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SHARING THE COST OF A NETWORK: CORE AND CORE ALLOCATIONS.

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Abstract: This paper discusses the core of the game corresponding to the standard fixed tree problem. We introduce the concept of a *weighted constrained egalitarian solution*. The core of the standard fixed tree game equals the set of all weighted constrained egalitarian solutions. The notion of *home-down* allocation is developed to create further insight in the local behavior of the weighted constrained egalitarian allocation. A similar and dual approach by the notion of *down-home* allocations gives us the class of weighted Shapley values. The constrained egalitarian solution is characterized in terms of a cost sharing mechanism.

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Keywords: Cooperative game theory, tree games, core, weighted constrained egalitarian solution

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1 Introduction

In this paper we consider cost sharing problems arising from *standard fixed tree enterprises*.[1]. There is a fixed and finite set of agents who are connected to a source through a fixed tree network. We seek to allocate the cost of this tree for cases where the connections within the network is costly. Many real-life situations can be modelled to fit in this general setting. For instance, consider the problem of allocating the maintenance cost of an irrigation network or a cablevision network, setting airport taxes for planes or setting dredging fees for ships. In a natural way each standard fixed tree problem gives rise to a *standard fixed tree game*, which relates each coalition of agents /players to the minimal expenses for connecting all its members to the source. This makes it possible to investigate this type of problems with techniques from cooperative game theory. Extensive study for essentially the same type of situations has resulted in a long list of papers (see Bird (1976), Claus and Granot (1976), Megiddo (1978), Galil (1980), Granot and Huberman (1981), (1984), Granot and Granot (1992), Granot and Maschler (1994), Granot *et al.* (1995), Maschler *et al* (1995)). The special case when the underlying structure of the game is a chain, is also known as the airport problem and considered by several authors (Littlechild (1974), Littlechild and Owen (1977), Littlechild and Thompson (1977), Dubey (1982), Sudhölter and Potters (1995), Aadland and Kolpin (1997)).

We are concerned with the core of the standard fixed tree game. It is a well known fact that the game under consideration is concave. The importance of this aspect and the implications of the tree structure become clear in Section 3, where we characterize the core in three ways and investigate its geometrical structure. As is known for concave games in general, the core is large and coincides with the set of weighted Shapley values (Monderer *et al.* (1992)). Similar to this result, in Section 4 we show that the core of the standard fixed tree game equals the set of weighted constrained egalitarian allocations. Here, a weighted constrained egalitarian allocation is the weighted adaptation of the constrained egalitarian solution of Dutta and Ray (1989), incorporating exogeneously given information about the impact of the individual players that is summarized by a vector of weights. In a way that is very natural, but particular to this model, weighted Shapley values and weighted constrained egalitarian allocations are duals of each other. We show that both sets of allocations can be seen as the result of a dynamical process of *locally* distributing the costs of the arcs forming the tree. While a weighted Shapley value is a *down-home* allocation in the sense that it is determined by splitting incremental

[1] Here we adopt the terminology as in Maschler *et al.* (1995)

costs from the source to the leafs (Section 6), a weighted constrained egalitarian allocation is of *home-down* type, splitting the incremental costs from the leafs to the source (Section 4). The above terminology is inspired by the *painting story* in Maschler *et al.* (1995), which describes a dynamic process of the distribution of costs. Monotonicity properties for both above mentioned classes of solutions can be obtained easily from this dynamic approach. In Section 5 we provide two characterizations for the constrained egalitarian solution as a cost sharing mechanism. It is shown that among the class of cost monotonic mechanisms sharing the core property it is, on one hand, the only mechanism that minimizes the range of cost shares, and, on the other hand, the only mechanism that maximizes Rawlsian welfare. A discussion of analogies of these results for weighted constrained egalitarian cost sharing mechanisms is postponed till the Appendix. But first, in Section 2 we will formally define the standard fixed tree problem, its game and introduce necessary notations.

2 The fixed tree connection problem: the model and its game

In this paper we consider a *fixed tree connection problem* $\mathcal{G} := \langle G, c, N \rangle$. Here $G = (V, E)$ is a tree, i.e. a directed connected graph without cycles, with vertex set V and arc set E. The set V contains a vertex which has a special meaning. We denote this vertex by r and refer to it as the *source*. The function $c : E \to \mathbb{R}_+$, called *cost function*, associates with each arc e a cost $c(e)$. It can be interpreted as the cost to maintain e. At each vertex there is exactly one player, the finite set of all players is denoted by $N = \{1, \dots, n\}$. The objective of the players is to maintain sufficiently many arcs such that by the corresponding network each finds himself connected to the source. We assume for simplicity that the source is not occupied and that only one arc leaves the source. Then $\mathcal{G}$ is referred to as simply a *maintenance problem* . In the sequel we identify vertices with players ($V = N \cup \{r\}$). For each vertex $i \in N$ there is a unique path from the source to vertex i. If that path, which we denote by $P(i)$, consists of the points $j_0 = r$, $j_1, \dots, j_q = i$, then j_{q-1} is called the *predecessor* $\pi(i)$ of vertex i. We denote by e_i the arc $(\pi(i), i)$. The *precedence relation* $(V, \preceq)$ on the set of vertices and/or players is defined by $i \preceq j$ if and only if $i \in P(j)$. Analogously we define the precedence relation $(E, \preceq)$ on the arcs. In this way, the arcs are considered to be directed away from the source. A *trunk* of $G = (V, E)$ is a set of vertices $T \subseteq N$, which is closed under the precedence relation defined above, i.e. if $i \in T$ and $j \preceq i$, then $j \in T$. An *outgoing arc* for a trunk

T is an element $e_i = (\pi(i), i) \in E$ such that $\pi(i) \in T$ but $i \notin T$. The *followers* of a vertex i constitute the set $F(i) = \{j \in N \mid i \preceq j\}$. A vertex i is called a *leaf* if $F(i) = \{i\}$. With each maintenance problem $\mathcal{G} = \langle G, c, N \rangle$ can be associated a *cost game* $(N, c_{\mathcal{G}})$, where the cost $c_{\mathcal{G}}(S)$ of each coalition S is defined as the minimal cost needed to connect all members of S to the source via a connected subgraph of (V, E), i.e.

$$c_{\mathcal{G}}(S) = \sum_{i \in T_S} c(e_i) \quad \text{for all} \quad \emptyset \neq S \subseteq N \tag{1}$$

where $T_S = \{i \in N \mid \exists\, j \in S \text{ with } i \preceq j\}$, and $c_{\mathcal{G}}(\emptyset) = 0$. T_S is the smallest trunk containing S.

Remark The previous definition is similar to that of the *standard tree enterprise* in Granot *et al.* (1996), only in their model they permit a vertex being occupied by more than one player. However, our results can be generalized to this kind of situations. Also the assumption that the source is not occupied, can be relaxed. As Granot *et al.* (1996) pointed out, we can always add a zero-cost arc from a new unoccupied source to the original source without changing the associated cost game. Also the requirement that there is only one arc leaving the source is not essential for any of our results.[2]

Granot *et al.* (1996) prove that the cost game associated to a maintenance problem is concave. This result also follows from the next proposition which deals with the representation of the cost game with respect to the basis $\{(N, u_S^*)\}_{S \subseteq N}$ of the duals of the unanimity games. Here, the game (N, u_S^*) is the concave simple game defined by

$$u_S^*(T) = \begin{cases} 1 & \text{if } S \cap T \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

[2] Suppose that more than one arc leaves the source. In the obvious way we define the associated cost game $c_{\mathcal{G}}$. Then $c_{\mathcal{G}}$ can be *decomposed* (Shapley (1971)) into $p \geq 2$ *components*, each component being itself a maintenance problem . That is, there exists a partition $\{N_1, \ldots, N_p\}$ of N into p nonempty subsets such that $c_{\mathcal{G}}(S) = c_{\mathcal{G}}(S \cap N_1) + \ldots + c_{\mathcal{G}}(S \cap N_p)$ for all $S \subseteq N$. The restrictions of $c_{\mathcal{G}}$ to each element of the partition are called *components* and they correspond to the various subtrees emanating from the source. The solution concepts of our concern are consistent in the sense that the restriction to each coalition N_j of the solution for N equals the solution for the corresponding component. For instance, it is verified (Shapley (1971), Granot and Huberman (1981)) that the core of the game $c_{\mathcal{G}}$, formally defined at the beginning of the next section, is simply the Cartesian product of the cores of the various components.

Proposition 2.1 *Let $\mathcal{G} = \langle G, c, N \rangle$ be a maintenance problem . Then the associated cost game $(N, c_{\mathcal{G}})$ can be represented as*

$$c_{\mathcal{G}} = \sum_{i \in N} c(e_i) u^*_{F(i)} \tag{2}$$

where $F(i)$ is the set of followers of vertex i in the tree G.

Proof Let S be a nonempty coalition. It follows from expression (1) that S has to pay the cost of arc e_i in E if and only if there is a player j in S such that $j \in F(i)$.□

3 The core of a standard fixed tree connection problem

The problem we are concerned with is, given a maintenance problem $\mathcal{G} = \langle G, c, N \rangle$, to divide the construction cost of the tree $c_{\mathcal{G}}(N)$ among the players. A *vector of cost shares* is by definition a vector $x \in \mathbb{R}^N$ which is *efficient*, i.e., $\sum_{i \in N} x_i = c_{\mathcal{G}}(N)$. Here x_i represents the amount player i has to pay according to x. The *core* of a cost game (N, k) is the set

$$core(k) := \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in S} x_i \leq k(S) \text{ for all } S \subseteq N, \sum_{i \in N} x_i = k(N) \right\}.$$

If $x \in core(k)$, then no coalition S has an incentive to split off if x is the proposed vector of cost shares. The purpose of this section is to determine the structure of the core of the cost game corresponding to a maintenance problem . The first part of the section deals with alternative expressions of the core, while the last part is devoted to its geometric properties. There is an easy way to characterize the core of the game $(N, c_{\mathcal{G}})$. We show that the core consists of all those allocations according to which each agent has to make at least a zero contribution and for which the core inequalities are met for those coalitions being trunks. For convenience, we first introduce some additional notations. For any vector $x \in \mathbb{R}^N_+$ we denote $\sum_{i \in S} x_i$ by $x(S)$ for all $S \subseteq N$. In the same fashion, let $c(S) := \sum_{i \in S} c(e_i)$ for all $S \subseteq N$.

