carrier is a subset of I. Further, the *stability region* $U(I)$ (of player two) is defined as

$$U(I) := \{q \in \Delta(N) \mid \Delta(I) \subset BR_1(q)\}.$$

Similarly we can define sets $\Delta(J)$ and $U(J)$ for a subset J of N.

Theorem 1. *The set of equilibria of the game* (A, B) *equals the union over all* $I \subset M$ *and* $J \subset N$ *of the sets*

$$(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I)).$$

Proof. (a) Assume that a strategy pair (p^*, q^*) is an element of a set $(\Delta(I) \cap U(J)) \times (\Delta(J) \cap U(I))$ for some subset I of M and subset J of N. We will only show that p^* is a best reply against q^*.

Since q^* is an element of $U(I)$, we know that any strategy in $\Delta(I)$ is a best reply against q^*. However, p^* *is* an element of $\Delta(I)$ by assumption. Hence, p^* is a best reply against q^*.

(b) Conversely, let (p^*, q^*) be an equilibrium. Take $I = C(p^*)$ and $J = C(q^*)$. We will show that p^* is an element of $\Delta(I) \cap U(J)$.

Obviously p^* is an element of $\Delta(I)$. So we only need to show that p^* is also an element of $U(J)$. In other words, we need to show that each strategy $q \in \Delta(J)$ is a best reply against p^*. To this end, take a $q \in \Delta(J)$. Since q^* is a best reply against p^* we know by Lemma 1 that there exists a weight vector $\mu = (\mu_t)_{t \in T}$ such that q^* is a best reply against p^* according to the criterion associated with $B(\mu)$. In other words, the vector $p^* B(\mu)$ attains its maximum over $\Delta(N)$ in q^*. However, since q^* is an element of the relative interior of $\Delta(J)$, $p^* B(\mu)$ must also attain its maximum in q by Lemma 2. Hence, q is a best reply against p^* according to $B(\mu)$, and, again by Lemma 1, q is a best reply against p^*. $\triangleleft$

Clearly the sets $\Delta(I)$ and $\Delta(J)$ are polytopes for all subsets I of M and J of N. So, from the previous Theorem it follows that the set of equilibria of the game (A, B) is a finite union of polytopes as soon as the sets $U(I)$ and $U(J)$ are polytopes. Unfortunately this need not be the case. In the next section we will provide a counterexample.

4. An Example

We will give a fairly elaborate analysis of the counterexample. This is done because the calculations involved in the determination of best replies and stability regions for this game are exemplary for such calculations in general.

There are two players in the game. Both players have three pure strategies. The pure strategies of player one are called T, M, and B, the pure strategies of player two are called L, C, and R. Further, player one has two criteria and player two has only one criterion. The payoff for player two according to his criterion is always zero. The payoff matrix A of player one is

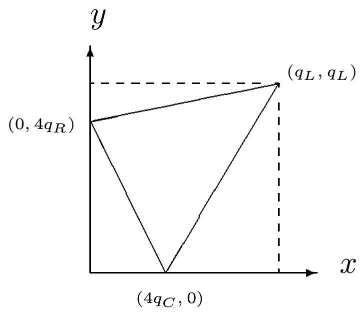
$$\begin{bmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (4,0) & (0,0) \\ (0,0) & (0,0) & (0,4) \end{bmatrix}.$$

Player one is the row player and player two is the column player. The first digit in an entry gives the evaluation by player one of the occurence of that particular entry according to his first criterion. The second digit gives the evaluation according to his second criterion.

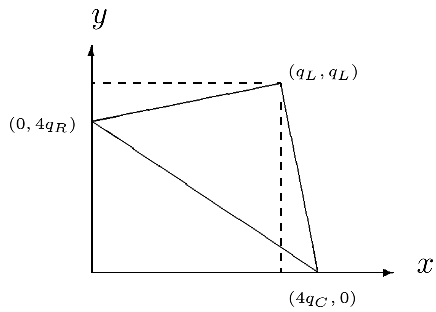
Since player two is completely indifferent it is immediately clear that a strategy pair (p^*, q^*) is an equilibrium if and only if p^* is an element of $BR_1(q^*)$. In other words, the set of equilibria equals the graph of the best reply correspondence BR_1. In order to calculate this graph we will first compute the areas in the strategy space of the second player where the best reply correspondence BR_1 is constant. In other words, we need to compute the stability regions of player two.

