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Average Convexity in Communication Situations

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Abstract

In this paper we study inheritance properties of average convexity in communication situations. We show that the underlying graph ensures that the graph-restricted game originating from an average convex game is average convex if and only if every subgraph associated with a component of the underlying graph is the complete graph or a star graph. Furthermore, we study inheritance of (average) convexity of the associated potential games.

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1 Introduction

A communication situation is a cooperative game with communication restrictions. The communication possibilities are modelled by means of an undirected (communication) graph. *Myerson* (1977) was the first to study these communication situations. He introduced the graph-restricted game and he provided an axiomatization of the Shapley value of these games. This value is usually referred to as the Myerson value.

Posterior papers have analyzed properties of graph-restricted games. *Owen* (1986) shows that superadditivity of a game implies superadditivity of the graph-restricted game. *Nouweland* and *Borm* (1991) show that if the communication graph is cycle-complete and the cooperative game convex, then the graph-restricted game is convex.[1]

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[1]*Nouweland* and *Borm* (1991) refer to the graph-restricted game as the *point game*.

Furthermore, they show that if the communication graph is not cycle-complete then there exists a convex cooperative game such that the graph-restricted game is not convex. In this paper we will study the class of graphs which ensure that if the underlying game is average convex then the graph-restricted game is average convex.

Average convexity was introduced by *Iñarra* and *Usategui* (1993). They study the necessary and sufficient conditions for the Shapley value of a game in characteristic function form to lie in the core. As a by-product, they introduce the class of average convex games and show that the Shapley value of an average convex game lies in the core. Since a subgame of an average convex game is average convex as well, it holds that the Shapley value of a subgame belongs to the core of this subgame. *Sprumont* (1990) showed that the extended Shapley value of an average convex game is a population monotonic allocation scheme (PMAS).[2]

Marín-Solano and *Rafels* (1996) study connections between convex and average convex games. They show that the Shapley values for a game and all its subgames lie in the corresponding cores if and only if the associated potential game (cf. *Hart* and *Mas-Colell* (1989)) is average convex. Furthermore, they show that the extended Shapley value is a population monotonic allocation scheme if and only if the associated potential game is convex.

In this paper we show that inheritance of average convexity of a game by the graph-restricted game is guaranteed if and only if every subgraph associated with a component of the underlying graph is the complete graph or a star graph. Furthermore, we study the relation between (average) convexity of an associated potential game and (average) convexity of the potential game associated with the graph-restricted game. We find that except the complete graphs, there is essentially no graph that ensures inheritance of convexity of the potential game associated with a cooperative game by the potential game associated with the graph-restricted game. Finally, we find with respect to potential games that inheritance of average convexity is ensured for the same class of graphs that ensure inheritance of average convexity of the underlying cooperative game by the graph-restricted game.

The plan of this paper is as follows. Section 2 provides some notations and definitions. Section 3 deals with the inheritance of average convexity of a game by the graph-restricted game. Inheritance of convexity and average convexity of the corresponding potential games is studied in section 4. In section 5 we conclude with a remark.

[2] *Sprumont* (1990) refers to an average convex game as a quasi-convex game.

2 Notation

A cooperative game is a pair (N, v), where $N = \{1, \ldots, n\}$ denotes the set of players and $v : 2^N \rightarrow \mathbb{R}$ the characteristic function. If no confusion can arise we sometimes refer to a game by its characteristic function. A cooperative game (N, v) is convex if for all $i \in N$ and all $T_1 \subseteq T_2 \subseteq N$ with $i \in T_1$ it holds that[3]

$$v(T_1) - v(T_1 \backslash \{i\}) \leq v(T_2) - v(T_2 \backslash \{i\}).$$

So, a game is convex if the marginal contribution of a player to any coaltion is less than his marginal contribution to a larger coalition.

Iñarra and *Usategui* (1993) introduced the class of average convex games. A game (N, v) is average convex if for all $T_1 \subseteq T_2 \subseteq N$ it holds that

$$\sum_{i \in T_1} [v(T_1) - v(T_1 \backslash \{i\})] \leq \sum_{i \in T_1} [v(T_2) - v(T_2 \backslash \{i\})] .$$

So, a game is average convex if for any coalition the sum (average) of marginal contributions for the players in this coalition is less than the sum (average) of marginal contributions for the same players in a larger coalition. Note that convexity implies average convexity. *Marín-Solano* and *Rafels* (1996) remark that average convex games are superadditive.

A communication graph is a pair (N, L) where the set of vertices N represents the set of players and the set of edges L represents the set of bilateral (communication) links. Two players i and j are directly connected iff $\{i, j\} \in L$. Two players i and j are connected (directly or indirectly) iff $i = j$ or there exists a path between players i and j. The notion of connectedness induces a partition of the player set into communication components, where two players are in the same communication component if and only if they are connected. The set of communication components will be denoted by N/L. The component $C \in N/L$ containing player $i \in N$ will be denoted by $C_i(L)$. Furthermore, denote the subgraph on the vertices in coalition $S \subseteq N$ by $(S, L(S))$, where $L(S) = \{\{i, j\} \in L \mid \{i, j\} \subseteq S\}$, and the partition of S into communication components according to graph $(S, L(S))$ by S/L.

Myerson (1977) studied communication situations (N, v, L) where (N, v) is a cooperative game and (N, L) a communication graph. He introduced the *graph-restricted game* (N, v^L), where

$$v^L(S) = \sum_{C \in S/L} v(C) , \quad \text{for all } S \subseteq N.$$

[3] $S \subseteq N$ denotes that S is a subset of N, $S \subset N$ denotes that S is a strict subset of N.

So, a coalition is split into communication components and the value of this coalition in the graph-restricted game is then defined as the sum of the values of the communication components in the original game. The Shapley value of the game (N, v^L) is usually referred to as the *Myerson value* of communication situation (N, v, L). The Myerson value will be denoted by $\mu(N, v, L)$.

3 Average convexity

In this section we describe the class of communication graphs for which average convexity of the original game implies average convexity of the graph-restricted game.

First consider the class of communication graphs for which convexity of the original game is inherited by the graph-restricted game (see *Nouweland* and *Borm* (1991)).

Definition 3.1 A graph (N, L) is *cycle-complete* if the following holds: if $(x_1, x_2, \ldots, x_k, x_1)$ is a cycle in the graph then $\{x_i, x_j\} \in L$ for all $i, j \in \{1, \ldots, k\}$ with $i \neq j$.

So, for every cycle the complete graph on the vertices forming this cycle is a subgraph of (N, L). Note that all cycle-free graphs and the complete graph are cycle-complete.

The following theorem follows straightforward from example 3 of *Nouweland* and *Borm* (1991). They construct for each communication graph that is not cycle-complete a convex game such that the corresponding graph-restricted game is not convex. We will show that these graph-restricted games are not even average convex.

