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Envy-free Allocations of Indivisible Objects: an Algorithm and an Application

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Abstract: This paper studies envy-free allocations for economies with indivisible objects, quasi-linear utility functions, and an amount of money. We give a polynomially bounded algorithm for finding envy-free allocations. Connectedness of envy-graphs, which are used in the algorithm, characterizes the extreme points of the polytopes of sidepayments corresponding with envy-free allocations. As an application, the existence result of envy-free allocations provides a proof of the total balancedness of permutation games.

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1 Introduction

In a lot of economic situations a finite number of objects and an amount of money have to be fairly allocated to a finite set of agents. An allocation assigns to every agent a bundle consisting of some objects and some amount of money. In this paper we study the class of envy-free allocations, as introduced by Foley (1967). An allocation is envy-free if everyone likes his own bundle at least as well as that of anyone else.

A general formulation of the allocation problem for divisible objects has been described by Varian (1974). He proved the existence of fair allocations (cf. Schmeidler and Yaari (1969)), i.e. allocations that are Pareto-efficient and envy-free, under the assumption that there are no weakly Pareto-efficient allocations which all individuals regard as indifferent.

The case with indivisible objects has been addressed by Svensson (1983) and Maskin (1987). They show that if there is enough money, in a suitable sense, fair allocations exist. A more general model, one without restrictions on the number of people and objects and that allows for

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undesirable objects and negative amounts of money, has been considered by Alkan, Demange, and Gale (1991). Their proof of the existence of fair allocations is constructive and can be used to provide an exponentially bounded algorithm for finding a fair allocation in our quasi-linear model. The algorithm is based on a result that can be proved from linear programming duality or by a direct combinatorial argument.

Finally, Aragones (1995) provides a polynomially bounded algorithm that yields envy-free allocations in economies with the same number of agents as indivisible objects, a fixed amount of money, and in which every individual has a quasi-linear utility function. It is assumed that each individual consumes one of the objects and an amount of money. Aragones requires an initial, arbitrary, Pareto-efficient allocation of the objects. The Pareto-efficient allocation induces a directed, weighted graph, where nodes correspond with agents, and the weight of an arc designates the extent to which an agent envies another agent under the allocation. Then, as Aragones shows, the search for an envy-free allocation reduces to finding a path with maximal sum of envies starting from each of the nodes. The envy-free allocation that follows is also Pareto-efficient, since in her model envy-freeness implies Pareto-efficiency.

In this paper, we present another polynomially bounded algorithm that yields an envy-free allocation for the model of Aragones. Starting with an arbitrary feasible allocation we construct a directed graph with nodes that correspond with the objects, and arcs that represent indifference (weak arcs) or strict envy (strong arcs). The algorithm eliminates all of the strong arcs; consequently, an envy-free allocations results.

The elimination of all of the strong arcs is obtained by two procedures. The first procedure is applied when there is a cycle containing a strong arc. In this procedure all of the agents in the cycle are transferred one node in the direction of the cycle, which has the effect that the number of strong arcs strictly decreases. The second procedure is applied when there is no cycle with a strong arc. This procedure changes the sidepayments among the agents such that a strong arc disappears or a cycle with a strong arc appears.

Our algorithm is, just as Aragones's algorithm, polynomially bounded. A difference, however, is that Aragones needs a Pareto-efficient initial allocation, while we do not impose any condition on the initial allocation; but if we do start with a Pareto-efficient allocation of the objects, the allocation of objects is not changed by the algorithm. Aragones considers directed, weighted graphs where nodes correspond with agents, and where the weight of an arc designates the envy of agent towards another agent. By contrast, we consider directed graphs with two different kinds of arcs and with nodes that correspond with the objects.

For every Pareto-efficient allocation of the objects, the set of sidepayments that give an envy-free allocation is a polytope. We show that connectedness of the undirected envy-graphs characterizes the extreme points of these polytopes. It is an easy device to recognize and to construct extreme envy-free allocations.

As an application of the model, we give a proof of the total balancedness of the class of permutation games, as introduced by Tijs, Parthasarathy, Potters, and Rajendra Prassad (1984). Although there are already some proofs, e.g. Tijs *et al.* (1984) and Curiel and Tijs (1986), another proof is given, using the existence result of envy-free allocations in our model.

The work is organized as follows. Section 2 deals with definitions and the formal description of our model. In section 3 the algorithm is presented. Extreme envy-free allocations are considered in section 4. Finally, section 5 provides a proof of the total balancedness of permutation games.

2 Definitions and the model

The model and also most of the notation are the same as in Aragones (1995). An economy is represented by an ordered pair $E = (F, M)$, where M is a real number representing the available amount of an infinitely divisible object, which we call money. F describes the fundamentals of the economy E and is given by $F = (N, Q, U)$, where $N = \{1, \ldots, n\}$ is a finite set of agents and $Q = \{\alpha_1, \ldots, \alpha_n\}$ the set of indivisible objects, and U the *valuation matrix* which will be defined next. Each agent $i \in N$ is assumed to be endowed with a quasi-linear utility function $u_i : Q \times \mathbb{R} \to \mathbb{R}$:

$$u_i(\alpha_j, x) = u_{i\alpha_j} + x \qquad (\alpha_j \in Q, x \in \mathbb{R}),$$

where $u_{i\alpha_j}$ can be any real number. The number $u_i(\alpha_j, x)$ is interpreted as the utility that agent $i \in N$ derives when he receives an object $\alpha_j \in Q$ and an amount of money $x \in \mathbb{R}$. Now, we define the valuation matrix U by $U_{ij} := u_{i\alpha_j}$ $(i, j \in N)$ [1].