First of all note that if player two plays strategy $q = (q_L, q_C, q_R)$ and player one plays his pure strategy e_T, the payoff for player one is $e_T A q = (q_L, q_L)$. This is a point on the line $x = y$ when plotted in the xy-plane. Similarly, $e_M Aq = (4q_C, 0)$ is a point on the line $y = 0$ and $e_B Aq = (0, 4q_R)$ is a point on the line $x = 0$. Now there are five possible situations as is depicted below.

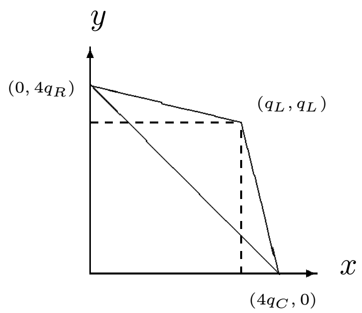
Situation I



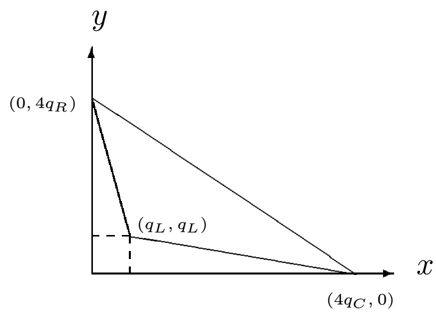
Situations II and III



Situation IV

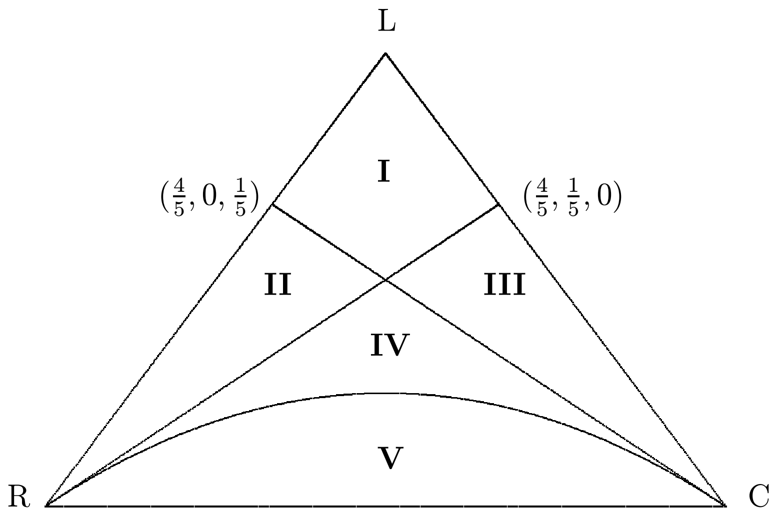


Situation V



In situation **I** both $e_M Aq$ and $e_B Aq$ are dominated by $e_T Aq$. In situation **II** $e_T Aq$ dominates $e_B Aq$, but does not dominate $e_M Aq$. (Situation **III** is the symmetric situation with the roles of the second and third pure strategy of player one interchanged.) In situation **IV** $e_T Aq$ is itself undominated and dominates neither $e_M Aq$ nor $e_B Aq$, and **V** depicts the situation in which $e_T Aq$ is dominated by some convex combination of $e_M Aq$ and $e_B Aq$.

Now if we calculate exactly where in the strategy space of player two these five situations occur we get the picture below. The boldface Roman numbers in the various areas in this picture correspond to the Roman numbers assigned to the situations depicted above. Notice that an area in the strategy space of player two corresponding to one of the five situations above is necessarily of full dimension by the graphics above. Further, one cannot jump from situation **V** to situations **I**, **II** or **III** without crossing the area where situation **IV** occurs (except on the boundary of the strategy space).