Theorem 3.1 Let (N, L) be a communication graph that is not cycle-complete. Then there exists a convex game (N, v) such that the graph-restricted game is not average convex.

Proof: Since (N, L) is not cycle-complete, there is a cycle $(x_1, \ldots, x_k, x_1)$ in (N, L) and $i, j \in \{1, \ldots, k\}$ with $i < j-1$, $\{x_i, x_j\} \notin L$ and $\{x_m, x_j\} \in L$ for all $m \in \{i+1, \ldots, j-1\}$. Consider the convex game (N, v) where $v(S) = |S| - 1$ for all $S \subseteq N$, $S \neq \emptyset$. Define $T_1 = \{x_i, x_{i+1}, x_j\}$ and $T_2 := \{x_1, \ldots, x_k\}$. Then,

$$\sum_{l \in T_1} \left[v^L(T_1) - v^L(T_1 \backslash \{l\}) \right] = 1 + 1 + 2 > 1 + 1 + 1 = \sum_{l \in T_1} \left[v^L(T_2) - v^L(T_2 \backslash \{l\}) \right].$$

Hence, the game (N, v^L) is not average convex. □

Now, the following result follows directly from theorem 3.1.

Corollary 3.1 Let (N, L) be a communication graph. Then the following two statements are equivalent:

(i) The graph (N, L) is cycle-complete.

(ii) For all convex (N, v) the graph-restricted game (N, v^L) is average convex.

Proof: Follows directly from theorem 3.1 above and theorem 1 of *Nouweland* and *Borm* (1991), which states that if (N, L) is cycle-complete and (N, v) convex, then (N, v^L) is convex and hence average convex. □

The corollary above states that cycle-completeness of the underlying graph is necessary for convexity of the cooperative game to guarantee average convexity of the graph-restricted game. Hence, cycle-completeness is a necessary condition on the underlying graph to guarantee that *average* convexity of the cooperative game is inherited by the graph-restricted game. The following example shows that this condition is not sufficient.

Example 3.1 Consider the following communication situation: $N = \{1, 2, 3, 4\}$, $L = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 2\}\}$ (see figure 1) and v such that

$$v(S) = \begin{cases} 0 & , |S| = 1 \text{ or } S \in \{\{1,2\}, \{1,3\}, \{2,4\}\} \\ 6 & , S \in \{\{1,4\}, \{3,4\}, \{2,3\}\} \\ 9 & , |S| = 3 \\ 16 & , S = N \end{cases} .$$

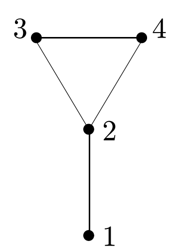

Figure 1: The graph (N, L).

Note that the communication graph (N, L) is cycle-complete. It can be checked that the game (N, v) is average convex. Note that (N, v) is not convex since

$$v(\{2, 3\}) - v(\{3\}) = 6 > 3 = v(\{2, 3, 4\}) - v(\{3, 4\}).$$

The characteristic function of the graph-restricted game is given by

$$v^L(S) = \begin{cases} 0 & , |S| = 1 \text{ or } S \in \{\{1,2\},\{1,3\},\{1,4\},\{2,4\}\} \\ 6 & , S \in \{\{3,4\},\{2,3\},\{1,3,4\}\} \\ 9 & , S \in \{\{1,2,3\},\{1,2,4\},\{2,3,4\}\} \\ 16 & , S = N \end{cases} .$$

Let $T_1 = \{1,2,4\}$ and $T_2 = N$ then

$$\begin{aligned} \sum_{i \in T_1} \left[v^L(T_1) - v^L(T_1 \backslash \{i\}) \right] &= 9 + 9 + 9 = 27 \\ &> 24 = 7 + 10 + 7 = \sum_{i \in T_1} \left[v^L(T_2) - v^L(T_2 \backslash \{i\}) \right]. \end{aligned}$$

Hence, (N, v^L) is not average convex.

The example above shows that we can find cycle-complete graphs for which average convexity of a game need not be inherited by the associated graph-restricted game. The graph in the example above contains a cycle. Hence, we wonder whether cycle-freeness might be sufficient to guarantee inheritance of average convexity. The following example shows cycle-freeness is not sufficient either.

Example 3.2 Consider the following communication situation: $N = \{1,2,3,4\}$, $L = \{\{1,2\},\{2,3\},\{3,4\}\}$ (see figure 2) and v such that

$$v(S) = \begin{cases} 0 & , |S| = 1 \text{ or } S \in \{\{1,3\},\{1,4\},\{2,3\}\} \\ 8 & , S \in \{\{1,2\},\{2,4\},\{3,4\},\{1,2,3\},\{1,3,4\}\} \\ 14 & , S \in \{\{1,2,4\},\{2,3,4\}\} \\ 19 & , S = N \end{cases} .$$

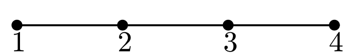

Figure 2: The graph (N, L).

It can be checked that (N, v) is average convex. However, since

$$v(\{1,2,3\}) - v(\{2,3\}) = 8 > 5 = v(N) - v(\{2,3,4\})$$

the game (N, v) is not convex.

The characteristic function of the graph-restricted game is given by

$$v^L(S) = \begin{cases} 0 & , |S| = 1 \text{ or } S \in \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\} \\ 8 & , S \in \{\{1,2\},\{3,4\},\{1,2,3\},\{1,3,4\},\{1,2,4\}\} \\ 14 & , S \in \{\{2,3,4\}\} \\ 19 & , S = N \end{cases}.$$

With $T_1 = \{2,3,4\}$ and $T_2 = N$, we find

$$\begin{aligned} \sum_{i \in T_1} \left[v^L(T_1) - v^L(T_1 \backslash \{i\})\right] &= 6 + 14 + 14 = 34 \\ &> 33 = 11 + 11 + 11 = \sum_{i \in T_1} \left[v^L(T_2) - v^L(T_2 \backslash \{i\})\right]. \end{aligned}$$

Hence, (N, v^L) is not average convex.

The examples above indicate that the set of graphs that ensure that for an average convex game the corresponding graph-restricted game is also average convex is restricted. However, the following set of graphs ensure inheritance of average convexity.

Definition 3.2 A graph (N, L) is a *star graph* if there exists $i \in N$ such that $L = \{\{i,j\} \mid j \in N \backslash \{i\}\}$.

In a star graph there is one central player who is directly connected with all other players and there are no links in which this player is not involved. The following theorem shows that for a star graph the average convexity of a game implies average convexity of the corresponding graph-restricted game.

Theorem 3.2 Let (N, v, L) be a communication situation where the underlying game (N, v) is average convex and the underlying communication graph (N, L) a star graph. Then the graph-restricted game (N, v^L) is also average convex.

