Characterizations of the Egalitarian Solution for Convex Games

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Characterizations of the Egalitarian Solution for Convex Games[1]

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Abstract: The egalitarian solution for TU-games as introduced by Dutta and Ray [3] is s-tudied. Two characterizations of the restriction of this solution to the class of convex games are given, using weak variants of the reduced game properties of Hart and Mas-Colell [6] and Davis and Maschler [5]. The other properties are a stability property, inspired by Selten [8], and a property restricting maximum payoffs. Further, a dual egalitarian solution is introduced and it is proved that for a convex game the egalitarian allocation is equal to the dual egalitarian allocation for its dual concave game.

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1 Introduction

Dutta and Ray [3] introduced the egalitarian solution as a solution concept for TU-games. This solution unifies the two conflicting concepts of individualistic utility maximization and the social goal of equality. Under certain conditions it is non-empty, and then its outcome is unique, namely it is the Lorenz maximal element of the set of payoffs satisfying core-like participation constraints. We refer to Dutta and Ray [3] for the details. For convex games Dutta and Ray [3] describe an algorithm to locate the unique egalitarian solution, and they show, in addition, that it is in the core. Dutta [2] characterizes the egalitarian solution over the class of convex games. Dutta and Ray [4] consider a parallel concept, the S-constrained egalitarian solution. Arin and Iñarra [1] introduce a solution concept that coincides with the egalitarian solution for 2-person games. This solution concept is called the egalitarian set.

Dutta [2] characterized the egalitarian solution over the class of convex games. The main properties used are the reduced game properties due to Hart and Mas-Colell [6] and Davis and Maschler [5]. The egalitarian solution is the only solution concept satisfying either of the two reduced game properties and a prescriptive property on two person games.

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Here we provide two other characterizations. Both characterizations involve a stability property due to the concept of the equal division core from Selten [8] and a property restricting maximum payoffs. The first characterization involves in addition a *weaker* variant of the reduced game property of Hart and Mas-Colell [6], whereas the second characterization is obtained by making use of a weaker variant of the reduced game property of Davis and Maschler [5].

Further, a dual egalitarian solution is defined on the class of concave TU-games. It turns out that for a convex game the egalitarian allocation is equal to the dual egalitarian allocation for its dual (concave) game. Similar results hold for the Shapley value [9], the Prenucleolus [7], and the τ-value [10].

The work is organized as follows. Section 2 deals with notation and definitions regarding TU-games. Section 3 recalls the egalitarian solution for convex games. Two characterizations of this solution concept are presented. Finally, in section 4 a dual egalitarian solution is introduced and a duality result is proved.

2 Preliminaries

A cooperative game with transferable utilities (TU-game) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the player set and v the characteristic function, which assigns to every subset[3] S of N a value $v(S)$, with $v(\emptyset) = 0$. A game (N, v) is called *convex* if

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) \quad \text{for all } S, T \subseteq N,$$

and *concave* if

$$v(S \cup T) + v(S \cap T) \leq v(S) + v(T) \quad \text{for all } S, T \subseteq N.$$

The *core* of a game (N, v) is defined by

$$C(N, v) := \{x \in \mathbb{R}^N : x(N) = v(N), \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\},$$

and its *dual core* is defined by

$$C^*(N, v) := \{x \in \mathbb{R}^N : x(N) = v(N), \text{ and } x(S) \leq v(S) \text{ for all } S \subseteq N\}.$$

The *dual game* of (N, v) is the game (N, v^*), given by

$$v^*(S) := v(N) - v(N \backslash S) \text{ for all } S \subseteq N.$$

It is easily shown that $C(N, v) = C^*(N, v^*)$.

Throughout this paper we will denote the *average worth* of coalition S in game (N, v) by

$$a(S, v) := \frac{v(S)}{|S|}.$$

[3] $S \subseteq N$ denotes that S is a subset of N and $S \subset N$ denotes that S is a strict subset of N.

3 Convex games and the egalitarian solution

In this section we will provide two logically independent characterizations of the egalitarian rule for convex games. To this end, let us first recall the algorithm of Dutta and Ray [3]. In every step of the algorithm a cooperative game is considered. The set of players in this game is the set of players that have not received a payoff yet. The largest coalition with the highest average worth is selected and the players in this coalition receive this average worth.

Let (N, v) be a convex TU-game. Define $N_1 := N$ and $v_1 := v$.