Proposition 3.1 *Let x be a vector of cost shares. Then x is a core element if and only if $x \geq 0$ and $x(T) \leq c(T)$ for each trunk T.*

Proof Trivially, if $x \in core(c_{\mathcal{G}})$, then $x \geq 0$ and $x(T) \leq c(T)$ for each trunk T. Conversely, let x be a nonnegative vector of cost shares such that $x(T) \leq c(T)$ for each trunk T. Let $S \subseteq N$ be a nonempty coalition. Then $S \subseteq T_S$ and therefore, according to (1), it holds $c_{\mathcal{G}}(S) = c(T_S) \geq x(T_S) \geq x(S)$. □

Let $e = (i, j)$ be an arc of G, we denote by $B_e = (V_e, E_e)$ the subtree of G generated by the set $F(j) \cup \{i\}$ of followers of j together with vertex i. B_e will be referred to as the *branch rooted at e*[3]. Then, the previous proposition can be rewritten in terms of the amount players outside a trunk have to pay, under a core element, as follows.

Proposition 3.2 *Let x be a vector of cost shares. Then x is a core element if and only if $x \geq 0$ and for each arc $e = (i, j) \in E$,*

$$\sum_{j \in V_e \setminus \{i\}} x_j \geq \sum_{e' \in E_e} c(e') \tag{3}$$

where $B_e = (V_e, E_e)$ is the branch rooted at e.

Proof The complement in V of $V_e \setminus \{i\}$ is a trunk. Therefore the result follows from efficiency and the application of the previous proposition. □

Remark The 'if' part of the above proposition appeared in fact in Granot *et al.* (1996).

The next proposition shows that every core element is obtained by means of splitting, arbitrarily, the cost of each arc among its users.

Proposition 3.3 *The vector x is a core element if and only if there exist $y^1, \ldots, y^n$ such that y^j is a point in the unit simplex in $\mathbb{R}^{F(j)}$ for all $j = 1, \ldots, n$ and*

$$x_i = \sum_{j \in P(i)} y_i^j c(e_j) \quad \textit{for all } i \in N. \tag{4}$$

Proof The core is additive on the cone of concave games (Dragan, Potters and Tijs, 1989). We will give a short outline of the proof. Let $\mathcal{G}^N$ be the class of TU games with player set N. First, for all $v, w \in \mathcal{G}^N$ it holds that $core(v) + core(w) \subseteq core(v + w)$. The Weber set $W(v)$ for $v \in \mathcal{G}^N$ is the convex hull of the $|N|!$ marginal vectors of v. W is subadditive as a multifunction, i.e. $W(v + w) \subseteq W(v) + W(w)$ for all

[3] According to the terminology introduced in Granot *et al.*, B_e is the branch at i in the direction of j if $e = (i, j)$

$v, w \in \mathcal{G}^N$. Furthermore on the class of concave games the Weber set and the core coincide (see Driessen (1988) or Ichiishi (1981)). So if $v, w \in \mathcal{G}^N$ are concave we also have the reversed inclusion, $core(v + w) = W(v + w) \subseteq W(v) + W(w) = core(v) + core(w)$, and consequently $core(v + w) = core(v) + core(w)$.
All the elements of the basis $\{(N, u^*_S)\}_{S \subseteq N}$ are concave, therefore it follows from the combination of the above result, the fact that costs are non-negative, and expression (2) that

$$core(c_{\mathcal{G}}) = \sum_{j \in N} c(e_j) core(u^*_{F(j)}). \tag{5}$$

Since $core(u^*_{F(j)})$ is the unit simplex in $\mathbb{R}^{F(j)}$, we are done. □

The following results show that every core element of a maintenance problem can be obtained by means of a *partition* of the original problem into various *subproblems*, each one of them being itself a maintenance problem . Conversely, at each core element x there is a unique finest partition $\mathcal{S}(x)$ into subproblems such that the restriction of x to each subproblem is a core element of the corresponding game. First we will formalize the notion of a subproblem.

Definition Let G be a tree. A subtree $G' = (V', E')$ is *rooted at* $r' \in V'$ if

(i) r' is the minimal element in V' w.r.t. $\preceq$
(ii) there is exactly one vertex in V' that has r' as predecessor.

We stress that, contrary to the usual terminology, the above definition of a subtree is nonstandard. According to our definition a subtree need not contain any leaf of the original tree whatsoever.

Definition A subtree $G' = (V', E')$ of G rooted at r' defines a *restricted connection problem* $\mathcal{G}' = \langle G', c', N' \rangle$ where c' is the restriction of c to E' and $N' = V' \backslash \{r'\}$.

Definition Let $\mathcal{S} = \langle G^1, \ldots G^p \rangle$ be an ordered collection of subtrees of G. Then, $\mathcal{S}$ is said to be a *partition of* G *into subtrees* if and only if the following conditions hold:

(i) For all $k = 1, \ldots, p,$ there exists $r_k \in V$ and $E_k \subseteq E$ such that $G^k = (S_k \cup \{r_k\}, E_k)$ is the subtree of G rooted at r_k
(ii) $\langle S_1, \ldots, S_p \rangle$ is a partition of the vertex set N.

Proposition 3.4

(i) *Let $\mathcal{S} = \langle G^1, \ldots G^p \rangle$ be a partition of G into subtrees. Then*

$$\prod_{k=1}^{p} core(c_{\mathcal{G}^k}) \subseteq core(c_{\mathcal{G}}) \tag{6}$$

where $(S_k, c_{\mathcal{G}^k})$ is the cost game corresponding to the restricted maintenance problem $\mathcal{G}^k = \langle G^k, c^k, S_k \rangle$.

(ii) *Let x be a core element for $c_{\mathcal{G}}$. Then there is a unique finest partition $\mathcal{S} = \langle G^1, \ldots G^p \rangle$ of G into subtrees such that $x \in \prod_{k=1}^{p} core(c_{\mathcal{G}^k})$.*

Proof Let $x = (y^1, \ldots, y^p)$ be an element of $\prod_{k=1}^{p} core(c_{\mathcal{G}^k})$. Then $x \geq 0$, and efficiency follows from

$$\sum_{i \in N} x_i = \sum_{k=1}^{p} \sum_{i \in S_k} y_i^k = \sum_{k=1}^{p} \sum_{i \in S_k} c^k(e_i) = \sum_{i \in N} c(e_i).$$

Then, according to Proposition 3.1 we need only prove that $x(T) \leq c(T)$ for each trunk T. Let T be a trunk of G. For any $k = 1, \ldots, p$, let T^k be the set of vertices $T \cap S_k$. Then $T^k \cup \{r_k\}$ is a trunk of $G^k = (S_k \cup \{r_k\}, E_k)$ for all $k \in \{1, \ldots, p\}$ for which $T^k \neq \emptyset$. Therefore

$$x(T) = \sum_{\substack{1 \leq k \leq p \\ T^k \neq \emptyset}} \sum_{i \in T^k} y_i^k \leq \sum_{\substack{1 \leq k \leq p \\ T^k \neq \emptyset}} \sum_{i \in T^k} c^k(e_i) = \sum_{i \in T} c(e_i).$$

This proves the first part, (i).

For the proof of (ii), let x be a vector of cost shares. Then, a trunk T will be referred to as an *autonomous* trunk at x if and only if $x(T) = c(T)$. We claim that there is a unique minimal autonomous trunk at x (with respect to inclusion). Since V itself is an autonomous trunk we need only to prove that the intersection of two autonomous trunks (which is obviously a trunk) is autonomous. The unique minimal autonomous trunk at x will then be the intersection of all autonomous trunks at x. Let T_1, T_2 be two autonomous trunks at x, then

$$\begin{aligned} x(T_1 \cap T_2) &\leq c(T_1 \cap T_2) = c_{\mathcal{G}}(T_1 \cap T_2) \leq c_{\mathcal{G}}(T_1) + c_{\mathcal{G}}(T_2) - c_{\mathcal{G}}(T_1 \cup T_2) = \\ &= x(T_1) + x(T_2) - c_{\mathcal{G}}(T_1 \cup T_2) \leq x(T_1) + x(T_2) - x(T_1 \cup T_2) = \\ &= x(T_1 \cap T_2). \end{aligned}$$

The second inequality follows from $(N, c_{\mathcal{G}})$ being a concave game, while the remaining ones are core inequalities. So $x(T_1 \cap T_2) = c(T_1 \cap T_2)$ and our claim is proved.

So, let T_1 be the unique minimal autonomous trunk at x (which exists by the above claim). Then, define $G^1 = (S_1 \cup \{r_1\}, E_1)$ to be the subtree generated by T_1 ($S_1 = T_1 \setminus \{r\}$ and $r_1 = r$). Obviously, $x^1 = (x_i)_{i \in S_1}$ is a core element of $(N, c_{\mathcal{G}^1})$. Let $l(T_1)$ be the set of outgoing arcs of T_1. Let $e = (i, j)$ be an arc in $l(T_1)$. Consider the branch $B_e = (V_e, E_e)$ rooted at e. Then, $x^e = (x_j)_{j \in V_e \setminus \{i\}}$ is a core element of the restricted problem defined by B_e. Certainly, $x^e \geq 0$ and efficiency follows from the subsequent conditions

$$\sum_{j \in V_e \setminus \{i\}} x_j \geq \sum_{e' \in E_e} c(e') \quad \text{for all } \; e = (i, j) \in l(T_1) \tag{7}$$

$$\sum_{e \in l(T_1)} \sum_{j \in V_e \setminus \{i\}} x_j = \sum_{e \in l(T_1)} \sum_{e' \in E_e} c(e'). \tag{8}$$

Let T^e be a trunk of B_e. Then, $T^e \cup T_1$ is a trunk of G, thus

$$x(T^e) + x(T_1) = x(T^e \cup T_1) \leq c(T^e \cup T_1) = c(T_1) + \sum_{i \in T^e} c(e_i) = x(T_1) + \sum_{i \in T^e} c(e_i).$$

Therefore, according to Proposition 3.1, x^e is a core element of the restricted problem.
Now, apply the previous reasoning to x^e as a core element of $(N, c_{\mathcal{G}^e})$. Define $G^2 = (S_2 \cup \{r_2\}, E_2)$ to be the subtree generated by T_2 ($S_2 = T_2 \setminus \{i\}$ and $r_2 = i$ if $e = (i, j)$), where T_2 is the unique minimal autonomous trunk at x^e of B_e. Then, select any outgoing arc of T_2 and repeat the previous reasoning until the corresponding autonomous trunk T^* were such that $l(T^*) = \emptyset$. Then, go down selecting the outgoing arcs which have not been selected previously. The process finishes when all the arcs have been selected and the collection of subtrees obtained at the end satisfies the desired conditions. □

Let x be a core element, then we refer to the partition $\mathcal{S}$ which satisfies the conditions of Proposition 3.4 (ii) as the *partition into subtrees induced by x*.