Proof: Without loss of generality assume that player 1 is the central player in the star graph. Let $T_1 \subseteq T_2 \subseteq N$. If $1 \notin T_1$ it is obvious that $T_1/L = \{\{j\} \mid j \in T_1\}$, and for all $i \in T_1$, $(T_1 \backslash \{i\})/L = \{\{j\} \mid j \in T_1 \backslash \{i\}\}$. If additionally, $1 \notin T_2$, we have

$$\begin{aligned} \sum_{i \in T_1} \left[v^L(T_1) - v^L(T_1 \backslash \{i\})\right] &= \sum_{i \in T_1} \left[\sum_{j \in T_1} v(\{j\}) - \sum_{j \in T_1 \backslash \{i\}} v(\{j\})\right] \\ &= \sum_{i \in T_1} v(\{i\}) \\ &= \sum_{i \in T_1} \left[\sum_{j \in T_2} v(\{j\}) - \sum_{j \in T_2 \backslash \{i\}} v(\{j\})\right] \\ &= \sum_{i \in T_1} \left[v^L(T_2) - v^L(T_2 \backslash \{i\})\right]. \end{aligned}$$

If $1 \in T_2$, we have

$$\begin{aligned}
\sum_{i\in T_1}\left[v^L(T_1)-v^L(T_1\backslash\{i\})\right] &= \sum_{i\in T_1}\left[\sum_{j\in T_1} v(\{j\})-\sum_{j\in T_1\backslash\{i\}} v(\{j\})\right] \\
&= \sum_{i\in T_1} v(\{i\}) \\
&\leq \sum_{i\in T_1}\left[v(T_2)-v(T_2\backslash\{i\})\right] \\
&= \sum_{i\in T_1}\left[v^L(T_2)-v^L(T_2\backslash\{i\})\right],
\end{aligned}$$

where, the inequality follows from superadditivity of (N, v).[4]

Now assume that $1 \in T_1$. This implies that $T_1/L = \{T_1\}$ and $(T_1\backslash\{i\})/L = \{T_1\backslash\{i\}\}$ for all $i \neq 1$, and $(T_1\backslash\{1\})/L = \{\{j\} \mid j \in T_1\backslash\{1\}\}$. Hence,

$$\begin{aligned}
&\sum_{i\in T_1}\left[v^L(T_1)-v^L(T_1\backslash\{i\})\right] \\
=&\sum_{i\in T_1}\left[v(T_1)-v^L(T_1\backslash\{i\})\right] \\
=&\sum_{i\in T_1\backslash\{1\}}\left[v(T_1)-v(T_1\backslash\{i\})\right]+v(T_1)-v^L(T_1\backslash\{1\}) \\
=&\sum_{i\in T_1}\left[v(T_1)-v(T_1\backslash\{i\})\right]+v(T_1\backslash\{1\})-\sum_{j\in T_1\backslash\{1\}} v(\{j\}). \qquad (1)
\end{aligned}$$

Analogously, we obtain

$$\sum_{i\in T_1}\left[v^L(T_2)-v^L(T_2\backslash\{i\})\right]=\sum_{i\in T_1}\left[v(T_2)-v(T_2\backslash\{i\})\right]+v(T_2\backslash\{1\})-\sum_{j\in T_2\backslash\{1\}} v(\{j\}) \qquad (2)$$

By average convexity of (N, v) we have

$$\sum_{i\in T_1}\left[v(T_1)-v(T_1\backslash\{i\})\right] \leq \sum_{i\in T_1}\left[v(T_2)-v(T_2\backslash\{i\})\right]. \qquad (3)$$

Furthermore,

$$\begin{aligned}
v(T_1\backslash\{1\})-\sum_{j\in T_1\backslash\{1\}} v(\{j\}) &= \left[v(T_1\backslash\{1\})+\sum_{j\in T_2\backslash T_1} v(\{j\})\right]-\sum_{j\in T_2\backslash\{1\}} v(\{j\}) \\
&\leq v(T_2\backslash\{1\})-\sum_{j\in T_2\backslash\{1\}} v(\{j\}), \qquad (4)
\end{aligned}$$

where the inequality follows from superadditivity of (N, v).

Combining equations (1), (2), (3), and (4) we have

$$\sum_{i\in T_1}\left[v^L(T_1)-v^L(T_1\backslash\{i\})\right] \leq \sum_{i\in T_1}\left[v^L(T_2)-v^L(T_2\backslash\{i\})\right].$$

[4] We remind the reader that average convexity implies superadditivity.

We conclude that (N, v^L) is average convex. $\square$

The main theorem of this section gives necessary and sufficient conditions on the communication graph to ensure that the graph-restricted game corresponding to an average convex game is average convex. Before we can prove this theorem we need some lemmas. The following lemma states that if a connected graph is cycle-complete, not cycle-free, and not complete, then we can find a subgraph which is similar to the graph of example 3.1.

Lemma 3.1 Let (N, L) be a connected graph that is (i) cycle-complete, (ii) not complete, and (iii) not cycle-free. Then there exist $x_1, x_2, x_3, x_4 \in N$ such that

$$L(\{x_1, x_2, x_3, x_4\}) = \Big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_2\}\Big\}.$$

Moreover, for such x_1, x_2, x_3, x_4 and for all (N, v) with $v(S) = v(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$ it holds that $v^L(S) = v^L(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$.

Proof: A set $S \subseteq N$ is called a *clique* in (N, L) if $\{i, j\} \in L$ for all $\{i, j\} \subseteq S$. A clique $S \subseteq N$ is called a *maximal clique* in (N, L) it there is no clique T with $T \supset S$.

Since (N, L) is not cycle-free it contains at least one cycle, and hence, $|N| \geq 3$. By cycle-completeness we then know that there is a clique containing at least three vertices. Let T be a maximal clique in (N, L) containing at least three vertices. Since (N, L) is not complete we have $T \subset N$. Because (N, L) is a connected graph there exist $i \in T$, $j \in N \backslash T$ with $\{i, j\} \in L$. T is a maximal clique, so there exists $k \in T$ with $\{k, j\} \notin L$. Cycle-completeness of (N, L) then implies that i must be the unique vertex in T directly connected with j. Define $x_1 = j$, $x_2 = i$, and let $x_3, x_4 \in T \backslash \{i\}$ with $x_3 \neq x_4$. Then $L(\{x_1, x_2, x_3, x_4\}) = \Big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_2\}\Big\}$.