STEP 1: Let S_1 be the largest coalition with the highest average worth in the game (N_1, v_1). Define

$$E_i(N, v) := a(S_1, v_1) \qquad \text{for all } i \in S_1.$$

STEP k: Suppose that $S_1, \ldots, S_{k-1}$ have been defined recursively and $S_1 \cup \cdots \cup S_{k-1} \neq N$. Define a new game with player set $N_k := N_{k-1} \backslash S_{k-1} = N \backslash (S_1 \cup \cdots \cup S_{k-1})$. For all subcoalitions $S \subseteq N_k$, define $v_k(S) := v_{k-1}(S_{k-1} \cup S) - v_{k-1}(S_{k-1})$. Convexity of (N_{k-1}, v_{k-1}) implies convexity of (N_k, v_k). Define S_k to be the largest coalition with the highest average worth in this game. Define

$$E_i(N, v) := a(S_k, v_k) \qquad \text{for all } i \in S_k.$$

It can be checked that in every step convexity ensures the existence of a largest coaliton with highest average worth. In at most n steps the algorithm ends, and the constructed allocation $E(N, v)$ is called the egalitarian solution of the game (N, v). Dutta and Ray [3] show that $E(N, v)$ is an element of the core of (N, v). Furthermore, they note that for each convex game (N, v) it holds that

$$E_i(N, v) > E_j(N, v), \text{ for all } i \in S_k,\ j \in S_{k+1}. \tag{1}$$

Our first characterization of the egalitarian solution for convex TU-games involves the properties equal division stability, bounded maximum payoff property, and HM max-consistency. We describe these properties below. Let $\mathcal{C}$ be the set of convex TU-games. A solution on $\mathcal{C}$ is a map ψ assigning to each convex game $(N, v) \in \mathcal{C}$ an element $\psi(N, v) \in \mathbb{R}^N$. Let (N, v) be a convex game. Given the solution ψ, define $S^m(N, v, \psi)$ (or S^m for short, if no confusion is possible) to be the set of players with the highest payoff. Formally,

$$S^m = S^m(N, v, \psi) := \operatorname*{argmax}_{j \in N} \psi_j(N, v).$$

A solution ψ on $\mathcal{C}$ satifies

- *equal division stability* (EDS) if for all games $(N, v) \in \mathcal{C}$ and all $S \subseteq N$ there exists $i \in S$ with

$$\psi_i(N, v) \geq a(S, v).$$

- *bounded maximum payoff property* (BMPP) if for all games $(N, v) \in \mathcal{C}$:

$$\sum_{i \in S^m} \psi_i(N, v) \leq v(S^m).$$

- *HM max-consistency* (HMMC) if for all games $(N, v) \in \mathcal{C}$, and all $i \in N \backslash S^m$:

$$\psi_i(N, v) = \psi_i(N \backslash S^m, v^{-S^m}),$$

  where v^{-S^m} is the reduced subgame[4] defined by

$$v^{-S^m}(T) := v(S^m \cup T) - \sum_{i \in S^m} \psi_i(S^m \cup T, v)$$

  for all subcoalitions $T \subseteq N \backslash S^m$.

(EDS) plays a role in the concept of equal division core from Selten [8]. (BMPP) states that the payoffs of the players receiving most is bounded, which might be desirable from a social point of view. (HMMC) is a weaker variant of the consistency property of Hart and Mas-Colell [6]. The following lemma shows that (EDS) and (BMPP) together imply an efficiency property.

Lemma 3.1 *If a solution ψ satisfies (EDS) and (BMPP) then for all $(N, v) \in \mathcal{C}$*

$$\sum_{i \in S^m} \psi_i(N, v) = v(S^m). \tag{2}$$

Proof. Let ψ be a solution that satisfies (EDS) and (BMPP). Let (N, v) be a convex game. By (BMPP),

$$\sum_{i \in S^m} \psi_i(N, v) \leq v(S^m).$$

Suppose

$$\sum_{i \in S^m} \psi_i(N, v) < v(S^m). \tag{3}$$

Since all players in S^m receive the same payoff we have for all $i \in S^m$

$$|S^m| \psi_i(N, v) < v(S^m) = |S^m| a(S^m, v).$$

Hence,

$$\psi_i(N, v) < a(S^m, v) \quad \text{for all } i \in S^m.$$

This contradicts ψ satisfying (EDS). So, equation (3) does not hold true. Hence equation (2) holds. $\square$

The property incorporated in equation (2) will be called *max-efficiency* (MEFF).

We have the following characterization.