For each economy we want to distribute the objects and the money among the agents in a feasible way, that is, we want each agent i to consume exactly one object $\sigma(i)$ and a certain amount of money x_i such that the sum of money distributed equals the amount of money available: $\sum_{i=1}^n x_i = M$. In view of the quasi-linear structure of the utility functions we may rescale the economy in such a way that, without loss of generality, $M = 0$. Henceforth, an economy is given by a triple (N, Q, U). Let $\mathcal{E}$ be the collection of all economies. Denote by Π_N the class of all bijections $N \to Q$.

Definition 2.1 *A feasible allocation for the economy* $E = (N, Q, U)$ *is a pair* $(\sigma, x) \in \Pi_N \times \mathbb{R}^n$ *such that* $\sum_{i=1}^n x_i = 0$.

Let $Z(E)$ be the set of feasible allocations for the economy E. A *solution* is a correspondence $\varphi : \mathcal{E} \to Z(\mathcal{E})$ that associates with each economy E a non-empty subset $\varphi(E)$ of $Z(E)$. We are interested in so called envy-free solutions, which satisfy the following notion of equity: no agent prefers the bundle of any other agent to his own.

Definition 2.2 *(Foley (1967)) Let* $E = (N, Q, U)$ *be an economy. A feasible allocation* $(\sigma, x) \in Z(E)$ *is envy-free if*

$$u_{i\sigma(i)} + x_i \geq u_{i\sigma(j)} + x_j \qquad \textit{for all } i, j \in N.$$

Let $F(E)$ be the set of envy-free allocations of E.

Another property that is often used in the selection of normatively appealing allocations is *Pareto-efficiency*. In our quasi-linear model a feasible allocation is Pareto-efficient if and only if there is no other feasible allocation which makes all agents strictly better off. The proof of the following proposition is omitted; it can be checked easily.

[1] The results are also applicable to the situation of fewer objects than agents by introducing null objects (worthless objects).

Proposition 2.3 *Let $E = (N, Q, U)$ be an economy. A feasible allocation $(\sigma, x) \in Z(E)$ is Pareto-efficient if and only if*

$$\sum_{i \in N} u_{i\sigma(i)} \geq \sum_{i \in N} u_{i\pi(i)} \quad \textit{for all } \pi \in \Pi_N. \tag{1}$$

Let $P(E)$ be the set of Pareto-efficient allocations of E. Since the requirement is only on the distribution of objects, we shall say that $\sigma \in \Pi_N$ is Pareto-efficient if condition (1) is satisfied. The proof of the next result for a more general class of preferences can be found in Svensson (1983) and Alkan *et al.* (1991).

Proposition 2.4 *For every economy $E \in \mathcal{E}$, $\emptyset \neq F(E) \subseteq P(E)$.*

In the next section, we turn to the main result of this paper. We will describe an algorithm that yields an envy-free (and thus Pareto-efficient) allocation, starting with an arbitrary feasible allocation.

3 The algorithm

Before we can describe the algorithm, we need to introduce envy-graphs. Given an economy $E = (N, Q, U)$ and a feasible allocation $(\sigma, x) \in Z(E)$, we make a directed graph G describing the envy between the agents in the following way. Let the objects be the nodes of the graph. To each node $\alpha \in Q$ we assign the pair $(\sigma^{-1}(\alpha), x_{\sigma^{-1}(\alpha)})$, meaning that agent $\sigma^{-1}(\alpha)$ receives the bundle $(\alpha, x_{\sigma^{-1}(\alpha)})$. Now, we introduce two kinds of directed arcs. There is a directed arc from node α to node β, $\beta \neq \alpha$ if the agent corresponding with node α, agent $\sigma^{-1}(\alpha)$, strictly prefers the bundle corresponding with node β to his own bundle. That is, if

$$u_{\sigma^{-1}(\alpha)\beta} + x_{\sigma^{-1}(\beta)} > u_{\sigma^{-1}(\alpha)\alpha} + x_{\sigma^{-1}(\alpha)}.$$

This kind of arc is referred to as a *strong* arc and will be depicted in the graph as shown in figure 1.

Figure 1: a strong arc.

The other situation in which two nodes α and β are connected by a directed arc is if agent $\sigma^{-1}(\alpha)$ is indifferent between having the bundle corresponding with node β and having his own bundle. That is, if

$$u_{\sigma^{-1}(\alpha)\beta} + x_{\sigma^{-1}(\beta)} = u_{\sigma^{-1}(\alpha)\alpha} + x_{\sigma^{-1}(\alpha)}.$$

We refer to this kind of arc as a *weak* arc and we will depict it in the graph as shown in figure 2.