Example 3.5 Consider the 10 player connection problem as is graphically depicted in Figure 1. The different vertices are depicted as circles. Each of the encircled numbers corresponds to a the location of the corresponding player. The special vertex, the source, is depicted by the triangle below. Furthermore, the arcs in the treenetwork are represented by the line segments connecting the different players/vertices. The cost of a specific link is put next to the corresponding line segment.

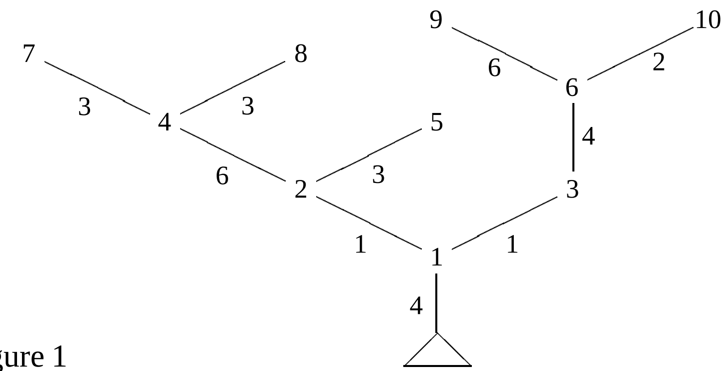


Figure 1

The partition into subtrees induced by the core element $x = (2,2,2,4,3,3,4,4,6,3)$ corresponds to the partition of the player set $\langle\{1,2,3\},\{4,7,8\},\{5\},\{6,10\},\{9\}\rangle$. Note that player 2 plays the role of the source for both the connection problems induced by the player sets $\{1,2,3\}$ and $\{5\}$ respectively. We stress, that he is not considered as a player in either case. The same counts for the players 3 and 6, who are now the local sources for the problems associated with the player sets $\{6,10\}$ and $\{9\}$ respectively, but neither of them are included as a player in the description of the corresponding connection problems.

We conclude this section with a geometric study of the core. This study is based on the properties of the core of convex (concave) games (Shapley (1971), Ichiishi (1983), Monderer, Samet and Shapley (1992)). Because it is concave, the game corresponding to a maintenance problem has a full dimensional core. (Shapley (1971)). We show that the core has but two *types* of faces: faces associated to nontrivial partitions into subtrees, which we refer to as faces of *type* I, and faces associated to groups of players who are paying nothing, which we refer to as faces of *type* II. From Monderer *et al.* (1992) it follows that if (N, v) is a concave game, then for each $\sigma = \langle S_1, \ldots, S_p\rangle$, ordered partition of N, the set $F_\sigma = \{x \in core(v) \mid x(\cup_{l=1}^{k} S_l) = v(\cup_{l=1}^{k} S_l) \text{ for all } 1 \leq k \leq p\}$ is a nonempty face of $core(v)$ of dimension $n - k$ at most.

Lemma 3.6 *An allocation x is in the face of $core(c_{\mathcal{G}})$ if and only if x verifies one of the following conditions:*

(I) *There exists an autonomous trunk $T \neq V$ at x*

(II) $x(T) < c(T)$ *for all nontrivial trunks T and there exists a nonempty coalition $S \subseteq N \setminus \{i \in N \mid i$ is a leaf $\}$ such that $x(S) = 0$.*

Proof 1) Let $\mathcal{S}(x) = \langle G^1, \dots G^p \rangle$, where $G^l = (S_l \cup \{r_l\}, E_l)$ for all $l = 1, \dots p$, be the partition of G into subtrees induced by x.
If x verifies condition (I), then $\mathcal{S}(x)$ is a nontrivial partition and x belongs to the nonempty face of $core(c_{\mathcal{G}})$ defined as $F_\sigma = \{x \in core(c_{\mathcal{G}}) \mid x(\cup_{l=1}^k S_l) = c_{\mathcal{G}}(\cup_{l=1}^k S_l) \text{ for all } 1 \le k \le p\}$ where $\sigma = \langle S_1, \dots S_p \rangle$ is the ordered partition of the player set N defined by $\mathcal{S}(x)$.
If x verifies condition (II), then $x(N \setminus S) = x(N) = c_{\mathcal{G}}(N) = c_{\mathcal{G}}(N \setminus S)$, therefore x belongs to the nonempty face of $core(c_{\mathcal{G}})$ defined as $F_\sigma = \{x \in core(c_{\mathcal{G}}) \mid x(\cup_{l=1}^k T_l) = c_{\mathcal{G}}(\cup_{l=1}^k T_l) \text{ for all } 1 \le k \le 2\}$ where $\sigma = \langle T_1, T_2 \rangle$ is the ordered partition of the player set N defined as $T_1 = N \setminus S$ and $T_2 = S$.
2) Now we will show that if x does not satisfy neither (I) nor (II) then x is in the relative interior of $core(c_{\mathcal{G}})$. Let $L \neq N$ be a nonempty coalition. Then we claim that $x(L) < c_{\mathcal{G}}(L)$. Denote by T_L the trunk $\{i \in N \mid \exists\ j \in L \text{ such that } j \preceq i\}$. Then two cases are possible.

Case 2.1. : If $T_L \neq N$, then $x(L) \le x(T_L) < c(T_L) = c_{\mathcal{G}}(L)$, where the first inequality follows from $x \ge 0$, according to Proposition 3.1 and the second one follows from the negation of (I).
Case 2.2. : If $T_L = N$, then L contains all the leafs of the tree and $N \setminus L \neq \emptyset$ is contained in $N \setminus \{i \in N \mid i \text{ is a leaf }\}$. Therefore, it follows from the negation of (I) and (II) that $x(T_L \setminus L) \neq 0$. Then, taking into account that x is a nonnegative vector it holds that $x(L) < x(T_L) = c_{\mathcal{G}}(L)$. □

Example 3.7 Let $\mathcal{G}^1$ be the subtree defined in Example 3.5. Then, the core of the cost game $(N, c_{\mathcal{G}^1})$ is represented below. Here,

$$F_1 = \{x \in \mathbb{R}^3 \,|\, x_3 = 1\,,\ x_1 + x_2 = 5\,,\ 0 \le x_1 \le 4\,,\ 0 \le x_2 \le 5\}$$

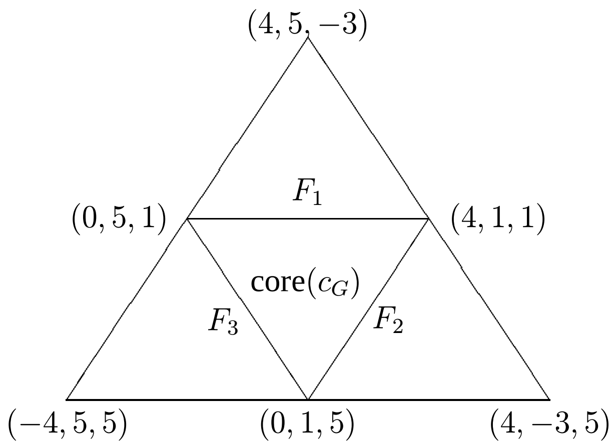
is the face of type (I) associated to the nontrivial partition into subtrees $\mathcal{S}_1 = \langle G_1^1, G_1^2 \rangle$, where

$$G_1^1 = (\{1, 2\} \cup \{r\}, \{e_1, e_2\})$$
$$G_1^2 = (\{3\} \cup \{r_2\}, \{e_3\}) \text{ with } r_2 = 1$$

and $F_2 = \{x \in \mathbb{R}^3 \,|\, x_2 = 1\,,\ x_1 + x_3 = 5\,,\ 0 \le x_1 \le 4\,,\ 0 \le x_3 \le 5\}$ is the face of type (I) associated to the nontrivial partition into subtrees $\mathcal{S}_2 = \langle G_2^1, G_2^2 \rangle$, where

$$G_2^1 = (\{1, 3\} \cup \{r\}, \{e_1, e_3\})$$
$$G_2^2 = (\{2\} \cup \{r_2\}, \{e_2\}) \text{ with } r_2 = 1$$

and $F_3 = \{x \in \mathbb{R}^3 \,|\, x_1 = 0\,,\, x_2 + x_3 = 6\,,\, 0 \leq x_2 \leq 5\,,\, 0 \leq x_3 \leq 5\}$ is a face of type (II). The below figure shows the core, and in particular its faces, laying inbedded in the imputation set $I(c_{\mathcal{G}}) := \{x \in \mathbb{R}^N | x(N) = c_{\mathcal{G}}(N), x_i \leq c_{\mathcal{G}}(i)$ for all $i \in N\}$, which is the set of all individually rational and efficient allocations for the game $c_{\mathcal{G}}$. Here it equals the convex hull of the vectors $(4, 5, -3)$, $(-4, 5, 5)$ and $(4, -3, 5)$.



4 Weighted constrained egalitarian allocations and the core: a dynamic approach

The *constrained egalitarian solution* of Dutta and Ray (1989) is a solution concept for TU games which combines commitment for egalitarianism and promotion of individual interests in a consistent manner. In convex games, it selects the unique core allocation which Lorenz-dominates all core allocations and can be computed using a fairly simple algorithm. According to the constrained egalitarian solution players are treated in a symmetric way. However, in many situations, this seems unrealistic. For a discussion on examples where lack of symmetry is present, the reader is referred to Kalai and Samet (1988) and Shapley (1981). In this section we analyze the constrained egalitarian solution for standard fixed tree connection problems, as well as weighted generalizations. Asymmetries between players are represented by a weight vector $\omega \in \mathbb{R}^N_+$. For any maintenance problem $\mathcal{G}$ and weight vector $\omega \in \mathbb{R}^N_+$, the weighted constrained egalitarian solution captures the idea that the vector of proportional cost shares with respect to ω should be chosen whenever it determines a core element of $(N, c_{\mathcal{G}})$. Throughout this section we will fix a maintenance problem $\mathcal{G} = \langle G, c, N\rangle$. We will restrict ourselves to *admissible vectors of weights with respect to* $\mathcal{G}$, i.e. vectors $\omega \in \mathbb{R}_+^N$ such that for all $i \in N$ with $c(e_i) > 0$ there exists a follower $j \in F(i)$ of player i with strictly positive weight, i.e. $\omega_j > 0$. The set of all those weights for $\mathcal{G}$ is denoted by $\mathcal{W}(\mathcal{G})$. The admissibility condition will express in the sequel no more than the idea that for any arc of the tree there is at least one user that can be held responsible for the corresponding cost.