[4] With a slight abuse of notation we write $(S^m \cup T, v)$ for the restriction of the game (N, v) to the set of players $S^m \cup T$. It is obvious that the restricted game is convex as well.

Theorem 3.1 *A solution ψ satisfies (EDS), (BMPP), and (HMMC) if and only if $\psi = E$.*

Proof. First we show that E satisfies the properties. Since E assigns to every convex game a core element, it satisfies (EDS). It follows from (1) that every player in S_1 receives the maximum payoff and that all other players receive less than this maximum. Since these players divide $v(S_1)$ it follows that E satisfies (BMPP). Since E satisfies the reduced game property of Hart and Mas-Colell [6] it satisfies (HMMC), a weaker variant of the reduced game property of Hart and Mas-Colell [6].

Now suppose that a solution ψ satisfies the properties. We prove that $\psi = E$. By lemma 3.1 it follows that ψ satisfies (MEFF). The proof will be by induction on the number of players.

Clearly, for convex games (N, v) with $|N| = 1$ we have that $\psi(N, v) = v(\{1\}) = E(N, v)$ by (MEFF). Suppose that for some $p \geq 2$ we have $\psi(N, v) = E(N, v)$ for all convex games (N, v) with $|N| \leq p - 1$. We prove that $\psi(N, v) = E(N, v)$ also holds for all convex games (N, v) with $|N| = p$.

Let (N, v) be a convex game with $|N| = p$. Let S_1 be the largest coalition that maximizes the average worth function $a(\cdot, v)$. First we will show that $a(S_1, v) = a(S^m, v)$. Since ψ satisfies (EDS) there exist $i \in S_1$ with $\psi_i(N, v) \geq a(S_1, v)$. Then for all $j \in S^m$ we have

$$a(S^m, v) = \psi_j(N, v) \geq \psi_i(N, v) \geq a(S_1, v),$$

where the equality follows by definition of S^m and (MEFF). The first inequality follows by definition of S^m. Since the definition of S_1 implies $a(S_1, v) \geq a(S^m, v)$ we conclude

$$a(S^m, v) = a(S_1, v). \tag{4}$$

Again by definition of S_1 this implies $S^m \subseteq S_1$. We will show that $S^m = S_1$.

We need the following lemma. The proof is relegated to appendix A.

Lemma 3.2 *For all $T \subseteq N \backslash S^m$ it holds that $v^{-S^m}(T) = v(S^m \cup T) - v(S^m)$.*

It follows from lemma 3.2 that the reduced game $(N \backslash S^m, v^{-S^m})$ is convex.
We continue with the proof of $S^m = S_1$. Suppose $S^m \subset S_1$. For $T = S_1 \backslash S^m \neq \emptyset$ lemma 3.2 gives

$$v^{-S^m}(S_1 \backslash S^m) = v(S_1) - v(S^m),$$

where the equality follows since $S^m \subseteq S_1$. But then

$$\frac{v(S_1)}{|S_1|} = \frac{v(S_1) - v(S^m) + v(S^m)}{|S_1 \backslash S^m| + |S^m|} = \frac{v^{-S^m}(S_1 \backslash S^m) + v(S^m)}{|S_1 \backslash S^m| + |S^m|}.$$

From this and (4) it follows that

$$a(S_1, v) = \frac{v(S_1)}{|S_1|} = \frac{v^{-S^m}(S_1 \backslash S^m)}{|S_1 \backslash S^m|}. \tag{5}$$

Now, using the convexity of the reduced game $(N\backslash S^m, v^{-S^m})$ it follows that

$$\psi_i(N\backslash S^m, v^{-S^m}) = \psi_i(N,v) < a(S_1,v) = \frac{v^{-S^m}(S_1\backslash S^m)}{|S_1\backslash S^m|} \text{ for all } i \in S_1\backslash S^m, \tag{6}$$

where the first equality follows from (HMMC), the strict inequality from (4) and the definition of S^m, and the second equality from (5). Inequality (6) contradicts with (EDS) of $S_1\backslash S^m$ in the reduced game $(N\backslash S^m, v^{-S^m})$. Hence, the assumption $S^m \subset S_1$ is false. Since $S^m \subseteq S_1$, this completes the proof of $S^m = S_1$.