Figure 2: a weak arc.

The graph G is called an envy-graph. Note that different allocations may yield the same envy-graph.

Example 3.1 Consider the economy with $N = Q = \{1, \ldots, 6\}$ and valuation matrix U given by

$$U = \begin{bmatrix} 1 & 0 & 7 & 6 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 7 & 0 & 0 \\ 1 & 0 & 6 & 6 & 1 & 4 \\ 0 & 0 & 1 & 6 & 5 & 0 \\ 1 & 1 & 1 & 0 & 0 & 5 \end{bmatrix}.$$

For instance, $U_{13} = 7$ means that agent 1 derives utility 7 when he receives object 3. Figure 3 depicts the envy-graph corresponding to the feasible allocation $(\sigma, \underline{0})$, where $\sigma(i) = i$ for all $i \in N$ and $\underline{0} = (0, 0, 0, 0, 0, 0)$. $\diamond$

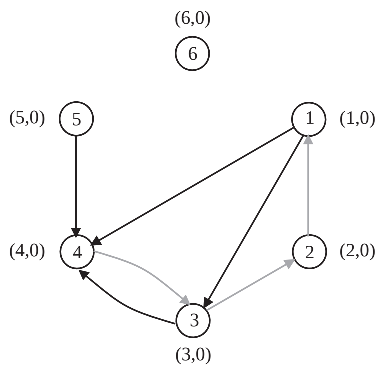

Figure 3: G, an envy-graph.

Clearly, a feasible allocation is envy-free if and only if the corresponding envy-graph has no strong arcs. Assume that we have a feasible allocation (σ, x) that is not envy-free. Let G be the envy-graph corresponding with (σ, x). We want to eliminate the strong arcs of G, and maintain feasibility at the same time. For that purpose we introduce two procedures called the *permutation procedure* and the *sidepayment procedure*.

Let us first describe the permutation procedure. This procedure is applied in case there is a strong arc in G that is contained in a directed cycle. So, suppose that this is indeed the case and let c be such a cycle. All of the agents in c are transferred one node in the direction of the cycle, while all of the other agents and all of the amounts of money are kept in place. Clearly, the new allocation that we get in this way is still feasible. What is more, lemma 3.2 tells us that the corresponding envy-graph G^* has at least one strong arc less than G.

Lemma 3.2 *Let G be an envy-graph. If G contains a cycle with at least one strong arc, then the permutation procedure applied to this cycle yields an envy-graph G^* with at least one strong arc less than G.*

Proof. First, note that arcs from nodes outside the cycle to nodes inside the cycle do not change; agents' envy depends only on the bundles, not on the particular agents whom the bundles are assigned to. Second, it is obvious that arcs between nodes outside the cycle do not change. Third, the arcs from nodes inside the cycle to nodes outside the cycle weaken. That is, they may disappear or turn from strong to weak. This follows since the players in the cycle get better bundles than in the previous situation. Finally, we consider the case of arcs between nodes inside the cycle. The transfer in the direction of the cycle leads to a weak gain for all of the agents that correspond with nodes inside the cycle. Hence, the number of agents a particular agent (strictly) envies does not increase. Note that the agent that corresponds with the starting point of the considered strong arc in G envies strictly fewer agents after the permutation procedure. We conclude that there are fewer strong arcs in the new envy-graph G^* than in G. $\square$

Example 3.3 illustrates that also the number of strong arcs from nodes inside the cycle to nodes outside the cycle may decrease strictly.

Example 3.3 We apply the permutation procedure to the cycle $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ in the graph of figure 3. This gives the graph depicted in figure 4. Two strong arcs disappear. Note that one strong arc is a strong arc from a node inside the cycle to a node outside the cycle. $\diamond$

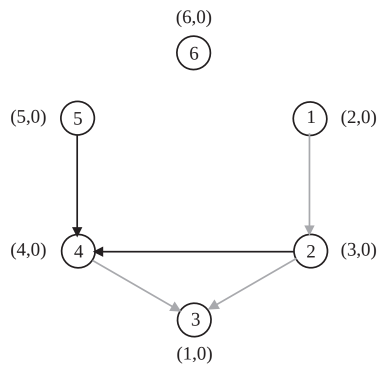

Figure 4: G^*, the result of the permutation procedure.

If the envy-graph G contains strong arcs, but none of them is contained in a cycle, then we apply the sidepayment procedure. We take a strong arc in G and try to construct a cycle which contains the strong arc. The construction is based on adding money to and subtracting money from some specific bundles. It turns out that either this can be done (without getting new strong arcs), or a strong arc is eliminated.