Let (N, v) be a cooperative game with $v(S) = v(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$ and let $T \subseteq N$. Then,

$$v^L(T) = \sum_{C \in T/L} v(C) = \sum_{C \in T/L} v(C \cap \{x_1, x_2, x_3, x_4\}). \tag{5}$$

Let $x_i, x_j \in C \in T/L$ with $\{i, j\} \subseteq \{1, 2, 3, 4\}$. Then x_i and x_j are connected directly or $\{x_i, x_j\} \in \{\{x_1, x_3\}, \{x_1, x_4\}\}$. If $\{x_i, x_j\} \in \{\{x_1, x_3\}, \{x_1, x_4\}\}$ then $\{\{x_i, x_2\}, \{x_2, x_j\}\} \subseteq L$. Since $\{x_i, x_j\} \notin L$ it follows by cycle-completeness that every path between x_i and x_j is via x_2. Since x_i and x_j both belong to component C this implies that $x_2 \in C$ as well. We conclude that for all $x_i, x_j \in C \in T/L$ there exists $D \in (T \cap \{x_1, x_2, x_3, x_4\})/L$ with $x_i, x_j \in D$.

Obviously, if x_i, x_j are connected via links in $L(T \cap \{x_1, x_2, x_3, x_4\})$ then they are connected in $(T, L(T))$. Hence $\{C \cap \{x_1, x_2, x_3, x_4\} \mid C \in T/L\} = (T \cap \{x_1, x_2, x_3, x_4\})/L$, which implies

$$\sum_{C \in T/L} v(C \cap \{x_1, x_2, x_3, x_4\}) = \sum_{D \in (T \cap \{x_1, x_2, x_3, x_4\})/L} v(D) = v^L(T \cap \{x_1, x_2, x_3, x_4\}). \quad (6)$$

Combining equations (5) and (6) gives the desired result. □

The following lemma shows that if a connected graph is cycle-free, but not a star-graph, then it contains a subgraph similar to the graph in example 3.2.

Lemma 3.2 Let (N, L) be a connected graph that is (i) not a star graph and (ii) cycle-free. Then there exist $x_1, x_2, x_3, x_4 \in N$ such that

$$L(\{x_1, x_2, x_3, x_4\}) = \Big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\Big\}.$$

Moreover, for such x_1, x_2, x_3, x_4 and for all (N, v) with $v(S) = v(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$ it holds that $v^L(S) = v^L(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$.

Proof: Since (N, L) is connected, cycle-free, and not a star graph it follows immediately that $|N| \geq 4$. Furthermore, since (N, L) is cycle-free we just have to show that there exist two vertices for which the shortest path connecting them consists of three links.

Since (N, L) is connected and cycle-free (a tree) we have $|L| = |N| - 1$. If we denote the degree of i by $\rho(i) = |\{j \mid \{i, j\} \in L\}|$, then we have $\sum_{i \in N} \rho(i) = 2|N| - 2$. Since (N, L) is not a star graph but cycle-free we have for all $i \in N$ that $\rho(i) \leq |N| - 2$. Then it readily follows that there exist $i, j \in N$, $i \neq j$, with $\rho(i) \geq 2$ and $\rho(j) \geq 2$.

Since (N, L) is a tree there exists a unique path between two players, which is consequently the shortest path between them. The path between i and j consists of at least one link. Since the degree of both i and j is at least 2, we can find a vertex k directly connected to i and a vertex l directly connected to j both not on the (shortest) path between i and j. Since (N, L) is cycle-free, the (shortest) path between l and k is via i and j and hence, we found a pair of vertices with the shortest path between them consisting of at least three links. Denote this path by $(x_1, \ldots, x_m)$. Then obviously, since $m \geq 4$, $L(\{x_1, x_2, x_3, x_4\}) = \Big\{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\Big\}$.

Cycle-freeness implies that (x_1, x_2, x_3, x_4) is the unique path from player x_1 to player x_4. Let $T \subseteq N$ and $x_i, x_j \in \{x_1, x_2, x_3, x_4\} \cap T$. Obviously, there exists $C \in T/L$ with $x_i, x_j \in C$ if and only if there exists $D \in (T \cap \{x_1, x_2, x_3, x_4\})/L$ with $x_i, x_j \in D$.

Hence, $\{C \cap \{x_1, x_2, x_3, x_4\} \mid C \in T/L\} = (T \cap \{x_1, x_2, x_3, x_4\})/L$. If (N, v) satisfies $v(S) = v(S \cap \{x_1, x_2, x_3, x_4\})$ for all $S \subseteq N$, then for all $S \subseteq N$

$$\begin{aligned} v^L(T) &= \sum_{C \in T/L} v(C) = \sum_{C \in T/L} v(C \cap \{x_1, x_2, x_3, x_4\}) \\ &= \sum_{D \in (T \cap \{x_1, x_2, x_3, x_4\})/L} v(D) = v^L(T \cap \{x_1, x_2, x_3, x_4\}). \end{aligned}$$

This completes the proof. □

The following theorem deals with average convexity inheritance in case the underlying graph is connected.

Theorem 3.3 Let (N, L) be a communication graph with N the unique component. Then the following two statements are equivalent:

(i) The communication graph (N, L) is a complete graph or a star graph.

(ii) For all average convex games (N, v) the graph-restricted game (N, v^L) is average convex.

Proof:

(i)⇒(ii) Let (N, v) be an average convex game. If (N, L) is complete then $v^L = v$ and hence (N, v^L) is average convex. If (N, L) is a star graph it follows from theorem 3.2 that (N, v^L) is average convex.

(ii)⇒(i) Assume (ii) holds. Since every convex game is average convex, we have that for all convex games (N, v) the graph-restricted game is average convex. Then by corollary 3.1 we know that (N, L) is cycle-complete. Now suppose that (N, L) is not complete and not a star graph. We will show that then condition (ii) is violated. We will distinguish between two cases, (N, L) is not cycle-free and (N, L) is cycle-free.

First suppose that (N, L) is not cycle-free. From lemma 3.1 it follows that there exists $\{x_1, x_2, x_3, x_4\} \subseteq N$ with $L(\{x_1, x_2, x_3, x_4\}) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_2\}\}$. Without loss of generality assume $x_i = i$ for all $i \in \{1, 2, 3, 4\}$. Now construct the game (N, w) as follows: $w(S) = v(S \cap \{1, 2, 3, 4\})$ for all $S \subseteq N$, where v is the characteristic function of the game in example 3.1. From lemma 3.1 it follows for all $S \subseteq N$ that $w^L(S) = w^L(S \cap \{1, 2, 3, 4\}) = v^{L^1}(S \cap \{1, 2, 3, 4\})$, where L^1 denotes the graph of example 3.1. Using example 3.1 it is obvious that (N, w) is average convex but (N, w^L) is not.

Secondly suppose that (N, L) is cycle-free. From lemma 3.2 it follows that there exists $\{x_1, x_2, x_3, x_4\} \subseteq N$ with $L(\{x_1, x_2, x_3, x_4\}) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}\}$. Without loss of generality assume $x_i = i$ for all $i \in \{1, 2, 3, 4\}$. Now construct the game (N, z) as follows: $z(S) = v(S \cap \{1,2,3,4\})$ for all $S \subseteq N$, where v is the characteristic function of the game in example 3.2. From lemma 3.2 it follows for all $S \subseteq N$ that $z^L(S) = z^L(S \cap \{1,2,3,4\}) = v^{L^2}(S \cap \{1,2,3,4\})$, where L^2 denotes the graph of example 3.2. Using example 3.2 it is obvious that (N, z) is average convex but (N, z^L) is not. □

We will extend theorem 3.3 to graphs with more than one component. To do so, we need two more lemmas.