It remains to prove that indeed from $S^m = S_1$ it follows that $\psi(N,v) = E(N,v)$. Note first that (MEFF), the definition of S^m, and $S^m = S_1$ yield

$$\psi_i(N,v) = \frac{v(S^m)}{|S^m|} = E_i(N,v) \quad \text{ for all } i \in S^m. \tag{7}$$

Then, if $S^m = N$ we are done. If $S^m \neq N$ it holds that

$$\psi_i(N,v) = \psi_i(N\backslash S^m, v^{-S^m}) = E_i(N\backslash S^m, v^{-S^m}) = E_i(N,v) \quad \text{ for all } i \in N\backslash S^m, \tag{8}$$

where the first equality follows from (HMMC), the second equality from the induction hypothesis, and the third equality from $S^m = S_1$. The theorem now follows from (8) and (7). □

It follows from the following examples that the propertiess in theorem 3.1 are logically independent.

- The solution that equally divides the worth of the grand coalition to the players satisfies (HMMC) and (BMPP), but does not satisfy (EDS).

- Define the solution α by $\alpha_i(N,v) := \max_{S\in 2^N\backslash\{\emptyset\}} \frac{v(S)}{|S|}$ for all $i \in N$. The solution α satisfies (EDS) and (HMMC). Obviously, it is does not satisfy (BMPP).

- Define the solution β as follows. Let

  $$\delta := \frac{v(S_1)}{|S_1|} - \frac{v(S_1 \cup S_2) - v(S_1)}{|S_2|},$$

  and define

  $$\beta_i(N,v) := \begin{cases} E_i(N,v) & \text{if } i \in S_1; \\ E_i(N,v) + \frac{\delta}{2} & \text{otherwise.} \end{cases}$$

  The solution β satisfies (EDS) and (BMPP), but not (HMMC).

A second characterization is obtained by (EDS), (BMPP), and (DMMC), which is a weaker variant of the reduced game property of Davis and Maschler [5]. Formally, a solution ψ on $\mathcal{C}$ satisfies

- *DM max-consistency (DMMC) if for all games $(N, v) \in \mathcal{C}$, and all $i \in N \backslash S^m$:*

$$\psi_i(N, v) = \psi_i(N \backslash S^m, v_{-S^m}),$$

where v_{-S^m} is the reduced subgame defined by

$$v_{-S^m}(T) := \begin{cases} 0 & \text{if } T = \emptyset; \\ v(N) - \sum_{i \in S^m} \psi_i(N, v) & \text{if } T = N \backslash S^m; \\ \max_{Q \subseteq S^m} \{v(T \cup Q) - \sum_{i \in Q} \psi_i(N, v)\} & \text{if } \emptyset \subset T \subset N \backslash S^m. \end{cases}$$

Thus, our second characterization of the egalitarian solution is as follows.

Theorem 3.2 *A solution ψ satisfies (EDS), (BMPP), and (DMMC) if and only if $\psi = E$.*

Proof. Since E satisfies the reduced game property of Davis and Maschler [5] it follows that E satisfies (DMMC), a weaker variant of the reduced game property of Davis and Maschler [5]. The proof is obtained from the proof of theorem 3.1 by replacing v^{-S^m} by v_{-S^m} and lemma 3.2 by

Lemma 3.3 *For all $T \subseteq N \backslash S^m$ it holds that $v_{-S^m}(T) = v(S^m \cup T) - v(S^m)$.*

The proof of lemma 3.3 can be found in appendix B. □

It follows from the same examples above that the properties in theorem 3.2 are logically independent.

4 The dual egalitarian solution

In this section, we will introduce a dual egalitarian solution on the class of concave games. It turns out that for a given convex game the egalitarian allocation is equal to the dual egalitarian allocation for its dual concave game. This result is in the vein of the duality result regarding the core (see section 2). Similar results can easily be proved for the Shapley value [9], the Prenucleolus [7], and the τ-value [10].

Let us start with the definition of the dual egalitarian solution E^*. In every step of the algorithm a cooperative game is considered. The set of players in this game is the set of players that have not received a payoff yet. The largest coalition with the lowest average worth is selected and the players in this coalition receive this average worth.

Let (N, w) be a concave game. Define $N_1 := N$ and $w_1 := w$.