Let us describe the sidepayment procedure in full detail. Consider a strong arc, from, say, node α to node β. We label the nodes in the graph, and thus the bundles, as follows. Node α gets label $\oplus$, node β gets label $\ominus$. Let $\bar{Q} := Q \backslash \{\alpha, \beta\}$. A node $\gamma \in \bar{Q}$ gets label $\ominus$ if there is a directed path of weak arcs from β to γ, and it gets label $\oplus$ if there is a directed path of weak arcs from γ to α. Because there is no cycle with a strong arc, it is not possible that a node gets label $\ominus$ as well as label $\oplus$. The remaining nodes, that have neither label $\ominus$ nor label $\oplus$, get label $\odot$. Let r and s be the number of nodes having label $\ominus$ and label $\oplus$, respectively. Note that $r, s > 0$, since α and β get label $\oplus$ and $\ominus$, respectively. Every bundle with label $\ominus$ will 'pay' an amount $x > 0$ of money, and every bundle with label $\oplus$ will 'get' an amount $y > 0$ of money such that $sy = rx$. Let $\hat{x} > 0$ be the minimum value of x for which a new weak arc emerges. This minimum exists, because there are only a finite number of agents and because all of the agents have a quasi-linear utility function. Let $\hat{y} = \frac{r\hat{x}}{s}$. Clearly, the new allocation is feasible, since the sum of sidepayments equals $s\hat{y} - r\hat{x} = s\frac{r\hat{x}}{s} - r\hat{x} = 0$.

The following properties tell us what happens with the envy between the nodes. For convenience we adopt the following way of speaking. We say, for instance, that the envy of $(\ominus, \oplus)$ increases, if for every agent corresponding with a node labeled $\ominus$, we have an increasing extent of envy with respect to the bundles that are labeled $\oplus$.

Property 3.4 *The sidepayment procedure does not change the envy of* $(\oplus, \oplus)$, $(\ominus, \ominus)$, *and* $(\odot, \odot)$.

Proof. This immediately follows from the fact that equally labeled bundles undergo the same change of money. $\Box$

Property 3.5 *The sidepayment procedure reduces the envy of* $(\oplus, \odot)$, $(\oplus, \ominus)$, *and* $(\odot, \ominus)$.

Proof. This follows from the fact that $\hat{x}, \hat{y} > 0$ and the fact that $\oplus$-labeled, $\ominus$-labeled, and $\odot$-labeled bundles get a positive, negative, and a zero amount of money, respectively. $\Box$

Property 3.6 *The sidepayment procedure increases the envy of* $(\ominus, \odot)$, $(\ominus, \oplus)$, *and* $(\odot, \oplus)$. *There emerges, however, no new strong arcs between these nodes.*

Proof. The first part is clear. The second follows from the way $\hat{x}$ is chosen. $\Box$

Note that from property 3.5 it follows that there may disappear a strong arc different from the strong arc in consideration. This is illustrated in example 3.7.

Example 3.7 We apply the sidepayment procedure to the strong arc 2 → 4 in the graph of figure 4. Nodes 1 and 2 get label $\oplus$, nodes 3 and 4 get label $\ominus$ and nodes 5 and 6 get label $\odot$ (see figure 5). We find $\hat{x} = 2$. The first new weak arc that emerges is the one from node 4 to node 6. Note that the strong arc 5 → 4 disappears. The new envy-graph is depicted in figure 6. $\diamond$

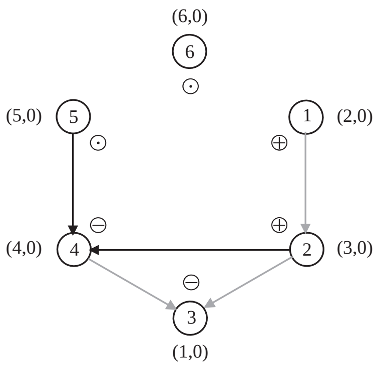

Figure 5: labeling.

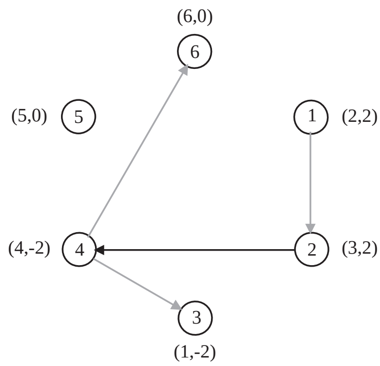

Figure 6: result of the sidepayment procedure.

From properties 3.4, 3.5, and 3.6 it follows that in the sidepayment procedure we do not get any new strong arcs.

If the sidepayment procedure results in a decrease of the number of strong arcs, then we have a new feasible allocation with at least one strong arc less. If the sidepayment procedure yields a cycle containing a strong arc, then we can apply the permutation procedure and get a decrease of the number of strong arcs as well. The following lemma states that one of the two conditions is satisfied after at most $n-1$ times of applying the sidepayment procedure to the strong arc under consideration.

Lemma 3.8 *Let G be an envy-graph. If G contains at least one strong arc and if there is no cycle in G containing a strong arc, then after at most $n-1$ times of applying the sidepayment procedure to a fixed strong arc leads either to the elimination of a strong arc, or to the emergence of a cycle containing a strong arc.*

Proof. Consider a strong arc and apply the sidepayment procedure to this strong arc. If a strong arc turns weak, or if a weak arc from a $\ominus$-labeled to a $\oplus$-labeled node emerges then we are done. So, suppose this is not the case. Then, by properties 3.4, 3.5, and 3.6, a new weak arc from a $\ominus$-labeled to a $\odot$-labeled node or from a $\odot$-labeled to a $\oplus$-labeled node emerges. This implies that the number of $\ominus$-labeled and $\oplus$-labeled nodes increases strictly. Since there are in G at most $n-2$ nodes that have label $\odot$, the lemma follows. $\square$

We have proved now that the following algorithm yields an envy-free allocation in a finite number of steps.