Stability regions of player two

The boundary line between areas **I** and **II** and areas **III**, **IV** and **V** is given by the equality $q_L = 4q_R$. Similarly, $q_L = 4q_C$ is the boundary between areas **I** and **III** and areas **II**, **IV** and **V**.

Finally, it can be seen in the graphics above that the boundary between area **V** and the others is exactly the set of strategies where $e_T Aq$ is an element of the line segment between $e_M Aq$ and $e_B Aq$. This means that it is the set of strategies for which (q_L, q_L) satisfies the linear equation $q_R x + q_C y = 4q_C q_R$. Hence it must be the set of strategies that satisfy the quadratic equation

$$q_L q_R + q_L q_C = 4q_C q_R$$

(except the solution $(q_L, q_C, q_R) = (1, 0, 0)$ of this equation). This gives us enough information to write down the stability regions of player two.

$$\begin{array}{rcl}
U(\{T\}) & = & \mathbf{I} \cup \mathbf{II} \cup \mathbf{III} \cup \mathbf{IV} \\
U(\{M\}) & = & \mathbf{II} \cup \mathbf{IV} \cup \mathbf{V} \\
U(\{B\}) & = & \mathbf{III} \cup \mathbf{IV} \cup \mathbf{V} \\
U(\{T, M\}) & = & \mathbf{II} \cup \mathbf{IV} \\
U(\{T, B\}) & = & \mathbf{III} \cup \mathbf{IV} \\
U(\{M, B\}) & = & \mathbf{V} \\
U(\{T, M, B\}) & = & \mathbf{IV} \cap \mathbf{V}
\end{array}$$

Note the essential differences with the structure of stability regions for bimatrix games. For a bimatrix game we would for example have the equality

$$U(\{M, B\}) = U(\{M\}) \cap U(\{B\}).$$

The example shows that this is no longer true for multicriteria games. In this case the set

$$U(\{M\}) \cap U(\{B\}) = \mathbf{IV} \cup \mathbf{V}$$

subdivides into the areas **IV**, on whose relative interior

$$\Delta(\{T, M\}) \cup \Delta(\{T, B\})$$

is the set of best replies, and **V**, on whose relative interior the set of best replies is indeed $\Delta(\{T, M, B\})$. An area like **IV** simply cannot occur for bimatrix games.

The second essential difference, and the main one in this section, is the fact that $U(\{T, M, B\})$ is a quadratic curve. This means that the subset

$$\Delta(\{T, M, B\}) \times U(\{T, M, B\})$$

of the set of equilibria cannot be written as a finite union of polytopes. This concludes the example.

5. Multicriteria games of size $\mathbf{2 \times n}$

The previous example shows that, in case at least one of the players has more than one criterion, the set of equilibria may have a quadratic component as soon as both players have at least three pure stategies. So, in the multicriteria case it is necessary to have (at least) one player who has exactly two pure strategies to guarantee that the set of equilibria is indeed a finite union of polytopes. So assume w.l.o.g. that player one's set of pure strategies M equals $\{T, B\}$. In this section we will show that this assumption is also sufficient, i.e., under this assumption the set of equilibria is indeed a finite union of polytopes. A complication though is that we only have a polyhedral description of those polytopes when player two has only one criterion.

STABILITY REGIONS OF PLAYER TWO

In this special case the analysis of the dominance relation on the possible payoff vectors for player one for a fixed strategy q of player two is quite straightforward. Since player one has only two pure strategies e_T and e_B, the set of possible payoff vectors is a line segment (or a singleton in case $e_T Aq = e_B Aq$) in $\mathbb{R}^S$. Given this observation it is easy to check

Lemma 3. *The following two statements are equivalent.*

(i) *$e_T Aq$ is dominated by pAq for some $p \in \Delta(M)$*

(ii) *$e_T Aq$ is dominated by $e_B Aq$.*

Given this lemma we can show that each stability region of player two is a finite union of polytopes. Two cases are considered.