Step 1: Let T_1 be the largest coalition with the *lowest* average worth in the game (N, w_1). Define

$$E_i^*(N, w) := a(T_1, w_1) \qquad \text{for all } i \in T_1.$$

Step k: Suppose that $T_1, \ldots, T_{k-1}$ have been defined recursively and $T_1 \cup \cdots \cup T_{k-1} \neq N$. Define a new game with player set $N_k := N \backslash (T_1 \cup \cdots \cup T_{k-1})$. For all subcoalitions $T \subseteq N_k$, define $w_k(T) := w_{k-1}(T_{k-1} \cup T) - w_{k-1}(T_{k-1})$. The game (N_k, w_k) is concave since (N_{k-1}, w_{k-1}) is concave. Define T_k to be the largest coalition with the lowest average worth in this game. Define

$$E_i^*(N, w) := a(T_k, w_k) \qquad \text{for all } i \in T_k.$$

The following lemma shows that concavity of the game (N_k, w_k) ensures that there is a largest coalition with the lowest average worth in the game (N_k, w_k). The proof is skipped.

Lemma 4.1 *For a concave game (N, w) the collection $L(w)$ consisting of the empty set and the coalitions with lowest average worth is a lattice. Hence, there is a unique largest coalition with the lowest average worth.*

Although in section we considered the egalitarian solution for convex games only it is defined on a larger set of games. However, the egalitarian solution does not exist for all TU-games. Specifically, if the game is concave and not additive then the egalitarian solution does not exist. Hence, the dual egalitarian solution is not simply the egalitarian solution for concave games, since the dual egalitarian solution exists for all concave games.

Dutta and Ray [3] already noted that for the E-algorithm it holds that in every next step the payoff given to an agent is strictly less. The following lemma describes a similar result for E^*.

Lemma 4.2 *Let (N, w) be a concave game. For $i \in T_k$ and $j \in T_{k+1}$ it holds that $E_j^*(N, w) > E_i^*(N, w)$.*

Proof.

$$\begin{aligned} E_j^*(N, w) &= \frac{v_{k+1}^*(T_{k+1})}{|T_{k+1}|} = \frac{v_k^*(T_k \cup T_{k+1}) - v_k^*(T_k)}{|T_{k+1}|} \\ &> \frac{\frac{v_k^*(T_k)}{|T_k|}\left(|T_k| + |T_{k+1}|\right) - v_k^*(T_k)}{|T_{k+1}|} = \frac{v_k^*(T_k)}{|T_k|} = E_i^*(N, w), \end{aligned}$$

where the inequality follows since $\frac{v_k(T_k \cup T_{k+1})}{|T_k \cup T_{k+1}|} > \frac{v_k(T_k)}{|T_k|}$ by definition of T_k and $T_k \cap T_{k+1} = \emptyset$. □

We now turn to the main result in this section. It can be formulated concisely as follows: for every convex game (N, v) it holds that $E(N, v) = E^*(N, v^*)$. To prove this we need some lemmas.

We start with lemma 4.3, which states that $E^*(N, v^*)$ is an element of the dual core of (N, v^*).

Lemma 4.3 *For a convex game (N, v) it holds that $E^*(N, v^*) \in C^*(N, v^*)$.*

Proof. Let $T_1, \ldots, T_q$ be the sets that are generated by the E^*-algorithm. Then,

$$E_i^*(N, v^*) = \frac{1}{|T_1|} v^*(T_1) \quad \forall i \in T_1,$$

and for $2 \leq k \leq q$,

$$E_i^*(N, v^*) = \frac{1}{|T_k|} \left[v^* \left(\bigcup_{r=1}^{k} T_r \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right] \quad \forall i \in T_k.$$

Further, for all $1 \leq k \leq q$ and all $T \subseteq \bigcup_{r=k}^{q} T_r$, $T \neq \emptyset$,

$$\frac{1}{|T|} \left[v^* \left(\bigcup_{r=1}^{k-1} T_r \cup T \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right] \geq \frac{1}{|T_k|} \left[v^* \left(\bigcup_{r=1}^{k} T_r \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right], \tag{9}$$

since by definition T_k minimizes this expression.

First, we prove that $E^*(N, v^*)$ is efficient:

$$\begin{aligned}
\sum_{i \in N} E_i^*(N, v^*) &= \sum_{i \in T_1} E_i^*(N, v^*) + \sum_{k=2}^{q} \sum_{i \in T_k} E_i^*(N, v^*) \\
&= v^*(T_1) + \sum_{k=2}^{q} \left[v^* \left(\bigcup_{r=1}^{k} T_r \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right] \\
&= v^* \left(\bigcup_{r=1}^{q} T_r \right) \\
&= v^*(N).
\end{aligned}$$