Algorithm 3.9 *(Envy-free allocation by elimination of strong arcs)*
Let a feasible allocation be given. Consider its envy-graph.

Step 1. If there are no strong arcs, then we have an envy-free allocation. Stop.
Otherwise go to step 2.
Step 2. If there is a cycle containing a strong arc,
then apply the permutation procedure and go to step 1.
Otherwise fix a strong arc and go to step 3.
Step 3. Apply the sidepayment procedure to the fixed arc and go to step 4.
Step 4. If a strong arc is eliminated, go to step 1.
If a cycle with a strong arc appears, go to step 2.
Otherwise go to step 3.

The complete algorithm is illustrated by figure 3 through figure 8.

Example 3.10 We apply the sidepayment procedure to the arc $2 \rightarrow 4$ in the graph of figure 6. The result is depicted in figure 7. Now, we apply the permutation procedure to the cycle $2 \rightarrow 4 \rightarrow 1 \rightarrow 2$ in the graph of figure 7. The resulting allocation, which is depicted in figure 8, consists of the allocation of objects $\sigma = (3, 2, 4, 1, 5, 6)$ and sidepayments $x = (-2.4, 2.6, -2.4, 2.6, 0, -0.4)$. This allocation is envy-free. $\diamond$

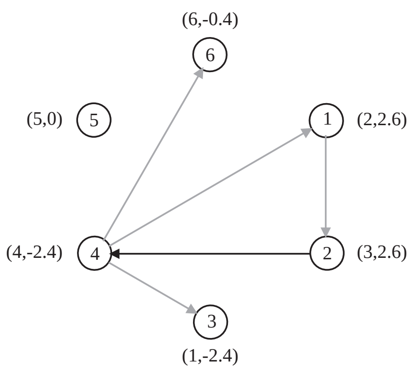

Figure 7

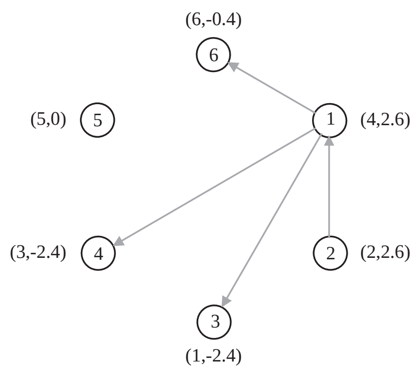

Figure 8: an envy-free allocation.

Finally, we discuss some aspects of the algorithm. To discuss the computational complexity of the algorithm, let an *action* be defined as the application of one of the two procedures. We make the following observations. First, the graph corresponding with the initial allocation has at most $2\binom{n}{2}$ strong arcs. Second, it takes at most $(n-1)$ actions to reduce the number of strong arcs by one. Hence, the total number of actions is at most $2\binom{n}{2}(n-1) = n(n-1)^2 \simeq n^3$. The computation of $\hat{x}$ in the sidepayment procedure requires $\mathcal{O}(n^2)$ operations. So, the algorithm is bounded in a polynomial way.

Once there is a Pareto-efficient allocation of objects, this particular allocation of objects does not change during the rest of the algorithm. This easily follows from proposition 2.3 and the fact that whenever an exchange of objects is carried out the sum of utilities increases strictly.

It is easy to check that the algorithm is still applicable to the model with the more general utility functions

$$u_i(\alpha_j, x) = u_{i\alpha_j} + g(x) \qquad (i \in N, \alpha_j \in Q, x \in \mathbb{R}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and satisfies

$$\lim_{x\to\infty} g(x) = \infty \quad \text{and} \quad \lim_{x\to-\infty} g(x) = -\infty.$$

The allocation of objects σ of figure 8 is the first allocation of objects in the example that is Pareto-efficient. Starting with σ and sidepayments equal to zero, the algorithm of Aragones gives the envy-free allocation (σ, y), where $y = (-2, 3, -2, 3, -1, -1)$. The vector of sidepayments y differs from the vector of sidepayments $x = (-2.4, 2.6, -2.4, 2.6, 0, -0.4)$ found by our algorithm.

4 Extreme envy-free allocations

In this section we will consider the set of envy-free allocations, and show its connection with the envy-graphs we introduced in the previous section. Every Pareto-efficient allocation of the objects gives rise to a polytope of sidepayments which generate all envy-free allocations for this allocation of the objects. We will show that a vector of sidepayments of an envy-free allocation is an extreme point of the corresponding polytope if and only if the corresponding *undirected* envy-graph is connected.