The following lemma deals with the component additivity of graph-restricted games. It states that average convexity of a graph-restricted game is equivalent to average convexity of all the subgames associated with the components of the graph. The characteristic function v^L restricted to the subcoalitions of a coalition C will be denoted by $(v^L)_{|C}$.

Lemma 3.3 Let (N, v, L) be a communication situation. The graph-restricted game (N, v^L) is average convex if and only if for all $C \in N/L$ the game $(C, (v^L)_{|C})$ is average convex.

Proof: The only-if-part follows directly from the fact that for all $C \in N/L$ and all $T \subseteq C$ we have $v^L(T) = (v^L)_{|C}(T)$.

It remains to prove the if-part. Assume that the game $(C, (v^L)_{|C})$ is average convex for all for all $C \in N/L$. Let $T_1 \subseteq T_2 \subseteq N$. Then we have,

$$\begin{aligned}
\sum_{i \in T_1} \left[v^L(T_1) - v^L(T_1 \backslash \{i\}) \right] &= \sum_{i \in T_1} \sum_{C \in N/L} \left[v^L(T_1 \cap C) - v^L((T_1 \backslash \{i\}) \cap C) \right] \\
&= \sum_{C \in N/L} \sum_{i \in T_1} \left[v^L(T_1 \cap C) - v^L((T_1 \backslash \{i\}) \cap C) \right] \\
&= \sum_{C \in N/L} \sum_{i \in C \cap T_1} \left[v^L(C \cap T_1) - v^L((C \cap T_1) \backslash \{i\}) \right] \\
&\leq \sum_{C \in N/L} \sum_{i \in C \cap T_1} \left[v^L(C \cap T_2) - v^L((C \cap T_2) \backslash \{i\}) \right] \\
&= \sum_{i \in T_1} \left[v^L(T_2) - v^L(T_2 \backslash \{i\}) \right].
\end{aligned}$$

The first equality follows from the additive definition of the graph-restricted game, which implies that for all $S \subseteq N$, $v^L(S) = \sum_{C \in N/L} v^L(S \cap C)$. The third equality follows since $v^L(T_1 \cap C) - v^L((T_1 \backslash \{i\}) \cap C) = 0$ if $i \in T_1 \backslash C$. The inequality follows from the average convexity of the subgames $(C, (v^L)_{|C})$ and the notion that for all $T \subseteq C$ we have $v^L(T) = (v^L)_{|C}(T)$. The last equality follows similar to the first three equalities. □

The following lemma gives a relation between average convexity inheritance for all games with a fixed player set and average convexity inheritance for a subset of this player set.

Lemma 3.4 Let (N, L) be a communication graph that ensures that for every average convex game (N, v) the graph-restricted game (N, v^L) is average convex as well. Let $C \in N/L$. If (C, w) is average convex then $(C, w^{L(C)})$ is average convex.

Proof: Let (C, w) be an average convex game. Define the game (N, v) by $v(S) := w(S \cap C)$ for all $S \subseteq N$. Hence, all players in $N \backslash C$ are zero players. Obviously, (N, v) is average convex, which implies that (N, v^L) is average convex. Since the subgame of an average convex game is average convex and $(v^L)_{|C} = w^{L(C)}$ we conclude that $(C, w^{L(C)})$ is average convex. □

Using the lemmas above we can prove the main theorem of this section.

Theorem 3.4 Let (N, L) be a communication graph. Then the following two statements are equivalent:

(i) For all $C \in N/L$ it holds that $(C, L(C))$ is a complete graph or a star graph.

(ii) For all average convex games (N, v) the graph-restricted game (N, v^L) is average convex.

Proof: Suppose (i) holds. Let (N, v) be an average convex game. Then, since a subgame of an average convex is average convex, it holds that for all $C \in N/L$, $(C, v_{|C})$ is average convex as well. Since $(C, L(C))$ is a complete graph or a star graph for all $C \in N/L$, it follows by theorem 3.3 that for all $C \in N/L$, $(C, (v_{|C})^{L(C)})$ is average convex. Since $(v_{|C})^{L(C)} = (v^L)_{|C}$ it follows by lemma 3.3 that (N, v^L) is average convex. So, (ii) holds.

Suppose (ii) holds. Let $C \in N/L$. By lemma 3.4 we have that for all average convex (C, w) it holds that $(C, w^{L(C)})$ is average convex as well. By theorem 3.3, this implies that $(C, L(C))$ is a complete graph or a star graph and hence (i) holds.

This completes the proof. □

4 Potential games

In this section we will study inheritance of (average) convexity of the potential game associated with a specific game by the potential game associated with the graph-restricted game.

Potential games associated with cooperative games, were first introduced by *Hart* and *Mas-Colell* (1989). They define for every game (N, v) an associated potential game (N, Pv) defined by $Pv = \sum_{R \subseteq N} \frac{\lambda_R}{|R|} u_R$, where $v = \sum_{R \subseteq N} \lambda_R u_R$ is the unique linear decompostion of (N, v) into unanimity games.[5] Note that a cooperative game completely

[5] $u_R(S) = 1$ if $R \subseteq S$ and $u_R(S) = 0$ otherwise. See *Shapley* (1953).

determines its associated potential game and vice versa. For convenience we will sometimes refer to an associated potential game instead of to a potential game associated with some underlying cooperative game.

In *Marín-Solano* and *Rafels* (1996) it is shown that the associated potential game is average convex if and only if the Shapley values of the original game and all related subgames are in the corresponding cores. Furthermore, they showed that the extended Shapley value $(\Phi_i(S, v_{|S}))_{S \subseteq N,\ i \in S}$ is a population monotonic allocation scheme (PMAS) if and only if the associated potential game is convex.[6]

So, it is interesting to see whether (average) convexity of the potential game associated with a specific game is inherited by the potential game associated with the graph-restricted game. First we will focus on convexity. It follows from *Nouweland* and *Borm* (1991) that if (N, L) is cycle-complete and (N, Pv) convex, then $(N, (Pv)^L)$ is convex. The following example shows that this does not imply that $(N, P(v^L))$ is convex as well.