In order to adapt Dutta and Ray's algorithm, we need a few definitions. Let T be a trunk. Suppose we make the following changes to $\mathcal{G}$. First remove all vertices in T together with the source r, and all arcs in E that are incident to at least one player in T. Instead, create a new source r_T and new arcs (i, r_T) for all agents $i \in N\backslash T$ for which e_i is an outgoing arc of T in G. The rest of the tree remains unchanged. As a result of these steps of cutting the trunk, discarding it and bundling the remaining branches together, we get a new (contracted) tree G_T. A cost function c_T on the set of the corresponding set of arcs E_T is defined as follows. An arc $e \in E \cap E_T$ is as costly as before, $c_T(e) = c(e)$, while the cost of an arc of type (i, r_T) is given by $c_T((i, r_T)) = c(e_i)$. Then the *contraction of* $\mathcal{G}$ *by* T is defined by the triple $\mathcal{G}_T = \langle G_T, c_T, N\backslash T\rangle$. Note that though $\mathcal{G}_T$ fails standardness, in the sense that in general there is more than one arc connected with the source, it has all other characteristics of a maintenance problem . But this assumption will not be of importance for our results; we remind the reader of the fact that concavity of the game $(N, c_{\mathcal{G}_T})$ can be achieved without it. This follows from the possibility of *decomposing* the game as is pointed out in the footnote in Section 2.

Define for $\omega \in \mathcal{W}(\mathcal{G})$ the weight of a coalition $S \subseteq N$ as $\omega_S := \sum_{i\in S}\omega_i$. The *weighted average cost under* $\mathcal{G}$ *(with respect to* ω*)* of a nonempty coalition S is defined to be

$$\alpha_\omega(S) := \begin{cases} \dfrac{\sum_{i\in S} c(e_i)}{\omega_S} & \text{if } \omega_S > 0 \\ \infty & \text{if } \omega_S = 0. \end{cases}$$

Definition Let $\omega \in \mathcal{W}(\mathcal{G})$. Then, the ω-*constrained egalitarian* solution is defined as the allocation obtained at the end of the following algorithm.

Algorithm 4.1

STEP 0

Initialize by letting $k = 0$ and $\mathcal{G}^0 = \mathcal{G}$.

STEP 1

Let $k = k+1$. Define $\mu_k = \min\left\{ \alpha_\omega(T) \mid T \text{ is a trunk of } G^{k-1} \right\}$ to be the minimum weighted average cost under $\mathcal{G}^{k-1}$ (w.r.t ω). Let $T_k(\omega) \subseteq N^{k-1}$ be the unique maximal trunk of G^{k-1} (with respect to inclusion) of minimum weighted average cost[4]. Define $\xi_i(\omega) = \omega_i \mu_k$ for all $i \in T_k(\omega)$.

STEP 2

If $\cup_{l=1}^{k} T_l(\omega) = N \cup \{r\}$, then terminate. Otherwise, define by $\mathcal{G}^k = \langle G^k, c^k, N^k \rangle$ the contraction of $\mathcal{G}^{k-1}$ by $T_k(\omega)$ and repeat Step 1.

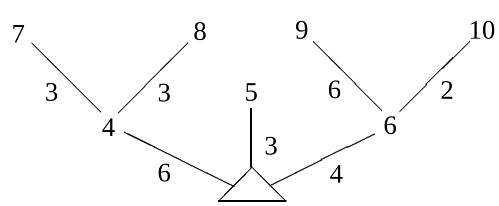
Obviously, this process must end after at most $m \leq n$ stages. Taking into account the special structure of the maintenance problem , the above algorithm yields the constrained egalitarian solution of Dutta and Ray in case of $\omega_i = 1$, for all $i \in N$. Let $\mathcal{T}(\omega) = \langle T_1(\omega), \ldots, T_p(\omega) \rangle$ define the partition of the player set induced by the algorithm. Then

$$\xi_i(\omega) = \omega_i \alpha_\omega(T_l(\omega)) \text{ for all } i \in T_l(\omega) \text{ and for all } l = 1, \ldots, p \tag{9}$$

$$\sum_{l=1}^{k} \sum_{i \in T_l(\omega)} \xi_i(\omega) = \sum_{i \in \cup_{l \leq k} T_l(\omega)} c(e_i) \tag{10}$$

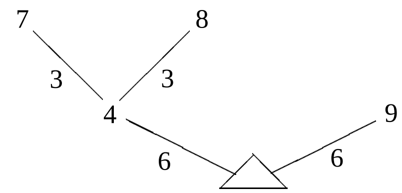
$$\frac{\xi_i(\omega)}{\omega_i} < \frac{\xi_j(\omega)}{\omega_j} \text{ for all } i \in T_l(\omega), \ j \in T_s(\omega) \text{ and } l < s. \tag{11}$$

Example 4.2 Let $\mathcal{G} = \langle G, c, N \rangle$ be the maintenance problem of Example 3.5 and $\omega \in \mathcal{W}(\mathcal{G})$ such that $\omega_i = 1$ for all $i \in N$. Then, the constrained egalitarian allocation for that problem is $\xi(\omega) = (2,2,2,4,3,3,4,4,6,3)$. In the first step we determine $\mu_1 = 2, T_1(\omega) = \{1,2,3\}$ and $\xi_i(\omega) = 2$ for all $i \in T_1(\omega)$. The next figures show the following steps,

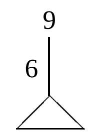


$\mu_2 = 3, T_2(\omega) = \{5, 6, 10\}$,
$\xi_i(\omega) = 3$ for all $i \in T_2(\omega)$

[4]Observe that the existence of $T_k(\omega)$ follows from the concavity of the cost game.



$\mu_3 = 4, T_3(\omega) = \{4, 7, 8\}$,
$\xi_i(\omega) = 4$ for all $i \in T_3(\omega)$



$\mu_4 = 6, T_4(\omega) = \{9\}$,
$\xi_i(\omega) = 6$ for all $i \in T_4(\omega)$

Observe that $\mathcal{T}(\omega)$ can be refined obtaining a partition of G into subtrees in such a way that the restriction of $\xi(\omega)$ to each subtree turns out to be the egalitarian allocation for the restricted problem. For this example, such a partition into subtrees is that one described in Example 3.5.

Now we give an algorithm for calculating any ω-constrained egalitarian allocation by means of taking a *dynamic* approach. Consider a maintenance problem $\mathcal{G} = \langle G, c, N \rangle$. Interpret the vertices as the villages of the different players and the arcs as the roads to the capital city of the region (source). The cost of a road is expressed as the number of days it takes (for one person) to paint the stripes on the road. The constrained egalitarian solution is determined as the time that each of the residents are painting provided that (i) every worker keeps working as long as the road from the capital to his residence has not been completed, (ii) every worker does his job on an unfinished segment between the capital and his home village, (iii) each worker starts painting at the same moment and (iv) all workers paint equally fast.[5] For any vector of admissible weights ω, the ω-constrained egalitarian solution is obtained by just prescribing different speeds to the workers, ω_i for player i. Instead of the individual time expenses, the cost share now is determined by the distance that an agent covers until his path is entirely painted. In this way, once players get to work in some group at one and the same road, each of them is charged for the fraction of the incurred cost corresponding to painting the unfinished part that is proportional to his weight. Due to the way of distributing costs we will interpret the weights as *contribution rates*. We will see that this dynamic approach amounts to calculating the individual cost shares in a finite number of stages; each of the different stages

[5]Our approach resembles the *painting story* as in Maschler *et al.* (1995), which resulted in an algorithm for calculating the nucleolus corresponding to a standard fixed tree game. Only, in our setting we can do without the *social obligation* condition.

corresponds to the actual status of the work procedings at the very moments that stripes on a specific road are realized. First we formally describe the algorithm and we prove its validity for calculating weighted constrained egalitarian allocations. Then we show that for any given fixed tree connection problem $\mathcal{G} = \langle G, c, N \rangle$, the set of all ω-constrained egalitarian allocations $\xi(\omega)$ when ω varies over $\mathcal{W}(\mathcal{G})$ equals the core of the associated cost game $(N, c_{\mathcal{G}})$. Once established this equivalence, we will study the properties of the map $\xi^{\mathcal{G}} : \mathcal{W}(\mathcal{G}) \to core(c_{\mathcal{G}})$, which assigns to each vector of contribution rates ω the *home-down allocation* $\xi(\omega)$ associated to it.

For a coalition S and admissible weight ω its total weight is defined by $\omega_S = \sum_{i \in S} \omega_i$. Let $x(e, k) \in [0, c(e)]$ be the part of the cost of arc e which is paid before stage k. Let $E_k \subseteq E$ be the subset of arcs whose cost is covered at stage k and let $E(k) = \cup_{j<k} E_j$ be the subset of arcs which have been paid before stage k. Let $e(i, k)$ be the arc to which player i contributes in stage k and let $S(e, k) = \{i \in N \,|\, e(i, k) = e\}$ be the set of players contributing to arc e in stage k. Let $K(i)$ denote the first stage in which i stops contributing.

Definition Let $x(\omega)$ be the allocation obtained at the end of the algorithm. We will refer to it as the *home-down allocation associated to* ω.