Let us first introduce some additional notation. Let $E = (N, Q, U)$ be an economy. Let $\sigma : N \to Q$ be a Pareto-efficient allocation of the objects. The set of sidepayments $x \in \mathbb{R}^n$ for which (σ, x) is an envy-free allocation is denoted by $S^\sigma \subset \mathbb{R}^n$. We give an explicit expression for the set S^σ. For $i \in \{1, \dots, n\}$, let $A_i \in M_{n\times[n(n-1)]}(\mathbb{R})$ be the matrix with columns $e^i - e^j$, $j \neq i$ (here, e^k is the k-th unit vector in $\mathbb{R}^n$). Furthermore, let e be the all-one vector of size n, i.e. $e = (1, \dots, 1)^\top \in \mathbb{R}^n$. Now we define the matrix $A \in M_{n\times[n(n-1)+2]}(\mathbb{R})$ by $A := [A_1 \cdots A_n \ e \ (-e)]$. Thus,

$$A = \begin{bmatrix} 1 & \cdots & 1 & -1 & 0 & \cdots & 0 & \cdots & 0 & 1 & -1 \\ -1 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & -1 & \cdots & 0 & \cdots & 0 & 1 & -1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & 1 & -1 \\ 0 & \cdots & -1 & 0 & 0 & \cdots & -1 & \cdots & 1 & 1 & -1 \end{bmatrix}.$$

For $i \in \{1, \dots, n\}$, let $c_i^\sigma \in \mathbb{R}^{n-1}$ be the vector with entries k_{ij}^σ, $j \neq i$, where k_{ij}^σ denotes the extent to which agent i envies $j \neq i$ under the allocation $(\sigma, \underline{0})$:

$$k_{ij}^\sigma := u_{i\alpha_{\sigma(j)}} - u_{i\alpha_{\sigma(i)}}.$$

Define $b^\sigma \in \mathbb{R}^{[n(n-1)+2]}$ by $(b^\sigma)^\top := ((c_1^\sigma)^\top, \cdots, (c_n^\sigma)^\top, 0, 0)$. Thus,

$$(b^\sigma)^\top = (k_{12}^\sigma, \dots, k_{1n}^\sigma, k_{21}^\sigma, k_{23}^\sigma, \dots, k_{n(n-1)}^\sigma, 0, 0).$$

Then,

$$S^\sigma = \{x \in \mathbb{R}^n : x^\top A \geq (b^\sigma)^\top\}.$$

Clearly, S^σ is a polytope. Let $ext(S^\sigma)$ be the set of its extreme points.

For an envy-graph G corresponding to an envy-free allocation (σ, x), we define the undirected envy-graph $\bar{G}$ to be the graph that results when we take the directions out of G. For example, the undirected envy-graph that results from figure 8 (see example 3.10) is depicted in figure 9.

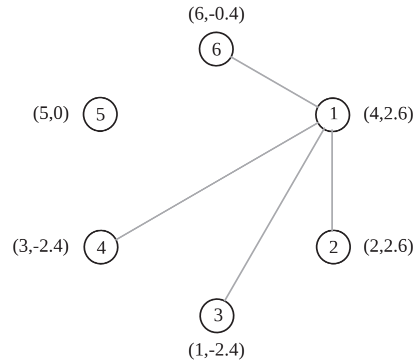

Figure 9: $\bar{G}$, an undirected envy-graph.

The following theorem shows that it is easy to ascertain whether a sidepayment vector $x \in \mathbb{R}^n$ of an envy-free allocation (σ, x) is an extreme point of the polytope S^σ. To prove the theorem, we need a lemma on the extreme points of S^σ. For $x \in S^\sigma$ define $tight(\sigma, x) := \{Ae^j : x^\top Ae^j = b^\sigma_j,\ 1 \leq j \leq n(n+1)+2\}$, the set of columns that give an equality in $x^\top A \geq (b^\sigma)^\top$, and $es.tight(\sigma, x) := tight(\sigma, x) \backslash \{e, -e\}$, the set of columns that are essentially tight (the last two columns of A yield an equality for *all* $x \in S^\sigma$). Finally, let W be the subspace of $\mathbb{R}^n$ spanned by the first $n(n-1)$ columns of A. Note that W is the orthogonal complement of $\{e\}$ in $\mathbb{R}^n$. Without proof the following lemma recalls a well-known result from linear algebra.

Lemma 4.1 *Let $E = (Q, N, U)$ be an economy. Let σ be a Pareto-efficient allocation of the objects, and let $x \in S^\sigma$. Then, the following statements are equivalent.*

$$\begin{array}{ll} (i) & x \in ext(S^\sigma) \\ (ii) & tight(\sigma, x) \text{ spans } \mathbb{R}^n \\ (iii) & es.tight(\sigma, x) \text{ spans } W \end{array}$$

We can now prove the following theorem.