Example 4.1 Consider the communication situation (N, v, L) with $N = \{1, 2, 3\}$,

$$v = 2u_{\{1,2\}} + 2u_{\{1,3\}} + 2u_{\{2,3\}} - 3u_N,$$

and $L = \{\{1, 2\}, \{2, 3\}\}$. Then we have

$$Pv = u_{\{1,2\}} + u_{\{1,3\}} + u_{\{2,3\}} - u_N.$$

So, (N, Pv) is convex. Since (N, L) is cycle-complete (in fact the graph is cycle-free, or even stronger, a star graph) we wonder whether $(N, P(v^L))$ is also convex. Note that we already know that $(N, (Pv)^L)$ is convex since (N, L) is cycle-complete and (N, Pv) convex.

Some calculations result in

$$v^L = 2u_{\{1,2\}} + 2u_{\{2,3\}} - u_N.$$

Hence, we find that

$$P(v^L) = u_{\{1,2\}} + u_{\{2,3\}} - \frac{1}{3}u_N.$$

Since

$$P(v^L)(\{1, 2\}) - P(v^L)(\{2\}) = 1 > \frac{2}{3} = P(v^L)(\{1, 2, 3\}) - P(v^L)(\{2, 3\})$$

it follows that $(N, P(v^L))$ is not convex.

[6] A vector $(x_{i,S})_{S \subseteq N,\ i \in S}$ is a PMAS of (N, v) if and only if for all $S \subseteq N$ it holds that (i) $\sum_{i \in S} x_{i,S} = v(S)$ and (ii) for all T with $S \subseteq T \subseteq N$ and all $i \in S$, $x_{i,S} \leq x_{i,T}$.

This example can easily be extended to show that for all graphs with at least one non-complete component, convexity of (N, Pv) need not be inherited by $(N, P(v^L))$. So, the example above shows that besides the complete graphs we cannot find connected graphs that ensure inheritance of convexity of the associated potential game in a communication situation. Since inheritance of average convexity of a cooperative game by the graph-restricted game is ensured for a subclass of the class of graphs that ensure inheritance of convexity, one could think that it will not be possible to find an interesting class of graphs that ensures inheritance of average convexity of the associated potential game. Surprisingly, inheritance of average convexity of the associated potential game is ensured for exactly the same class of graphs for which inheritance of average convexity of the underlying game is ensured.

We will use the following result of *Marín-Solano* and *Rafels* (1996).

Theorem 4.1 A cooperative game (N, v) is an average convex game if and only if its unanimity coordinates $(\lambda_R)_{R\subseteq N}$ satisfy

$$\sum_{R\subseteq T,\ R\cap S^c\neq\emptyset} |S\cap R|\lambda_R \geq 0 \quad , \quad \text{for any } S\subseteq T\subseteq N \tag{7}$$

where $S^c = N\backslash S$.

Using this we prove the following lemma.

Lemma 4.1 Let (N, v) be an average convex and zero-normalized cooperative game with $v = \sum_{R\subseteq N} \lambda_R u_R$. Then for $i \in N$:

$$\sum_{R\subseteq N:\ i\in R,\ |R|\geq 2} \frac{\lambda_R}{|R|+1} \geq 0.$$

Proof: Define the cooperative game $(N\cup\{d\}, z)$ by adding a dummy player $d \notin N$ to the game (N, v), i.e. $z(S) = v(S\cap N)$ for all $S \subseteq N\cup\{d\}$. Average convexity of (N, v) implies average convexity of $(N\cup\{d\}, z)$.Consider the star graph $(N\cup\{d\}, L)$ with d the central player, $L = \{\{j, d\} \mid j \in N\}$. By theorem 3.3 it follows that $(N\cup\{d\}, z^L)$ is average convex.

Denote the unanimity coordinates of $(N\cup\{d\}, z^L)$ by $(\mu_R)_{R\subseteq N\cup\{d\}}$. The potential game corresponding to an average convex game is average convex.[7] Since the unanimity

[7] Average convexity implies that the Shapley values of the game and all its subgames belong to the respective cores. This holds if and only if the associated potential game is average convex.

coordinates of $(N \cup \{d\}, P(z^L))$ are $(\frac{\mu_R}{|R|})_{R \subseteq N \cup \{d\}}$ it follows by theorem 4.1 with $S = \{i\}$ and $T = N \cup \{d\}$ that

$$\sum_{R: i \in R,\ R \subseteq N \cup \{d\}, |R| \geq 2} \frac{\mu_R}{|R|} \geq 0.$$

Since $\mu_R = 0$, if $d \notin R$ or $R \in \{\{j, d\} \mid j \in N\}$ (by zero-normalization of (N, w)) and $\mu_R = \lambda_{R \backslash \{d\}}$, otherwise, we conclude $\sum_{R \subseteq N:\ i \in R,\ |R| \geq 2} \frac{\lambda_R}{|R|+1} \geq 0$. □

Using the lemma above we can prove that if the underlying communication graph is a star graph, then average convexity of the potential game associated with a cooperative game implies average convexity of the potential game associated with the graph-restricted game.

Theorem 4.2 Let (N, v, L) be a communication situation where the associated potential game (N, Pv) is average convex and the communication graph (N, L) a star graph. Then the potential game $(N, P(v^L))$ corresponding to the graph-restricted game (N, v^L) is average convex.

Proof: First recall that (N, Pw) is average convex if and only if $\Phi(w_{|T}) \in C(w_{|T})$ for all $T \subseteq N$. Hence, it suffices to show $\Phi((v^L)_{|T}) \in C((v^L)_{|T})$ for all $T \subseteq N$.

Without loss of generality assume player 1 is the central player in the star graph. Let $T \subseteq N$. We will distinguish between two cases: (i) $1 \notin T$ and (ii) $1 \in T$.

(i) $1 \notin T$. For all $S \subseteq T$, $v^L(S) = \sum_{i \in S} v(\{i\})$ and $\Phi_i((v^L)_{|T}) = v(\{i\})$ for all $i \in T$ so, $\Phi((v^L)_{|T}) \in C((v^L)_{|T})$.

(ii) $1 \in T$. We will show that $\sum_{i \in S} \Phi_i((v^L)_{|T}) \geq (v^L)_{|T}(S)$ for all $S \subseteq T$. Let $S \subseteq T$. We will distinguish between two cases again: (ii-a) $1 \notin S$ and (ii-b) $1 \in S$.