Second, we prove the stability of $E^*(N, v^*)$, i.e. $\sum_{i \in T} E_i^*(N, v^*) \leq v^*(T)$ for all $T \subseteq N$. Let $K \subseteq N$. Define $K_r := K \cap T_r$ for all $1 \leq r \leq q$. Then,

$$\begin{aligned}
\sum_{i \in T} E_i^*(N, v^*) &= |K_1| \frac{v^*(T_1)}{|T_1|} + \sum_{k=2}^{q} |K_k| \frac{\left[v^* \left(\bigcup_{r=1}^{k} T_r \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right]}{|T_k|} \\
&\leq v^*(K_1) + \sum_{k=2}^{q} \left[v^* \left(\left(\bigcup_{r=1}^{k-1} T_r \right) \cup K_k \right) - v^* \left(\bigcup_{r=1}^{k-1} T_r \right) \right] \\
&\leq v^*(K_1) + \sum_{k=2}^{q} \left[v^* \left(\left(\bigcup_{r=1}^{k-1} K_r \right) \cup K_k \right) - v^* \left(\bigcup_{r=1}^{k-1} K_r \right) \right] \\
&= v^*(K).
\end{aligned}$$

Here the first inequality follows from (9), and the second inequality from the concavity of v^*. This completes the proof of the lemma. $\square$

Note that since $C^*(N, v^*) = C(N, v)$ it follows by lemma 4.3 that $E^*(N, v^*) \in C(N, v)$.

Next we prove that E^* satisfies two welfare properties, namely it is the element of the core with the highest minimum and the lowest maximum payoffs among all core elements. The same holds true for E.

Proposition 4.1 *Let (N, v) be a convex game. For all $x \in C(N, v)$ it holds that*

$$\min_{i \in N} x_i \leq \min_{i \in N} E_i^*(N, v^*) = \min_{i \in N} E_i(N, v), \text{ and}$$
$$\max_{i \in N} x_i \geq \max_{i \in N} E_i^*(N, v^*) = \max_{i \in N} E_i(N, v).$$

Proof. We only give the proof for E^*; the proof for E is similar. Let $T_1, \ldots, T_q$ be the sets that are subsequently generated by the E^*-algorithm. Let $x \in C(N, v)$.

Then,

$$\min_{i \in N} x_i \leq \min_{i \in T_1} x_i \leq \frac{x(T_1)}{|T_1|} \leq \frac{v^*(T_1)}{|T_1|} = \min_{i \in N} E_i^*(N, v^*),$$

where the last inequality follows from the fact that $x \in C(N, v) = C^*(N, v^*)$, and the equality from lemma 4.2. This proves the first part.

It also holds that,

$$\max_{i \in N} E_i^*(N, v^*) = \frac{v(T_q)}{|T_q|} \leq \frac{x(T_q)}{|T_q|} \leq \max_{i \in T_q} x_i \leq \max_{i \in N} x_i,$$

where the equality follows from lemma 4.2 and the first inequality from $x \in C(N, v)$. This proves the second part. □

To prove the main result in this section we need one lemma more.

Lemma 4.4 *Let (N, v) be a convex game. Let $S_1, \ldots, S_p$ and $T_1, \ldots, T_q$ be the sets that are subsequently generated by the E-algorithm and the E^*-algorithm, respectively.*
Then $S_p \subseteq T_1$. Additionally it holds that $E_i(N, v) = E_i^*(N, v^*)$ for all players $i \in S_p$.

Proof. From proposition 4.1 it follows that for all $i \in S_p$ and all $j \in T_1$ we have that

$$\frac{v(N) - v(N \backslash S_p)}{|S_p|} = E_i(N, v) = \max_{x \in C(N,v)} \min_{k \in N} x_k = E_i^*(N, v^*) = \frac{v(N) - v(N \backslash T_1)}{|T_1|}.$$

Then by lemma 4.1 it follows that $S_p \subseteq T_1$. □

The following theorem is the third main theorem of this paper.

Theorem 4.1 *Let (N, v) be a convex game. Then, $E(N, v) = E^*(N, v^*)$.*

Proof. Let $S_1, \ldots, S_p$ and $T_1, \ldots, T_q$ be the sets that are subsequently generated by the E-algorithm and the E^*-algorithm, respectively. We call these sets the E-sets and the E^*-sets, respectively. The proof is by induction on the number p of E-sets.

Suppose (N, v) is a convex game with $p = 1$. Then $S_p = S_1 = N$. By lemma 4.4 it immediately follows that $E_i(N, v) = E_i^*(N, v^*)$ for all players $i \in N$.