Theorem 4.2 *Let $E = (Q, N, U)$ be an economy and let (σ, x) be an envy-free allocation. The sidepayment vector $x \in \mathbb{R}^n$ is an extreme point of the polytope S^σ if and only if the undirected envy-graph corresponding with (σ, x) is connected.*

Proof. Suppose that for a sidepayment vector $x \in \mathbb{R}^n$ of an envy-free allocation (σ, x) the corresponding undirected envy-graph is connected. We show that $es.tight(\sigma, x)$ spans W. Let $i, j \in N$, $i \neq j$. We are done if $e^i - e^j$ is in the subspace spanned by $es.tight(\sigma, x)$. Since there is an undirected path in $\bar{G}$ from $\sigma(i)$ to $\sigma(j)$ there are $p(1), \ldots, p(z) \in N$ such that, $e^i - e^{p(1)}, e^{p(1)} - e^{p(2)}, \ldots, e^{p(z)} - e^j$ are in the subspace spanned by $es.tight(\sigma, x)$. Then the sum of these vectors, $e^i - e^j$, is also in the subspace spanned by $es.tight(\sigma, x)$. This proves the 'if' part.
To prove the 'only if' part, let the sidepayment vector $x \in \mathbb{R}^n$ of an envy-free allocation (σ, x) be an extreme point of the polytope S^σ. We show that the corresponding undirected envy-graph $\bar{G}$ is connected. So suppose, to the contrary, that $\bar{G}$ is not connected. Then there are $i, j \in N$, $i \neq j$ for which there is no undirected path from $\sigma(i)$ to $\sigma(j)$. Now, define for every $p \in N$

$$V(p) := \{\underline{0}\} \cup \{e^k - e^l \in es.tight(\sigma, x) \ : \ k, l \in N \text{ and there is an undirected path in } \bar{G} \text{ from } \sigma(p) \text{ to } \sigma(k)\}.$$

Note that if $e^k - e^l \in V(p)$, then $\sigma(k)$ and $\sigma(l)$ are directly connected in $\bar{G}$. Furthermore, let

$$\tilde{V} := \{\underline{0}\} \cup es.tight(\sigma, x) \backslash (V(i) \cup V(j)).$$

For $p \in N$, let $W(p)$ be the linear span of $V(p)$ and let $\tilde{W}$ be the linear span of $\tilde{V}$. Since $x \in ext(S^\sigma)$, it follows from lemma 4.1 that $W = \tilde{W} + W(i) + W(j)$. Hence, there are $\tilde{w} \in \tilde{W}, w^i \in W(i)$, and $w^j \in W(j)$ such that $e^i - e^j = \tilde{w} + w^i + w^j$. Since $\tilde{w}_i = \tilde{w}_j = w^i_j = w^j_i = 0$, it follows that

$$\begin{aligned} \| e^i - w^i \| &= \sqrt{< e^i, e^i - w^i > - < w^i, e^i - w^i >} \\ &= \sqrt{< e^i, e^j + \tilde{w} + w^j > - < w^i, e^j + \tilde{w} + w^j >} = 0. \end{aligned}$$

Hence, $e^i = w^i \in W(i)$. This implies that there are real numbers λ_{kl} $(k, l \in N, (e^k - e^l) \in W(i))$ such that,

$$e^i = \sum_{\substack{(k,l) \in N^2 : \\ (e^k - e^l) \in W(i)}} \lambda_{kl}(e^k - e^l). \tag{2}$$

It follows from (2) that

$$\sum_{\substack{(k,l) \in N^2 : \\ (e^k - e^l) \in W(i)}} \lambda_{kl} - \sum_{\substack{(l,k) \in N^2 : \\ (e^l - e^k) \in W(i)}} \lambda_{lk} = \begin{cases} 0 & \text{if } k \neq i; \\ 1 & \text{if } k = i. \end{cases} \tag{3}$$

But it also holds that,

$$\sum_{k \in N} \left(\sum_{\substack{l \in N : \\ (e^k - e^l) \in W(i)}} \lambda_{kl} - \sum_{\substack{l \in N : \\ (e^l - e^k) \in W(i)}} \lambda_{lk} \right) = \sum_{\substack{(r,s) \in N^2 : \\ (e^r - e^s) \in W(i)}} \lambda_{rs} - \sum_{\substack{(r,s) \in N^2 : \\ (e^r - e^s) \in W(i)}} \lambda_{rs} = 0. \tag{4}$$

Equation (4) contradicts with equation (3). This completes the proof. $\Box$

In the sequel we will say that an envy-free allocation (σ, x) is extreme if $x \in ext(S^\sigma)$.

From theorem 4.2 we learn that the allocation (σ, y) at the end of section 3 is extreme, whereas the allocation (σ, x) from example 3.10 is not. In general neither the algorithm of section 3 nor the algorithm of Aragones generates extreme allocations. However, given an envy-free allocation (σ, x), one can find an extreme point of the corresponding set S^σ in a straigthforward manner as follows. For this, suppose that x is not an extreme point of S^σ. By theorem 4.2 the corresponding undirected envy-graph $\bar{G}$ is not connected. Let $Q' \neq Q$ be one of the largest maximally connected[2] sets of nodes in $\bar{G}$. Again, we use a kind of sidepayment procedure to get a new envy-free allocation. Every node in $Q \backslash Q'$ will 'pay' an amount $x > 0$ of money. Every node in Q' will 'get' an amount $y > 0$ of money such that $q'y = (n - q')x$, where q' denotes the number of elements in Q'. Since the undirected envy-graph is not connected, we have $q' < n$. Let $\hat{x} > 0$ be the minimum value of x for which a new weak arc emerges. This minimum exists because there are only a finite number of agents and because all of the agents have a quasi-linear utility function. Let $\hat{y} = \frac{(n-q')\hat{x}}{q'}$. Clearly, the new allocation is feasible, since the sum of sidepayments is

$$q'\hat{y} - (n - q')\hat{x} = q'\frac{(n - q')\hat{x}}{q'} - (n - q')\hat{x} = 0.$$