(ii-a) $1 \notin S$. Hence, $v^L(S) = \sum_{i \in S} v(\{i\})$. Let $i \in S$. Consider $R \subseteq T$ with $i \in R$. Since $\Phi(v_{|R}) \in C(v_{|R})$ it holds that $\Phi_i(v_{|R}) \geq v(\{i\})$ and $\sum_{j \in R \backslash \{i\}} \Phi_j(v_{|R}) \geq v(R \backslash \{i\})$. Hence,

$$v(R) = \sum_{j \in R} \Phi_j(v_{|R}) \geq v(R \backslash \{i\}) + v(\{i\}). \tag{8}$$

$\Phi_i((v^L)_{|T})$ is a convex combination of $\{v^L(R) - v^L(R \backslash \{i\})\}_{R \subseteq T:\ i \in R}$. Since

$$v^L(R) - v^L(R \backslash \{i\}) = \begin{cases} v(R) - v(R \backslash \{i\}) & , \text{if } 1 \in R \\ v(\{i\}) - v(\emptyset) & , \text{if } 1 \notin R \end{cases},$$

it follows that $\Phi_i((v^L)_{|T})$ is a convex combination of $\{v(R) - v(R \backslash \{i\})\}_{R \subseteq T:\ i \in R}$. Using (8) we conclude $\Phi_i((v^L)_{|T}) \geq v(\{i\})$. Hence, $\sum_{i \in S} \Phi_i((v^L)_{|T}) \geq \sum_{i \in S} v(\{i\}) = (v^L)_{|T}(S)$.

(ii-b) $1 \in S$. Since (N, Pv) is average convex, we have $\Phi(v_{|T}) \in C(T, v_{|T})$, implying $\sum_{i\in S} \Phi_i(v_{|T}) \geq v(S) = v^L(S)$. Since the Shapley value is efficient, it suffices to show that $\Phi_i((v^L)_{|T}) \leq \Phi_i(v_{|T})$ for all $i \in T\backslash S$, since in that case

$$\sum_{i\in S} \Phi_i((v^L)_{|T}) = v(T) - \sum_{i\in T\backslash S} \Phi_i((v^L)_{|T}) \geq v(T) - \sum_{i\in T\backslash S} \Phi_i(v_{|T}) = \sum_{i\in S} \Phi_i(v_{|T}).$$

So, it remains to show that $\Phi_i((v^L)_{|T}) \leq \Phi_i(v_{|T})$ for all $i \in T\backslash S$. Denote the unanimity coordinates of (N, v) by $(\lambda_R)_{R\subseteq N}$ and the unanimity coordinates of (N, v^L) by $(\mu_R)_{R\subseteq N}$. Then[8]

$$\mu_R = \begin{cases} 0 & , \text{ if } 1 \notin R \text{ and } |R| \geq 2 \\ \lambda_R & , \text{ if } |R| = 1 \text{ or } |R| = 2 \text{ and } 1 \in R \\ \lambda_R + \lambda_{R\backslash\{1\}} & , |R| \geq 3 \text{ and } 1 \in R \end{cases} .$$

Let $i \in T\backslash S$, so $i \neq 1$. Then

$$\begin{aligned}
& \Phi_i(v_{|T}) - \Phi_i((v^L)_{|T}) \\
= & \sum_{R\subseteq T:\ i\in R} \frac{\lambda_R}{|R|} - \sum_{R\subseteq T:\ i\in R} \frac{\mu_R}{|R|} \\
= & \sum_{R\subseteq T:\ i\in R} \frac{\lambda_R}{|R|} - \lambda_{\{i\}} - \frac{\lambda_{\{i,1\}}}{2} - \sum_{R\subseteq T\backslash\{1\}:\ i\in R,\ |R|\geq 2} \frac{\lambda_R + \lambda_{R\cup\{1\}}}{|R|+1} \\
= & \sum_{R\subseteq T:\ i\in R} \frac{\lambda_R}{|R|} - \lambda_{\{i\}} - \sum_{R\subseteq T:\ \{i,1\}\subseteq R} \frac{\lambda_R}{|R|} - \sum_{R\subseteq T\backslash\{1\}:\ i\in R,\ |R|\geq 2} \frac{\lambda_R}{|R|+1} \\
= & \sum_{R\subseteq T\backslash\{1\}:\ i\in R,\ |R|\geq 2} \left(\frac{\lambda_R}{|R|} - \frac{\lambda_R}{|R|+1}\right) \\
= & \sum_{R\subseteq T\backslash\{1\}:\ i\in R,\ |R|\geq 2} \frac{\lambda_R}{|R|(|R|+1)}.
\end{aligned} \tag{9}$$

Theorem 4.1 implies that a game (N, w), where $w = \sum_{R\subseteq N} \nu_R u_R$ is average convex if and only if the the zero-normalization of this game (N, w^*), where $w^* = \sum_{R\subseteq N, |R|\geq 2} \nu_R u_R$, is average convex, since the unanimity coordinates of one-person coalitions appear with coefficient 0 in condition (7). Consider the game $(T\backslash\{1\}, w)$ with $w = \sum_{R\subseteq T\backslash\{1\}, |R|\geq 2} \frac{\lambda_R}{|R|} u_R$. This game is the zero-normalization of the average convex potential game $(T\backslash\{1\}, P(v_{|T\backslash\{1\}}))$, so $(T\backslash\{1\}, w)$ is average convex as well. Applying lemma 4.1 to $(T\backslash\{1\}, w)$ implies

$$\sum_{R\subseteq T\backslash\{1\}:\ i\in R,\ |R|\geq 2} \frac{\lambda_R/|R|}{|R|+1} \geq 0.$$

Hence expression (9) is non-negative. This completes the proof. □

[8]See *Owen* (1986) for relations between unanimity coordinates of (N, v) and unanimity coordinates of (N, v^L).

Remark 4.1 Let (N, v) be a cooperative game with an average convex associated potential game. Let $S \subseteq N$ and $T \subseteq N \backslash S$. Then

$$v(S \cup T) = \sum_{i \in S \cup T} \Phi_i(v_{|S \cup T}) = \sum_{i \in S} \Phi_i(v_{|S \cup T}) + \sum_{i \in T} \Phi_i(v_{|S \cup T}) \geq v(S) + v(T),$$

where the inequality follows since $\Phi(v_{|S \cup T}) \in C(S \cup T, v_{|S \cup T})$. So, average convexity of the potential game implies superadditivity of the underlying cooperative game.

In the remainder of this section we will show that the class of graphs that guarantee inheritance of average convexity of the associated potential game coincides with the class of graphs that guarantee inheritance of average convexity of the original game by the graph-restricted game.

The following lemma shows that for every graph that is not cycle-complete we can find a game with an average convex associated potential game, while the potential game corresponding to the graph-restricted game is not average convex.

Lemma 4.2 Let (N, L) be a communication graph that is not cycle-complete. Then there exists a game (N, v) with an average convex associated potential game such that the potential game associated with the graph-restricted game (N, v^L) is not average convex.

Proof: Since (N, L) is not cycle-complete there exists a cycle $(x_1, \ldots, x_k, x_1)$ and $i, j \in \{1, \ldots, k\}$, $i < j - 1$, with $\{x_i, x_j\} \notin L$. Define $v = u_{\{x_i, x_j\}}$. Hence, $Pv = \frac{1}{2} u_{\{x_i, x_j\}}$ is average convex, Pv is even convex. Since $\{x_i, x_j\} \notin L$ it holds that $v^L(\{x_i, x_j\}) = 0$.