Case 1. For $|I| = 1$. Assume for the moment that $I = \{T\}$. Then

$$\begin{aligned} U(I) &= \{q \in \Delta(N) \mid \Delta(\{T\}) \subset BR_1(q)\} \\ &= \{q \in \Delta(N) \mid e_T \in BR_1(q)\} \\ &= \{q \in \Delta(N) \mid e_T Aq \text{ is not dominated by } pAq \text{ for any } p \in \Delta(M)\} \\ &= \{q \in \Delta(N) \mid e_T Aq \text{ is not dominated by } e_B Aq\} \\ &= \bigcup_{s \in S} \{q \in \Delta(N) \mid e_T A_s q \geq e_B A_s q\} \end{aligned}$$

where the fourth equality follows from the previous Lemma. Clearly this last expression is a finite union of polytopes. By the same line of reasoning we get that $U(\{B\})$ is a finite union of polytopes.

Case 2. For $I = \{T, B\}$. Using Lemma 3 it is easy to check that $U(I)$ is the set of strategies q for which $e_T Aq$ does not dominate $e_B Aq$ and $e_B Aq$ does not dominate $e_T Aq$. So, $U(I) = U(\{T\}) \cap U(\{B\})$. Thus, since both $U(\{T\})$ and $U(\{B\})$ are finite unions of polytopes as we saw in Case 1, $U(I)$ is also a finite union of polytopes.

STABILITY REGIONS OF PLAYER ONE

Now that we have come this far, the only thing left to prove is that the stability region

$$U(J) = \{p \in \Delta(M) \mid \Delta(J) \subset BR_2(p)\}$$

is a finite union of polytopes for each set $J \subset N$ of pure strategies of player two. In order to do this we need to do some preliminary work.

Let the subset $V(J)$ of $\Delta(M) \times \mathrm{I\!R}^T$ be defined by

$$\begin{aligned} V(J) \;\; &:= \{(p,\mu) \mid \quad \Delta(J) \text{ is included in the set of best replies against } p \\ &\qquad\qquad\qquad \text{according to the criterion } B(\mu)\} \\ &= \{(p,\mu) \mid \quad \Delta(J) \text{ is included in the set of strategies where} \\ &\qquad\qquad\qquad \text{the vector } pB(\mu) \text{ attains its maximum over } \Delta(N)\}. \end{aligned}$$

Note that we allow $pB(\mu)$ to attain its maximum in points outside $\Delta(J)$ as well. We only require that $\Delta(J)$ is indeed a subset of the set of points where $pB(\mu)$ attains its maximum over $\Delta(N)$.

Further, let the projection $\pi\colon \mathrm{I\!R}^2 \times \mathrm{I\!R}^T \to \mathrm{I\!R}^2$ be defined by

$$\pi(p,v) := p \quad \text{for all } (p,v) \in \mathrm{I\!R}^2 \times \mathrm{I\!R}^T.$$

Now we can prove

Lemma 4. *The stability region $U(J)$ equals the projection $\pi\,(V(J))$ of the set $V(J)$.*

Proof. (a) Let p be an element of $U(J)$. We will show that p is also an element of $\pi\,(V(J))$.

Let q^* be an element of the relative interior of $\Delta(J)$. Since p is an element of $U(J)$ we know that q^* is a best reply to p. Then we know, by Lemma 1, that there is a weight vector $\mu = (\mu_t)_{t\in T}$ such that the vector $pB(\mu)$ attains its maximum over $\Delta(N)$ in q^*. So, since q^* is a relative interior point of $\Delta(J)$, $pB(\mu)$ also attains its maximum over $\Delta(N)$ in any other point of $\Delta(J)$ by Lemma 2. Therefore (p, μ) is an element of $V(J)$ and $p = \pi(p, \mu)$ is an element of $\pi\,(V(J))$.