Algorithm 4.3

STEP 0 :

Initialize by defining:
$k = 1$
$x(e, 1) = 0$ for all $e \in E$
$E(1) = \emptyset$
$e(i, 1) = e_i$ for all $i \in N$
$S(e_i, 1) = \{i\}$ for all $i \in N$

STEP 1 :

Given the contribution rates of the players contributing to an unfinished arc $e \in E \setminus E(k)$ in stage k, it would take $t(e, k) = \dfrac{c(e) - x(e, k)}{\omega_{S(e,k)}}$ units of time to finish paying for the arc e. Then, the shortest time needed to finish paying for an unfinished arc is considered to determine which fraction of each unfinished arc is constructed. That is, let $t(k) = \min\{t(e, k) \,|\, e \in E \setminus E(k)\}$, then $\omega_{S(e,k)} t(k)$ is the fraction of an unfinished arc $e \in E \setminus E(k)$ which is constructed at stage

k. Thus, the part of the cost of arc e which is paid before stage $k+1$ is given by $x(e,k+1) = x(e,k) + \omega_{S(e,k)} t(k)$ for all $e \in E \setminus E(k)$. Then, each player whose path has not been covered yet is charged according to his/her contribution rate. Let $E_k = \{e \in E \setminus E(k) \,|\, t(e,k) = t(k)\}$ be the subset of arcs whose cost is covered at stage k. Let $E(k+1) = E(k) \cup E_k$ be the subset of arcs which have been paid before stage $k+1$.

STEP 2 : Stop criterium.

If $E(k+1) = E$, then terminate. Let $K(i) = k$ be the finishing time for all players in $S(e,k)$, for all $e \in E_k$. Then, the home-down allocation associated to ω is $x_i(\omega) = \sum_{k=1}^{K(i)} \omega_i t(k)$ for all $i \in N$. Otherwise, let $e \in E_k$ be an arc whose cost has been paid at stage k. If $e' \in E(k+1)$ for all $e' \preceq e$, then the final allocation for a player i who has made a contribution to arc e at stage k is $x_i(\omega)$. Then, let $K(i) = k$ be the finishing time for all players in $S(e,k)$.

If there is an arc $e' \preceq e$ whose cost has not been paid yet, then all players in $S(e,k)$ start contributing to the arc $e' \notin E(k+1)$ preceding arc e which is closest to e. Then, calculate the set of players that start paying for e at stage $k+1$, $S(e,k+1)$. Let $k = k+1$ and repeat Step 1.

Clearly, the algorithm is well defined, i.e. it stops after at most $K \leq n$ stages. Let $x(\omega)$ be the home-down allocation associated to $\omega \in \mathcal{W}(\mathcal{G})$, then the following properties are satisfied.

$(C1)$ If $e_i \preceq e_j$, then $K(i) \leq K(j)$, that is, players closer to the source stop contributing earlier.

$(C2)$ For any $i,j \in N$, $K(i) \leq K(j)$ if and only if $\dfrac{x_i(\omega)}{x_j(\omega)} \leq \dfrac{\omega_i}{\omega_j}$.

$(C3)$ Let $A_k(\omega) = \{\, i \in N \mid K(i) = k \,\}$. Then, $\cup_{j \leq k} A_j(\omega) \cup \{r\}$ is a trunk of G for all $k = 1, \ldots, K$.

$(C4)$ $x_i(\omega) = \omega_i \dfrac{c(A_k(\omega))}{\omega_{A_k(\omega)}}$ for all $i \in A_k(\omega)$.

Observe that some sets in the ordered tuple $\langle A_1(\omega), \ldots, A_K(\omega)\rangle$ can be empty if no player has stopped paying at that stage. In the sequel we will refer to $\langle A_1(\omega), \ldots, A_K(\omega)\rangle$ as the *partition defined by the finishing time induced by* $x(\omega)$.

Example 4.4 The next example shows how the algorithm works. Let $\mathcal{G} = \langle G, c, N \rangle$ be the maintenance problem defined in Example 3.5. Then, the home-down allocation associated to ω with $\omega_i = 1$ for all $i \in N$, is calculated as follows. At stage $k = 1$, $x(e_i, 1) = 0$, $S(e_i, 1) = \{i\}$ and $t(e_i, 1) = c(e_i)$ for all $i \in N$. Then, $t(1) = 1$, $E_1 = \{e_2, e_3\}$ and $E(2) = E_1$. Let $k = 2$, then

i	$x(e_i, 2)$	$S(e_i, 2)$	$t(e_i, 2)$
1	1	$\{1, 2, 3\}$	1
2	1		
3	1		
4	1	$\{4\}$	5
5	1	$\{5\}$	2
6	1	$\{6\}$	3
7	1	$\{7\}$	2
8	1	$\{8\}$	2
9	1	$\{9\}$	5
10	1	$\{10\}$	1

Thus, $t(2) = 1$, $E_2 = \{e_1, e_{10}\}$ and $E(3) = \{e_1, e_2, e_3, e_{10}\}$. Therefore, $K(1) = K(2) = K(3) = 2$. Let $k = 3$, then

i	$x(e_i, 3)$	$S(e_i, 3)$	$t(e_i, 3)$
1	4		
2	1		
3	1		
4	2	$\{4\}$	4
5	2	$\{5\}$	1
6	2	$\{6, 10\}$	1
7	2	$\{7\}$	1
8	2	$\{8\}$	1
9	2	$\{9\}$	4
10	2		

Thus, $t(3) = 1$, $E_3 = \{e_5, e_6, e_7, e_8\}$ and $E(4) = \{e_1, e_2, e_3, e_5, e_6, e_7, e_8, e_{10}\}$. Therefore, $K(5) = K(6) = K(10) = 3$. Let $k = 4$, then

i	$x(e_i, 4)$	$S(e_i, 4)$	$t(e_i, 4)$
1	4		
2	1		
3	1		
4	3	$\{4, 7, 8\}$	1
5	3		
6	4		
7	3		
8	3		
9	3	$\{9\}$	3
10	2		

Thus, $t(4) = 1$, $E_4 = \{e_4\}$ and $E(5) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_{10}\}$. Therefore, $K(4) = 4$. Let $k = 5$, then $t(5) = 2$, $E_5 = \{e_9\}$ and $E(6) = E$. Thus, $K(9) = 5$ and $x(\omega) = (2, 2, 2, 4, 3, 3, 4, 4, 6, 3)$.

Proposition 4.5 *Let $\omega \in \mathcal{W}(\mathcal{G})$, then the home-down allocation $x(\omega)$ coincides with the ω-constrained egalitarian allocation.*

Proof Take $\omega \in \mathcal{W}(\mathcal{G})$ and let $\mathcal{T}(\omega) = \langle T_1(\omega), \ldots, T_p(\omega)\rangle$ be the ordered partition of the player set associated to $\xi(\omega)$, where $\xi(\omega)$ is the ω-constrained egalitarian allocation. Then, we claim that $\mathcal{T}(\omega)$ coincides with the ordered partition $\langle A_1(\omega), \ldots, A_K(\omega)\rangle$ defined by the finishing times induced by the home-down allocation $x(\omega)$.
The first nonempty set in $\langle A_1(\omega), \ldots, A_K(\omega)\rangle$, $A_1(\omega)$, equals the maximum trunk of G which is constructed fastest. Then, taking into account that the time needed to construct any trunk T of G is given by $\alpha_\omega(T) = \frac{c(T)}{\omega(T)}$, it holds that $T_1(\omega) = A_1(\omega)$ and therefore $x_i(\omega) = \xi_i(\omega)$ for all $i \in T_1(\omega) = A_1(\omega)$. Then contract $\mathcal{G}$ by $T_1(\omega)$. But $\langle A_{i+1}(\omega), \ldots, A_K(\omega)\rangle$ is, respectively, the ordered partition of the player set induced by both allocations for the contracted problem. Thus, repeated application of the previous reasoning to each contracted problem yields the claim. $\square$

We stress that different vectors of contribution rates may yield the same weighted constrained egalitarian solution. Take $\omega \in \mathcal{W}(\mathcal{G})$ and let $\mathcal{A} = \langle A_1(\omega), \ldots, A_K(\omega)\rangle$ be the partition of the player set induced by $\xi(\omega)$. For $\omega' \in \mathcal{W}(\mathcal{G})$ the home-down allocation $x(w')$ coincides with $\xi(\omega)$ precisely when the two following conditions are satisfied: 1) the partition of the player set induced by $x(\omega')$ is equal to $\mathcal{A}$ and 2) within each of the components of $\mathcal{A}$, the players must have the same relative weight. The latter condition is reflected by the statement that for each $A \in \mathcal{A}$ there

is a positive number λ_A with $\omega'_A = \lambda_A \omega_A$.
So if for all $k \in N$ we define $I(k)$ as the number such that $k \in A_{I(k)}(\omega)$, then a sufficient condition (though not necessary) for ω' in order to have the equality $x(\omega') = \xi(w)$ is that for all $i, j \in N$

$$\frac{\omega_i}{\omega_j} \leq \frac{\omega'_i}{\omega'_j} \text{ whenever } A_{I(i)}(\omega) \preceq A_{I(j)}(\omega).$$

Remark Notice that the above algorithm provides a way of calculating weighted constrained egalitarian solutions in polynomial time; its complexity is $\mathcal{O}(|N|^2)$.

We now are finished with all preparations for the main result in this section.

Theorem 4.6 *The core of the game $c_{\mathcal{G}}$ equals the set of all weighted constrained egalitarian allocations of $\mathcal{G}$, $\xi(\mathcal{W}(\mathcal{G}))$.*

Proof According to Algorithm 4.3 and Proposition 3.3 we have $\xi(\mathcal{W}(\mathcal{G})) \subseteq core(c_{\mathcal{G}})$. Conversely, let x be a core element. Then we have to show that there exists $\omega \in \mathcal{W}(\mathcal{G})$ for which $\xi(\omega) = x$. Let $\mathcal{S} = \langle G^1, \ldots, G^p \rangle$ be the partition into subtrees induced by x (Proposition 3.4), then two cases are possible:
Case (i) : If $\mathcal{S}$ is the trivial partition $\mathcal{S} = \langle G \rangle$, define $\omega_i = x_i$ for each player $i \in N$. In such a case,

$$\mu_1 = \min \{\, \alpha_\omega(T) \mid T \text{ is a trunk of } G \,\} = \alpha_\omega(N) = \frac{c(N)}{\omega_N}.$$

Therefore, $\xi_i(\omega) = \omega_i \alpha_\omega(N) = x_i$ for all $i \in N$.