The graph-restricted game (N, v^L) is determined by a set $\mathcal{W}$, where $S \in \mathcal{W}$ if and only if there exists $C \in S/L : \ \{x_i, x_j\} \subseteq C$. Note that if $T \supseteq S$ and $S \in \mathcal{W}$ then $T \in \mathcal{W}$. Since $v^L(S) = 1$, if $S \in \mathcal{W}$, and $v^L(S) = 0$, otherwise, this implies that (N, v^L) is monotonic.

Let Π be the set of all orders of N. Then the Shapley value of a player is the average over all orders in Π of the marginal contribution of this player to the set of players who precede him. Note that by monotonicity every marginal contribution of a player is non-negative and hence, every player receives a non-negative payoff. Since $v^L(N) = 1$ it follows that players x_i and x_j together receive at most one according to a specific order. Since they both receive zero if these two players are first and second in an order, we find $\Phi_{x_i}(v^L) + \Phi_{x_j}(v^L) < 1$.

By non-negativity of the payoffs and the efficiency of the Shapley value we have $\sum_{l=1}^{k} \Phi_{x_l}(v^L) \leq 1$, so $\sum_{l=1}^{k} \Phi_{x_l}(v^L) + \Phi_{x_i}(v^L) + \Phi_{x_j}(v^L) < 2$. This last expression implies

$$\sum_{l=i}^{j} \Phi_{x_l}(v^L) < 1$$

or

$$\sum_{l=j}^{k} \Phi_{x_l}(v^L) + \sum_{l=1}^{i} \Phi_{x_l}(v^L) < 1.$$

Since $v^L(\{x_i, x_{i+1}, \ldots, x_j\}) = v^L(\{x_j, \ldots, x_k, x_1, \ldots, x_i\}) = 1$ we find that $\Phi(v^L) \notin C(v^L)$ and thus $(N, P(v^L))$ is not average convex. □

The following two examples are based on examples 3.1 and 3.2.

Example 4.2 Let (N, w) be the 4-person game with

$$w = 12u_{\{1,4\}} + 12u_{\{2,3\}} + 12u_{\{3,4\}} + 9u_{\{1,2,3\}} + 9u_{\{1,2,4\}} - 9u_{\{1,3,4\}} - 9u_{\{2,3,4\}} - 8u_N.$$

Then (N, Pw) coincides with the game of example 3.1. Furthermore, let $L = \{\{1,2\},\{2,3\},\{3,4\},\{4,2\}\}$. Hence (N, L) is the graph of example 3.1. Some straightforward calculations show that $(N, P(w^L))$ is not average convex.

Example 4.3 Let (N, w) be the 4-person game with

$$w = 16u_{\{1,2\}} + 16u_{\{2,4\}} + 16u_{\{3,4\}} - 6u_{\{1,2,4\}} - 6u_{\{2,3,4\}} - 4u_N.$$

Then (N, Pw) coincides with the game of example 3.2. Furthermore, let $L = \{\{1,2\},\{2,3\},\{3,4\}\}$. So, (N, L) corresponds to the graph of example 3.2. Some straightforward calculations show that $(N, P(w^L))$ is not average convex.

Examples 4.2 and 4.3 will be used to show that every cycle-complete connected graph that is not a star graph nor the complete graph does not guarantee inheritance of average convexity of the potential game by the potential game corresponding to the graph-restricted game. Finally, we need the following lemmas.

Lemma 4.3 Let (N, v, L) be a communication situation. The game $(N, P(v^L))$ is average convex if and only if for all $C \in N/L$ the game $(C, (P(v^L))_{|C})$ is average convex.

Proof: Denote the unanimity coordinates of (N, v^L) by $(\mu_R)_{R\subseteq N}$. Then

$$(P(v^L))(S) = \sum_{R\subseteq S} \frac{\mu_R}{|R|} = \sum_{C\in N/L} \sum_{R\subseteq C\cap S} \frac{\lambda_R}{|R|} = \sum_{C\in N/L} (P(v^L)_{|C})(C\cap S)$$

for all $S \subseteq N$,where the second equality holds since $\mu_R = 0$ if R is not contained in a component $C \in N/L$. Now, the proof goes along the same lines as the proof of lemma 3.3 □

Lemma 4.4 Let (N, L) be a communication graph such that for every (N, v) with average convex (N, Pv) it holds that $(N, P(v^L))$ is average convex. Let $C \in N/L$. If (C, w) has an average convex associated potential game (C, Pw) then $(C, P(w^{L(C)}))$ is average convex.

Proof: Along the same lines as the proof of lemma 3.4. □

We can now prove the following theorem.

Theorem 4.3 Let (N, L) be a communication graph. Then the following two statements are equivalent:

(i) For all $C \in N/L$ it holds that $(C, L(C))$ is a complete graph or a star graph.

(ii) For all games (N, v) with an average convex associated potential game the graph-restricted game (N, v^L) has an average convex associated potential game.

Proof: The proof goes along the same lines as the proofs of theorems 3.3 and 3.4 using lemmas 3.1, 3.2, 4.2, 4.3, 4.4, theorem 4.2, and examples 4.2 and 4.3. □

5 Remark

Borm, *Owen* and *Tijs* (1992) define the link game (L, r) corresponding to a communication situation (N, v, L) where $r(A) = v^A(N)$ for all $A \subseteq L$.[9] The *position value* $\pi(N, v, L)$ is then obtained using the Shapley value of the link game in the following way:

$$\pi_i(N, v, L) = \sum_{l \in L_i} \frac{1}{2} \Phi_l(L, r),$$

where Φ denotes the Shapley value and $L_i = \{\{i, j\} \in L \mid j \in N\}$.

Nouweland and *Borm* (1991) find that the link game (L, r) corresponding to communication situation (N, v, L) is convex if (N, v) is convex and (N, L) cycle free. The following example shows that we cannot find a similar result for average convex games.

Example 5.1 Consider the communication situation (N, v, L) with $N = \{1, 2, 3\}$, v the characteristic function with $v(\{1, 2\}) = v(\{2, 3\}) = 2$, $v(N) = 3$, $v(S) = 0$ otherwise, and $L = \{\{1, 2\}, \{2, 3\}\}$. Denote $a = \{1, 2\}$ and $b = \{2, 3\}$. Then we have $r(\{a\}) = r(\{b\}) = 2$ and $r(\{a, b\}) = 3$. So, (L, r) is not average convex, although (N, v) is average convex and (N, L) is cycle-free. Note that in fact (N, L) is a star graph. Furthermore, note that adding the link $\{1, 3\}$ to the set L will also result in a link game that is not average convex.

[9] In *Borm*, *Owen* and *Tijs* (1992) (L, r) is referred to as the *arc game*.

The example above can easily be extended to show that there exists no graph (N, L) with at least one component containing at least two links which guarantees that average convexity of (N, v) is inherited by (L, r).

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