(b) Conversely, let $p = \pi(p, \mu)$ be an element of $\pi\,(V(J))$ and let q be an element of $\Delta(J)$. Then we know that the vector $pB(\mu)$ attains its maximum over $\Delta(N)$ in q. Again by Lemma 1, this means that q is a best reply against p. Hence, since q was chosen arbitrarily in $\Delta(J)$, p is an element of $U(J)$. $\triangleleft$

Now it is straightforward to show

Theorem 2. *For a multicriteria game of size $2 \times n$ the stability region $U(J)$ is a finite union of polytopes.*

Proof. Observe that the set $V(J)$ is the collection of points $(p, \mu) \in \mathrm{I\!R}^2 \times \mathrm{I\!R}^T$ that satisfy the system of polynomial (in)equalities

$$\begin{aligned}
p_i &\geq 0 \quad i = 1, 2 \\
p_1 + p_2 &= 1 \\
\mu_t &\geq 0 \quad \text{for all } t \in T \\
\sum_{t \in T} \mu_t &= 1 \\
\sum_{t \in T} \mu_t p B_t e_j &\geq \sum_{t \in T} \mu_t p B_t e_k \quad \text{for all } j \in J \text{ and } k \in N.
\end{aligned}$$

Therefore, $V(J)$ is a semi-algebraic set. Furthermore, by the previous Lemma, $U(J)$ is the set of vectors $p \in \mathrm{I\!R}^2$ such that there exists a $\mu \in \mathrm{I\!R}^T$ for which

$$(p, \mu) \in V(J).$$

Hence, by the Theorem of Tarski and Seidenberg (see e.g. Blume and Zame (1994) for a clear discussion of this Theorem) $U(J)$ is also a semi-algebraic set. Further,

$U(J)$ is compact, since $V(J)$ is compact and π is continuous. So, $U(J)$ is the union of a finite collection $\{S_\alpha\}_{\alpha \in A}$ of sets S_α in $\Delta(M)$ and each S_α is described by a finite number of polynomial inequalities

$$p_{\alpha,k}(x) \geq 0 \quad (k = 1, \dots, m(\alpha)).$$

However, $\Delta(M)$ is a line segment in $\mathbb{R}^2$. So the set of points in $\Delta(M)$ that satisfies one particular inequality is the finite union of (closed) line segments (singletons also count as line segments). So, since each S_α is the intersection of such finite unions, S_α is itself the finite union of closed line segments. Therefore, since $U(J)$ is the finite union over all sets S_α, it is the finite union of closed line segments. Hence, $U(J)$ is a finite union of polytopes. $\triangleleft$

THE CASE $|T| = 1$

In this case we have a complete polyhedral description of the polytopes involved in the union. Notice that we already know that the sets $\Delta(I)$ and $\Delta(J)$ are polytopes, and the sets $U(I)$ and $U(J)$ are finite unions of polytopes. We will now show that a polyhedral description of all these polytopes can be found.

For the polytopes $\Delta(I)$, $\Delta(J)$ this polyhedral description is trivial. For $U(I)$ we saw in Case 1 below Lemma 3 that it is the finite union of polytopes of the form

$$\{q \in \Delta(N) \mid e_T A_s q \geq e_B A_s q\}.$$

So, in Case 1 the polytopes involved in the union are already given by linear inequalities. This implies that also in Case 2 we can find the linear inequalities that describe the polytopes involved. Finally, for $J \subset N$, we get

$$\begin{aligned} U(J) &= \{p \in \Delta(M) \mid \Delta(J) \subset BR_2(p)\} \\ &= \{p \in \Delta(M) \mid pBe_j \geq pBe_k \text{ for all } j \in J \text{ and } k \in N\}. \end{aligned}$$

The assumption that $|T| = 1$ is used in the second equality. The last expression in the display now shows that $U(J)$ is itself a polytope that can be written as the solution set of a finite number of linear inequalities. This concludes the argumentation.

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