Case (ii) : If $\mathcal{S} = \langle G^1, \ldots, G^p \rangle$, where $G^k = (S_k \cup \{r_k\}, E_k)$ for each $k = 1, \ldots, p$, is a nontrivial partition , then consider the precedence relation $\preceq$ over $\langle S_1, \ldots, S_p \rangle$ defined as follows,

$$S_l \preceq S_t \Leftrightarrow \exists\, i \in S_l \text{ such that } S_t \subseteq F(i).$$

Then, define $\omega_i = \alpha_l x_i$ for each player $i \in S_l$, for all $l = 1, \ldots, p$, with $\alpha = (\alpha_1, \ldots, \alpha_p)$ being a strictly positive vector ($\alpha_l > 0$ for all l) which satisfies that $\alpha_l \geq \alpha_t$ for all l, t such that $S_l \preceq S_t$. Then, we will show that $\xi(\omega) = x$. Because our choice of α it holds that if S_l precedes S_t, then the cost of S_l ($\sum_{i \in S_l} c(e_i)$) is covered by the players in S_l before the cost of S_t ($\sum_{i \in S_t} c(e_i)$) is covered by the players in S_t. Thus, $\sum_{i \in S_l} \xi_i(\omega) = \sum_{i \in S_l} c(e_i) = \sum_{i \in S_l} x_i$ for all $l = 1, \ldots, p$. Suppose next that $\xi(\omega) \neq x$. Since $\sum_{i \in S_l} \xi_i(\omega) = \sum_{i \in S_l} x_i$ for all $l = 1, \ldots, p$, there must exists S_t and $i, j \in S_t$ such that

$$\xi_i(\omega) < x_i \text{ and } \xi_j(\omega) > x_j. \tag{12}$$

Since $\omega_r = \alpha_t x_r$ for each $r \in S_t$, then $\dfrac{\xi_i(\omega)}{\xi_j(\omega)} < \dfrac{\omega_i}{\omega_j}$.

By condition $(C2)$ satisfied by $\xi(\omega) = x(\omega)$, player i stops paying before player j at the home-down allocation $\xi(\omega)$, i.e., $K(i) < K(j)$. Let $F_t(j) = \{r \in S_t \,|\, j \preceq r\}$, i.e., the set of followers of j in S_t. It follows from condition $(C1)$ that $i \notin F_t(j)$. Moreover, since it must be $K(j) \leq K(r)$ for all $r \in F_t(j)$, then $\xi_r(\omega) \geq x_r$ for all $r \in F_t(j)$. Let now j^* be the nearest player to the root on the path $P(j)$ who stops paying at the same stage as player j. Then

$$K(j^*) = K(j) \implies \frac{\xi_{j^*}(\omega)}{\xi_j(\omega)} < \frac{x_{j^*}}{x_j}. \tag{13}$$

Since $\xi_j(\omega) > x_j$, then it follows from (13) that $\xi_{j^*}(\omega) > x_{j^*}$. Then, following the same argument as before,

$$\xi_r(\omega) \geq x_r \text{ for each } r \in F_t(j^*). \tag{14}$$

Let S_t^- be the union of all sets in $\langle S_1, \ldots, S_p \rangle$ which strictly precede S_l, then $S_t^-, S_t^- \cup S_t$ are autonomous trunks at $\xi(\omega)$. Let $i^* = \pi(j^*)$, i.e. the predecessor of player j^*, then i^* is not the root because $i \notin F_t(j^*)$ and G is a standard tree and, since nobody in $F_t(j^*)$ contributes to any arc on the path $P(i^*) \neq \emptyset$ at the allocation $\xi(\omega)$, then $S_t^- \cup (S_t \setminus F_t(j^*))$ is an autonomous trunk at $\xi(\omega)$ too. Therefore,

$$\sum_{i \in S_t \setminus F_t(j^*)} \xi_i(\omega) = \sum_{i \in S_t \setminus F_t(j^*)} c(e_i).$$

Thus, it follows from (12) and (14) that

$$\sum_{i \in F_t(j^*)} x_i < \sum_{i \in F_t(j^*)} \xi_i(\omega) = \sum_{i \in F_t(j^*)} c(e_i).$$

Then,

$$\sum_{i \in S_t \setminus F_t(j^*)} x_i > \sum_{i \in S_t \setminus F_t(j^*)} c(e_i).$$

Hence,

$$\sum_{i \in S_t^-} x_i + \sum_{i \in S_t \setminus F_t(j^*)} x_i > \sum_{i \in S_t^-} c(e_i) + \sum_{i \in S_t \setminus F_t(j^*)} c(e_i).$$

violating the core constraint corresponding to the trunk $S_t^- \cup (S_t \setminus F_t(j^*))$. $\square$

Remark Aadland and Kolpin (1997) have introduced a solution concept for fixed tree problems when the tree is a chain, the *restricted average cost share rule*, which turns out to be the constrained egalitarian solution of Dutta and Ray (1989).

Remark The parametric class of solutions for standard fixed tree problems introduced by van Gellekom and Potters (1997) contains the constrained egalitarian solution as an extreme.

5 The constrained egalitarian solution as a cost sharing mechanism

In this section we consider the class of maintenance problems corresponding to a fixed set of agents N and a fixed tree network $G = (V, E)$. The class of all cost functions $c : E \to \mathbb{R}_+$ is denoted by $\mathcal{C}$. A *cost sharing mechanism* is a mapping $\xi : \mathcal{C} \to \mathbb{R}_+^N$, relating each cost function $c \in \mathcal{C}$ to a vector of cost shares $\xi(c) \in \mathbb{R}_+^N$. The *constrained egalitarian rule* is defined to be the cost sharing mechanism ξ^e which assigns to each $c \in \mathcal{C}$ the constrained egalitarian solution $\xi^e(G, c, N)$ for $\mathcal{G} = \langle G, c, N\rangle$. Analogously, for $\omega \in \mathcal{W}(\mathcal{G})$ the *ω-constrained egalitarian mechanism* is defined to be the cost sharing mechanism ξ^ω which assigns to each $c \in \mathcal{C}$ the corresponding ω-constrained egalitarian solution for $\mathcal{G} = \langle G, c, N\rangle$. But we will focus only on the constrained egalitarian cost sharing mechanism; a treatment of the generalized class of all weighted constrained egalitarian cost sharing mechanisms is postponed to the Appendix.

Suppose that, given the cost function $c \in \mathcal{C}$, the arcs in E get (weakly) more expensive. Then, consistently, a reasonable cost sharing mechanism will increase (weakly) all the individual cost shares. Nobody should benefit from an increase of cost of the entire network. When a cost sharing mechanism ξ is consistent with this idea, then it is called *cost monotonic*. More formally, ξ is cost monotonic iff for all $c, c' \in \mathcal{C}$, $c' \geq c$ implies $\xi(c') \geq \xi(c)$. Furthermore, if a cost sharing mechanism ξ generates only stable outcomes in the sense that $\xi(c) \in core(c_{\mathcal{G}})$ for $\mathcal{G} = \langle G, c, N\rangle$ and all $c \in \mathcal{C}$, then it is said to satisfy the *core property*. It is clear from the dynamic approach described in the previous section that the (weighted) constrained egalitarian rule satisfies both above properties.

Let $\langle T_1, \ldots, T_p\rangle$ be the partition of the player set N associated to $\xi^e(c)$ induced by Algorithm 4.1. Let $t(i)$ be the number such that $i \in T_{t(i)}$. Then conditions (9), (10) and (11) can be rewritten as follows:

$$\xi_i^e(c) = \xi_j^e(c) \Leftrightarrow t(i) \leq t(j) \tag{15}$$

$$\sum_{l=1}^{k} \sum_{i \in T_l} \xi_i^e(c) = \sum_{i \in T(k)} c(e_i) \text{ for all } k = 1, \ldots, p \tag{16}$$

where $T(k)$ is the trunk of G defined as $\cup_{l=1}^{k} T_l$.

Theorem 5.1 *The constrained egalitarian cost share rule is the unique cost share rule which minimizes the range of the cost shares among those rules satisfying cost monotonicity and core property.*

Proof 1) The constrained egalitarian rule minimizes the range of cost shares among the rules verifying cost monotonicity and the core property. Certainly, let φ be any cost share mechanism satisfying those properties, then we will show that for all $c \in \mathcal{C}$ the following inequalities are satisfied:

$$\max\{\, \varphi_i(c) \,|\, i \in N \,\} \geq \max\{\, \xi_i^e(c) \,|\, i \in N \,\} \tag{17}$$

$$\min\{\, \varphi_i(c) \,|\, i \in N \,\} \leq \min\{\, \xi_i^e(c) \,|\, i \in N \,\} \tag{18}$$

where $\xi^e(c)$ is the constrained egalitarian solution of $\mathcal{G} = \langle G, c, N \rangle$.
On the contrary, suppose that inequality (17) is not satisfied. Then condition (15) implies

$$\max\{\, \varphi_j(c) \,|\, j \in T_p \,\} < \xi_i^e(c) \text{ for all } i \in T_p.$$

Therefore, it follows from efficiency and condition (16) that

$$\sum_{i \in T(p-1)} c(e_i) = \sum_{i \in T(p-1)} \xi_i^e(c) < \sum_{i \in T(p-1)} \varphi_i. \tag{19}$$

where $T(p-1)$ is the trunk $\cup_{l=1}^{p-1} T_l$, contradicting the core property. Then (17) holds. A similar reasoning gives inequality (18).

2) We next establish uniqueness. Let φ be any cost allocation rule satisfying cost monotonicity and core property such that $range(\varphi(c)) = range(\xi^e(c))$ for all $c \in \mathcal{C}$. We will prove that $\varphi_i(c) = \xi_i^e(c)$ for all $i \in T_k$ and for all $k = 1, \ldots, p$ by backward induction on the index k.
If $range(\varphi(c)) = range(\xi^e(G, c, N))$, then in view of inequalities (17) and (18) it has to be $\max\{\, \varphi_i(c) \,|\, i \in N \,\} = \max\{\, \xi_i^e(c) \,|\, i \in N \,\}$. Therefore, $\varphi_i(c) \leq \xi_i^e(c)$ for all $i \in T_p$. Now, suppose that $\varphi_i(c) < \xi_i^e(c)$ for some $i \in T_p$. Then

$$\sum_{i \in T(p-1)} c(e_i) < \sum_{i \in T(p-1)} \varphi_i(c), \tag{20}$$

which contradicts the core property. Thus, $\varphi_i(c) = \xi_i^e(c)$ for all $i \in T_p$. Suppose that $\varphi_i(c) = \xi_i^e(c)$ for all $i \in T_l$, for all $l = k, \ldots, p$.