Now suppose that for some $p \geq 2$ it holds that $E(N,v) = E^*(N,v^*)$ for all convex games (N,v) with at most $p-1$ E-sets. We prove that $E(N,v) = E^*(N,v^*)$ for all convex games (N,v) with exactly p E-sets. So, let (N,v) be a convex game with p E-sets.

Again, by lemma 4.4, $E_i(N,v) = E_i^*(N,v^*)$ for all players $i \in S_p$. It remains to prove that $E_i(N,v) = E_i^*(N,v^*)$ for all players $i \in N \backslash S_p$. Now, note that

$$E_i(N,v) = E_i(N \backslash S_p, v) = E_i^*(N \backslash S_p, (v_{|N \backslash S_p})^*) \text{ for all } i \in N \backslash S_p. \tag{10}$$

Here, the first equality can be readily verified, and the second equality follows from the induction hypothesis. By (10) we are done if we show that

$$E_i^*(N \backslash S_p, (v_{|N \backslash S_p})^*) = E_i^*(N,v) \text{ for all } i \in N \backslash S_p. \tag{11}$$

First, we will prove that $S_p = T_1$. By lemma 4.4 $S_p \subseteq T_1$. Suppose $S_p \subset T_1$. From the proof of lemma 4.4 it follows that

$$\frac{v(N) - v(N \backslash S_p)}{|S_p|} = \frac{v(N) - v(N \backslash T_1)}{|T_1|}. \tag{12}$$

Since $v(N \backslash S_p) - v(N \backslash T_1) = [v(N) - v(N \backslash T_1)] - [v(N) - v(N \backslash S_p)]$ and $|T_1 \backslash S_p| = |T_1| - |S_p|$, (12) implies

$$\frac{v(N \backslash S_p) - v(N \backslash T_1)}{|T_1 \backslash S_p|} = \frac{v(N) - v(N \backslash T_1)}{|T_1|}. \tag{13}$$

Using this, we can prove that the following two assertions hold true.

(i) $\dfrac{v(N \backslash S_p) - v(N \backslash (S \cup S_p))}{|S|} \geq \dfrac{v(N \backslash S_p) - v(N \backslash T_1)}{|T_1 \backslash S_p|}$ for all $S \subseteq N \backslash S_p,\ S \neq \emptyset$;

(ii) If $S \not\subseteq T_1 \backslash S_p,$ then $\dfrac{v(N \backslash S_p) - v(N \backslash (S \cup S_p))}{|S|} > \dfrac{v(N \backslash S_p) - v(N \backslash T_1)}{|T_1 \backslash S_p|}$.

We prove that (i) holds true ((ii) can be proved similarly). Suppose that (i) is not true. Hence, there is a set $S \subseteq N \backslash S_p$, $S \neq \emptyset$ with

$$\frac{v(N \backslash S_p) - v(N \backslash (S \cup S_p))}{|S|} < \frac{v(N \backslash S_p) - v(N \backslash T_1)}{|T_1 \backslash S_p|}. \tag{14}$$

By (12), (13), and (14) it follows that

$$\frac{v(N \backslash S_p) - v(N \backslash (S \cup S_p))}{|S|} < \frac{v(N) - v(N \backslash S_p)}{|S_p|} \tag{15}$$

By (15) and again (12) we have

$$\begin{aligned}
\frac{v(N) - v(N \backslash (S \cup S_p))}{|S \cup S_p|} &= \frac{v(N) - v(N \backslash S_p) + v(N \backslash S_p) - v(N \backslash (S \cup S_p))}{|S_p| + |S|} \\
&< \frac{v(N) - v(N \backslash S_p)}{|S_p|} \\
&= \frac{v(N) - v(N \backslash T_1)}{|T_1|}.
\end{aligned}$$

This is a contradiction with the definition of T_1. So, (i) holds true.

From (i) and (ii) we conclude that $T_1 \backslash S_p$ is the first E^*-set when we apply the E^*-algorithm to the concave game $(N \backslash S_p, v)$. Hence, for all $i \in T_1 \backslash S_p$

$$\begin{aligned} E_i^*(N \backslash S_p, (v_{|N \backslash S_p})^*) &= \frac{v(N \backslash S_p) - v(N \backslash T_1)}{|T_1 \backslash S_p|} \\ &= \frac{v(N) - v(N \backslash T_1)}{|T_1|}, \end{aligned} \tag{16}$$

where the last equality follows from (13). On the other hand, we have for $i \in T_1 \backslash S_p$ that

$$\begin{aligned} E_i^*(N \backslash S_p, (v_{|N \backslash S_p})^*) &= E_i(N \backslash S_p, v) \\ &> \frac{v(N) - v(N \backslash S_p)}{|S_p|} \\ &= \frac{v(N) - v(N \backslash T_1)}{|T_1|}. \end{aligned} \tag{17}$$

Here the first equality follows from the induction hypothesis, the inequality follows from the definition of S_p, and the second equality follows from (12).

We see that (16) and (17) give a contradiction. Hence, $S_p = T_1$.

The algorithm for E^* implies that equation (11) holds true. Combining (10) and (11) yields

$$E_i(N, v) = E_i^*(N, v) \text{ for all } i \in N \backslash S_p,$$

completing the proof. □

Appendix A Proof of lemma 3.2

Let $T \subseteq N \backslash S^m$. We first show that

$$\sum_{i \in S^m} \psi_i(S^m \cup T, v) = v(S^m). \tag{18}$$

If $T = N \backslash S^m$, then (18) holds by (MEFF). From now on assume $T \subset N \backslash S^m$.

Suppose there is a player $i \in S^m \cup T$ such that

$$\psi_i(S^m \cup T, v) > a(S^m, v). \tag{19}$$

Define

$$T^m := \operatorname*{argmax}_{j \in S^m \cup T} \psi_j(S^m \cup T, v). \tag{20}$$

By (MEFF),

$$\sum_{j \in T^m} \psi_j(S^m \cup T, v) = v(T^m). \tag{21}$$

Now,

$$a(T^m, v) = \psi_j(S^m \cup T, v) > a(S^m, v) \geq \psi_j(N, v) \text{ for all } j \in T^m \tag{22}$$

The strict inequality follows from (19), the equality from (20) and (21), and the second inequality from the definition of S^m and (MEFF). Equation (22) contradicts (EDS) of T^m in the game (N, v). Hence, there is no player $i \in S^m \cup T$ such that $\psi_i(S^m \cup T, v) > a(S^m, v)$. Hence,

$$\psi_j(S^m \cup T, v) \leq a(S^m, v) \quad \text{for all } j \in S^m \cup T. \tag{23}$$

By inequality (23) and (EDS): $a(L, v) \leq a(S^m, v)$ for all $L \subseteq S^m \cup T$. Hence,

$$S^m \in \operatorname*{argmax}_{\emptyset \neq L \subseteq S^m \cup T} a(L, v). \tag{24}$$

From the induction hypothesis it follows that

$$E_j(S^m \cup T, v) = \psi_j(S^m \cup T, v) \quad \text{for all } j \in S^m \cup T. \tag{25}$$

From (24) and (25) one deduces (18).

Now, (18) implies that for all $T \subseteq N \backslash S^m$ it holds that

$$v^{-S^m}(T) = v(S^m \cup T) - v(S^m). \tag{26}$$

This completes the proof of the lemma. □

Appendix B Proof of lemma 3.3

Let $T \subseteq N \backslash S^m$. We prove

$$v_{-S^m}(T) = v(S^m \cup T) - v(S^m). \tag{27}$$

If $T = \emptyset$,

$$v_{-S^m}(T) = 0 = v(S^m) - v(S^m). \tag{28}$$

If $T = N \backslash S^m$, then by (MEFF) and the definition of the reduced game

$$\begin{aligned} v_{-S^m}(T) &= v(N) - \sum_{i \in S^m} \psi_i(N, v) \\ &= v(S^m \cup T) - v(S^m). \end{aligned}$$

It remains to consider T with $\emptyset \subset T \subset N \backslash S^m$. Let $Q \subseteq S^m$. Note that

$$\sum_{i \in Q} \psi_i(N, v) = |Q| a(S^m, v) = |Q| a(S_1, v) \geq |Q| a(Q, v) = v(Q). \tag{29}$$

Here, the first equality follows from $Q \subseteq S^m$, the second equality from equation (4), and the inequality from the definition of S_1. Then,

$$\begin{aligned} v(Q \cup T) - \sum_{i \in Q} \psi_i(N, v) &\leq v(Q \cup T) - v(Q) \\ &\leq v(S^m \cup T) - v(S^m), \end{aligned} \tag{30}$$

where the first inequality follows from (29) and the second inequality from the convexity of v. We conclude that (27) holds true. $\square$

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