Let $\langle T_1, \ldots, T_p \rangle$ be the partition of the player set N associated to $\xi^e(c)$ induced by Algorithm 4.1. Suppose that for any $k \in \{1, \ldots, p-1\}$, we lower the cost of an arcs e_i for $i \in T_l$ and $l \geq k+1$ by

$$\varepsilon_l := \frac{\sum_{i \in T_l} c(e_i)}{|T_l|} - \frac{\sum_{i \in T_k} c(e_i)}{|T_k|}.$$

The costs of other arcs remain the same. Let $c^k : E \to \mathbb{R}$ be the function that gives for each arc the remaining cost. We claim that $\langle G, c^k, N \rangle$ defines a maintenance problem , such that $\langle T_1^k, \ldots, T_k^k \rangle$ with $T_l^k = T_l$ for all $l \leq k-1$ and $T_k^k = \cup_{l \geq k} T_l$ defines the partition of the player set N induced by $\xi^e(c^k)$ and in addition

$$\begin{aligned} &\xi_i^e(c^k) = \xi_i^e(c) \quad \text{for all} \quad i \in T_l \quad \text{and} \quad l \leq k \\ &\xi_i^e(c^k) = \frac{\sum_{i \in T_k} c(e_i)}{|T_k|} \quad \text{for all} \quad i \in T_l \quad \text{and} \quad l \geq k+1. \end{aligned} \tag{21}$$

This claim is verified as follows. Let φ be the cost allocation defined by the right-hand side in the above expression (21). Then we have to show that $\varphi = \xi^e(c^k)$. First, we will show that φ is a core element of $(N, c_{\mathcal{G}^k})$. First of all note that $\varphi_i \geq 0$ for all $i \in N$ and it is efficient as follows from

$$\begin{aligned} \sum_{i \in N} c^k(e_i) &= \sum_{i \in T(k)} c(e_i) + |\cup_{l>k} T_l| \frac{\sum_{i \in T_k} c(e_i)}{|T_k|} = \\ &= \sum_{l=1}^{k} \sum_{i \in T_l} \xi_i^e(c) + \sum_{l>k} \sum_{i \in T_l} \varphi_i = \sum_{i \in N} \varphi_i \end{aligned}$$

where the second equality follows from condition (16). Then, according to Proposition 3.1 it is enough to prove that $\sum_{i \in T} \varphi_i \leq \sum_{i \in T} c^k(e_i)$ for all trunks T. Let T be any trunk of G, then we are left two cases,

Case (i) : $T \subseteq \cup_{l \leq k} T_l$, then $\sum_{i \in T} \varphi_i = \sum_{i \in T} \xi_i^e(c) \leq \sum_{i \in T} c(e_i) = \sum_{i \in T} c^k(e_i)$.

Case (ii) : $T \cap (\cup_{l>k} T_l) \neq \emptyset$, then $T = T' \cup T''$, where $T' = T \cap (\cup_{l \leq k} T_l)$ and $T'' = T \setminus T'$, and

$$\sum_{i \in T''} \varphi_i = |T''| \cdot \frac{\sum_{i \in T_k} c(e_i)}{|T_k|}. \tag{22}$$

Also

$$
\begin{aligned}
\sum_{i\in T''} c^k(e_i) &= \textstyle\sum_{i\in T''} c(e_i) - \displaystyle\sum_{l>k} |T''\cap T_l|\cdot \varepsilon_l \\
&= \sum_{i\in T''} c(e_i) - \sum_{l>k} |T''\cap T_l|\cdot \frac{\sum_{i\in T_l} c(e_i)}{|T_l|} + |T''|\cdot \frac{\sum_{i\in T_k} c(e_i)}{|T_k|} \\
&= \sum_{i\in T''} c(e_i) - \sum_{i\in T''} \xi_i^e(c) + |T''|\cdot \frac{\sum_{i\in T_k} c(e_i)}{|T_k|}.
\end{aligned}
\tag{23}
$$

Furthermore it holds that

$$
\sum_{i\in T} \varphi_i = \sum_{i\in T'} \xi_i^e(c) + |T''|\cdot \frac{\sum_{i\in T_k} c(e_i)}{|T_k|}. \tag{24}
$$

Thus, combining expressions (22), (23) and (24) and taking into account the fact that $\xi^e(c)$ is a core element of $(N, c_{\mathcal{G}})$, it holds

$$
\begin{aligned}
\sum_{i\in T} \varphi_i &= \sum_{i\in T'} \xi_i^e(c) + \sum_{i\in T''} \xi_i^e(c) + \sum_{i\in T''} c^k(e_i) - \sum_{i\in T''} c(e_i) \\
&\leq \sum_{i\in T} c(e_i) + \sum_{i\in T''} c^k(e_i) - \sum_{i\in T''} c(e_i) \\
&= \textstyle\sum_{i\in T'} c(e_i) + \sum_{i\in T''} c^k(e_i) \\
&= \textstyle\sum_{i\in T} c^k(e_i).
\end{aligned}
$$

Now we will show that the partition generated by Algorithm 4.1 for calculating the constrained egalitarian solution of $\mathcal{G}^k$ coincides with the ordered partition $\langle T_1^k, \ldots, T_k^k\rangle$ defined above. Let T be a trunk of G, then

$$
\sum_{i\in T} c^k(e_i) \geq \sum_{i\in T} \varphi_i \geq \sum_{i\in T} \frac{\sum_{j\in T_1} c(e_j)}{|T_1|} = |T|\cdot \frac{\sum_{j\in T_1} c^k(e_j)}{|T_1|}.
$$

where the first inequality follows from the fact of φ being a core element of $(N, c_{\mathcal{G}^k})$. Therefore, the average cost (with respect to c^k) of $T_1 = T_1^k$ is minimum. Now repeated application of the previous reasoning to each contracted problem yields the claim (observe that $\varphi^1 = (\varphi_i)_{i\in N\setminus T_1}$ is a core element for the contracted problem as it was shown in the proof of Proposition 3.4). So this proves our claim.

So for c^{k-1} as defined above, it is verified that

$$
\begin{array}{rl}
(i) & c^{k-1} \leq c \\
(ii) & \xi_i^e(c) = \xi_i^e(c^{k-1}) \text{ for all } i \in T_l \text{ and all } l \leq k-1 \\
(iii) & \varphi_i(c^{k-1}) = \xi_i^e(c^{k-1}) \text{ for all } i \in T_l \text{ for all } l \geq k-1.
\end{array}
$$

Observe that condition (iii) follows from the fact that $\langle T_1^{k-1}, \ldots, T_{k-1}^{k-1} \rangle$ with $T_{k-1}^{k-1} = \cup_{l \geq k-1} T_l$ is the partition of the player set associated to $\xi^e(c^{k-1})$ generated by Algorithm 4.1 (see the above claim) and the application of the previous reasoning. Thus we derive

$$
\sum_{l \geq k} \sum_{i \in T_l} \varphi_i(c^{k-1}) = \sum_{l \geq k} \sum_{i \in T_l} \xi_i^e(c^{k-1}) \implies
$$

$$
\sum_{l=1}^{k-1} \sum_{i \in T_l} \varphi_i(c^{k-1}) = \sum_{l=1}^{k-1} \sum_{i \in T_l} \xi_i^e(c^{k-1}) = \sum_{l=1}^{k-1} \sum_{i \in T_l} \xi_i^e(c). \tag{25}
$$

The induction hypothesis gives

$$
\sum_{l \geq k} \sum_{i \in T_l} \varphi_i(c) = \sum_{l \geq k} \sum_{i \in T_l} \xi_i^e(c).
$$

So it holds that

$$
\sum_{i \in T(k-1)} \varphi_i(c) = \sum_{i \in T(k-1)} \xi_i^e(c). \tag{26}
$$

Therefore, together with expressions (25) and (26), cost monotonicity implies $\varphi_i(c) = \varphi_i(c^{k-1})$ for all $i \in T(k-1)$. But then the above conditions (i) and (ii) imply $\varphi_i(c) = \xi_i^e(c)$ for all $i \in T_{k-1}$. □

Suppose that getting connected is equally valued by the different agents. Then the constrained egalitarian cost share mechanism minimizes the range of cost shares among the class of cost monotonic mechanisms sharing the core property, simultaneously it maximizes Rawlsian welfare, that is measured by the opposite of the highest cost share. In fact, it is the only cost monotonic mechanism under this welfare consideration.

Theorem 5.2 *The constrained egalitarian cost share mechanism is the unique cost share mechanism that maximizes Rawlsian welfare among those mechanisms that satisfy cost monotonicity and which satisfy the core property.*

Proof The proof resembles that of Theorem 5.1 up to a high degree. First, the constrained egalitarian cost share mechanism selects the Lorenz maximal element in the core of a maintenance problem (Dutta and Ray (1989)), which implies Rawlsian maximality. Suppose φ satisfies also cost monotonicity and the core property therebye maximizing Rawlsian welfare. Then of course by assumption for any

$c \in \mathcal{C}$, $\max\{\varphi_i(c) \,|\, i \in N\} = \max\{\xi_i^e(c) \,|\, i \in N\}$. Now proceed along the same lines as in the proof of Theorem 5.1 after '2)' in order to see that φ equals ξ^e. □

In fact, the characterization results of the constrained egalitarian cost sharing mechanism for trees are similar to those for the restrictive average rule for chains as in Aadland and Kolpin (1997). Besides the fact that our results hold for a more general setting, Aadland and Kolpin needed an additional characterizing property which is satisfied by the constrained egalitarian cost share mechanism. The property in question is *ranking*, which requires that an agent with higher stand alone costs, should contribute (weakly) more.

One can trace easily the following independencies between the properties that we used above. Splitting the total costs equally between the players gives a cost monotonic mechanism that minimizes both the range of the weighted cost shares and the maximal weighted cost share. But the allocation need not be a core element. Furthermore there are mechanisms ξ that minimize the range of the cost shares subject to the core property but are not cost monotonic. A legitimate candidate would be the mechanism ξ that coincides with the constrained egalitarian solution for all problems except for the following 4-player problem in which $1 \prec 2 \prec 3 \prec 4, c_1 = 1, c_2 = c_3 = 2, c_4 = 3$ and $\xi(c)$ is given by $(1, 1\frac{1}{2}, 2\frac{1}{2}, 3)$. The mechanism that relates each cost function $c \in \mathcal{C}$ to the corresponding Shapley value for $c_{\mathcal{G}}$ with $\mathcal{G} = \langle G, c, N \rangle$, defines a cost monotonic mechanism for which the core property is satisfied, however it does not always minimize the range of cost shares or minimize the maximal cost share.

6 Weighted Shapley values and the core

In Section 4 we explained that each weighted constrained egalitarian allocation is a home-down allocation. Suppose that we systematically reverse the assignment rule in each step of the above allocation procedure and that in each step we first assign each of the disconnected players to the *furthest* unfinished arc on his path, which is the one that is closest to the source. Next, like before, each arc that becomes selected in this way is constructed by the agents assigned to it, and each agent i pays part of the incurred cost proportional to his weight ω_i. Repeat these steps until all players have the desired connections. As a result we get a vector of cost shares by this *bottom-up* procedure, which will be called the *down-home allocation* *corresponding to the vector of weights* ω.

For instance, consider Example 3.5 with equally weighted players. Then in order to determine the corresponding down-home allocation, we first assign all players to e_1. Then this arc is constructed and the corresponding costs are shared proportionally to the weights of the agents, which corresponds to the equal shares of $\frac{c(e_1)}{|N|} = \frac{4}{10}$ for each agent. At the second step the agents in $\{2,4,5,7,8\}$ and $\{3,6,9,10\}$ resp. are assigned to e_2 and e_3 resp. Agent 1 reached his home in this first step and does not contribute anymore. Next the procedure requires that agents in $\{3,6,9,10\}$ each pay $\frac{1}{4}$ and agents in $\{2,4,5,7,8\}$ pay $\frac{1}{5}$. Now the agents 2 and 3 are satisfied and their final cost shares are $\frac{4}{10} + \frac{1}{5}$ and $\frac{4}{10} + \frac{1}{4}$ respectively. At the beginning of the following step the players go on constructing, player 5 is situated at e_5, players 4,7 and 8 at e_4 and players 3,6,9 and 10 stick to e_3. This boils down to the additional payments of 3 for player 5, 2 for each of the players $4,7,8$ and $\frac{4}{3}$ for each player in $\{6,9,10\}$. Now agents $4,5,6$ are connected having contributed in total $2\frac{6}{10}, 3\frac{6}{10}$ and $1\frac{59}{60}$ respectively. The last step consists of letting each agent $i \in \{7,8,9,10\}$ construct it's own indispensible arc e_i at the cost of an additional payment $c(e_i)$. Then this results in the final payments $5\frac{6}{10}, 5\frac{6}{10}, 7\frac{59}{60}$ and $3\frac{59}{60}$ for agent $7,8,9$ and 10 respectively. It is widely known that the resulting allocation is nothing else than the Shapley value of the corresponding game $c_{\mathcal{G}}$.

In general, for any maintenance problem $\mathcal{G} = \langle G, c, N \rangle$, if we determine individual payments according to the above procedure with respect to an admissible vector of weights ω, we get as a result the *weighted Shapley value* for $c_{\mathcal{G}}$, that corresponds to the weight system $(\tilde{\omega}, \langle N \backslash T, T \rangle)$ where $T = \{i \in N \,|\, \omega_i > 0\}$ and $\tilde{\omega} \in \mathbb{R}^N_{++}$ is such that $\tilde{\omega}_T = \omega_T$ and $\tilde{\omega}_{N \backslash T}$ is arbitrarily chosen. For a more detailed discussion of weighted Shapley values we refer the reader to Kalai and Samet (1992). It is not difficult to see that by varying over all admissible weights, the corresponding set of weighted Shapley values does in fact not exhaust the core of the game $c_{\mathcal{G}}$. Below we describe how in a natural way the admissibility condition on weights can be relaxed, yielding *down-home allocations* each corresponding to some weighted Shapley value. Then with Monderer *et al.* (1992) in mind, it turns out that this extended class of allocations equals the core of $c_{\mathcal{G}}$.

Take a weight vector $\omega \in \mathbb{R}^N_{+}, \omega \neq 0$. Without admissibility condition on ω, we can actually take the same steps as in the above algorithm as long as arcs become selected that are used by at least one user i with strictly positive weight ω_i. Suppose that ω is not admissible for $\mathcal{G}$. Then by consequetively constructing arcs in each branch with at least one user with non-zero weight and doing nothing in all other branches, we end up with disconnected branches with all players of weight 0. Suppose the latter is the case, and let N_1 be the union of all players that are still disconnected after these operations. If we have additional exogeneously given information, that consists of a weight vector $\omega^1 \in \mathbb{R}_+^{N_1} \backslash \{0\}$, prescribing the relative impact of a player $i \in N_1$ compared to the others in N_1, then consistently we could proceed by applying the above techniques to the subproblems induced by each of the branches with the individual weights as in ω^1. Again the process terminates prematurely when the restriction of ω^1 to one of these branches is not admissible for one the problems induced by it. Then define N_2 as the set of those players in N_1 that are still not connected. Then we could proceed for those agents if only we have the disposal of another weight vector $\omega^2 \in \mathbb{R}_+^{N_2} \backslash \{0\}$, which is to summarize the relative impact of the remaining players in N_2. Again proceed with ω^2 for the remaining disconnected branches. In this way, by having sufficiently many weight vectors containing information about the relative impact of players, we end up with the whole constructed tree and an allocation of the corresponding total cost. Note that between two terminations of the procedure, we construct connected parts in the graph each determining a subtree of G; the cost for constructing arcs in such a tree are distributed among the players located here. If the set of players T of one such subtree is connected during phase t, i.e. the phase in which ω^t was used as a weight vector, then the restriction to T, w_T^t, is admissible for the restricted connection problem induced by the subtree corresponding to T, $G_T = (T \cup \{r_T\}, E_T)$. So the final allocation for the grand coalition N is determined by down-home allocations for different subproblems, which form a partition of $\mathcal{G}$ into subtrees. This gives rise to the following extension of the notion of down-home allocations.

Let $\langle S_1, \ldots, S_p \rangle$ be a partition of the player set N that induces a partition $\mathcal{S} = \langle G^1, \ldots, G^p \rangle$ of $\mathcal{G}$ into subtrees, such that for each $k \in \{1, \ldots p\}$, G^k is the tree corresponding to S^k and $\mathcal{G}^k$ it's restricted maintenance problem . Then a weight vector $\omega \in \mathbb{R}_+^N$ is called *admissible for the partition* $\langle S_1, \ldots, S_p \rangle$ if for each $k = 1, \ldots, p$, we have $\omega_{S^k} \in \mathcal{W}(\mathcal{G}^k)$. Such a weight represents the idea that agents in S_i have impact 0 compared to those in S_j whenever $i < j$. For each such a weight ω we define the down-home allocation as the combination of the down-home allocations for each of the problems $\mathcal{G}^k$ corresponding to ω_{S^k}.

It is tedious but not very hard to show that each such a down-home allocation is related in a natural way with a weighted Shapley value of $c_{\mathcal{G}}$. By extending the class of down-home allocations by enlargening the set of admissible weights we get all weighted Shapley values of $c_{\mathcal{G}}$. Then following the result of Monderer *et al.* (1992), stating that the core equals the set of weighted Shapley values it must be that it also equals the set of all down-home allocations.

7 Appendix

Throughout this section, fix a tree network $G = (V, E)$ and the set of players N of a maintenance problem $\mathcal{G}$. Denote by $\mathcal{C}$ the class of all cost functions $c : E \to \mathbb{R}_+$. Also, let there be given a *(positive) weight system*, i.e. a mapping $\omega : \mathcal{C} \to \mathbb{R}_+^N$ which relates each cost function c to a (positive) vector of admissible weights for $\mathcal{G} = \langle G, c, N\rangle$. Then, the *$\omega$-constrained egalitarian mechanism* is the cost sharing mechanism ξ^ω which relates each maintenance problem $\mathcal{G} = \langle G, c, N\rangle$ with its $\omega(c)$-constrained egalitarian allocation $\xi^\omega(c)$.

Theorem 7.1 *Suppose ω is positive. Then ω-constrained egalitarian mechanism minimizes the range of the weighted cost shares*

$$\max\left\{\frac{\xi_i(c)}{\omega_i(c)} \mid i \in N\right\} - \min\left\{\frac{\xi_i(c)}{\omega_i(c)} \mid i \in N\right\}$$

among those cost mechanisms which are cost monotonic and have the core property.

Theorem 7.2 *The ω-constrained egalitarian cost share mechanism minimizes the weighted maximal cost share $\max\left\{\frac{\xi_i(c)}{\omega_i(c)} \mid i \in N\right\}$ among those mechanisms which are cost monotonic and have the core property.*

The above two theorems are proved as straightforward adaptation of part 1) of the proof of Theorem 5.1.

If ω is not a positive weight system then the maximum and the minimum is taken over the subset of players with strictly positive weight and the class of cost sharing mechanisms has to be reduced to those mechanisms which have the core property and verify that $\xi_i(c) = 0$ for all $i \in N$ such that $\omega_i(c) = 0$ for all maintenance problems $\mathcal{G} = \langle G, c, N\rangle$. In order to obtain a characterization result which generalizes the result stated in Theorem 5.1 and Theorem 5.2 we have to restrict ourselves to *homogeneous* positive weight systems, i.e. weight systems ω that do not depend on the cost function, or $\omega(c) = \omega(c')$ for all $c, c' \in \mathcal{C}$.

Theorem 7.3 *Let ω be a positive homogeneous weight system. Then the ω-constrained egalitarian rule is the unique cost sharing mechanism which minimizes the weighted range of the cost shares among those mechanisms satisfying cost monotonicity and the core property.*

Theorem 7.4 *Let ω be a positive homogeneous weight system. Then the ω-constrained egalitarian rule is the unique cost sharing mechanism which minimizes*

$$\max\left\{\frac{\xi_i(c)}{\omega_i(c)} \,|\, i \in N, \omega_i(c) > 0\right\}$$

among those mechanisms satisfying cost monotonicity and the core property.

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