Furthermore, the new allocation is still envy-free, and the maximal number of nodes in a maximally connected set of nodes is increased by at least one. If the new undirected graph is not connected, we can apply this procedure once more. Since every time that we apply the procedure the maximal number of nodes in a maximally connected set of nodes is increased by at least one, it follows that after a finite number of applying the procedure we find an envy-free allocation (σ, z) where z is an extreme point of the set S^σ. Thus, we have an extension of the algorithm of section 3 that yields an envy-free allocation (σ, z) with $z \in ext(S^\sigma)$.

Example 4.3 We apply the procedure above to the envy-free allocation (σ, x) of example 3.10 (see also figure 9). We have $Q' = \{1, 2, 3, 4, 6\}$. Some calculation gives $\hat{x} = 5\hat{y} = 1\frac{1}{6}$, and the extreme envy-free allocation (σ, z) where $z = (-2\frac{1}{6}, 2\frac{5}{6}, -2\frac{1}{6}, 2\frac{5}{6}, -1\frac{1}{6}, -\frac{1}{6})$.

We conclude this section with the remark that for a Pareto-efficient allocation σ of the objects and every vector of sidepayments $x \in ext(S^\sigma)$, there is no vector of sidepayments $z \in S^\sigma$, $z \neq x$ for which the allocation (σ, z) gives the same undirected envy-graph. This follows from the connectedness of the undirected envy-graph corresponding with (σ, x) and the fact that for $z \in S^\sigma$ it holds that $\sum_{i \in N} z_i = 0$.

5 Permutation games

In this section the existence of envy-free allocations is used to prove that permutation games are totally balanced. First we recall the definition of permutation games.

[2]That is, there is no connected set $Q'' \neq Q'$ of nodes in $\bar{G}$ such that $Q' \subset Q''$.

Permutation games, introduced by Tijs, Parthasarathy, Potters, and Rajendra Prassad (1984), describe a situation in which n persons all have one job to be processed and one machine on which each job can be processed. No machine is allowed to process more than one job. Sidepayments between the players are allowed. If player i processes his job on the machine of player j the processing costs are c_{ij}. Let $N := \{1, \ldots, n\}$ be the set of players. The permutation game (N, c) with costs c_{ij} is the cooperative TU-game defined by

$$\begin{aligned} c(\emptyset) &:= 0 \quad \text{and} \\ c(S) &:= \min_{\pi_S \in \Pi_S} \sum_{i \in S} c_{i\pi_S(i)} \quad \text{for all } S \in 2^N \backslash \{\emptyset\}, \end{aligned}$$

where Π_S is the class of all S-permutations and 2^N the collection of all subsets of N.

Tijs *et al.* (1984) prove, using the Bondareva-Shapley theorem and the Birkhoff-von Neumann theorem on doubly stochastic matrices, that permutation games are totally balanced[3]. Curiel and Tijs (1986) give another proof, showing a relation between assignment games and permutation games. Here we give a proof of the total balancedness of permutation games by using the existence result of envy-free allocations in the model of section 2.

Theorem 5.1 *Permutation games are totally balanced.*

Proof. Let a cost matrix $C = \{c_{ij}\} \in M_{n\times n}(\mathbb{R})$ be given. Consider the permutation game (N, c). For every $T \in 2^N \backslash \{\emptyset\}$ the subgame (T, c_T) of (N, c) is also a permutation game. Consequently, it is sufficient to prove that (N, c) is balanced.

Let U be the valuation matrix defined by $U := -C$, and let E be the economy defined by $E := (N, N, U)$. From 2.4 it follows that there is a pair $(\sigma, x) \in F(E)$. Define

$$y_i := -u_{i\sigma(i)} - x_{\sigma(i)} + x_i \quad \text{for all } i \in N.$$

Then $y := (y_i)_{i\in N}$ is a core element. This can be seen rather easily. For the grand coalition N we have

$$\sum_{i\in N} y_i = \sum_{i\in N} (-u_{i\sigma(i)} - x_{\sigma(i)} + x_i) = \sum_{i\in S} u_{i\sigma(i)} = \min_{\pi_N \in \Pi_N} \sum_{i\in N} -u_{i\pi_N(i)} = c(N),$$

where the last but one equality follows from the Pareto-efficiency of (σ, x). For $S \in 2^N \backslash \{\emptyset\}$, with $\tau_S \in \Pi_S$ such that

$$c(S) = \min_{\pi_S \in \Pi_S} \sum_{i\in S} -u_{i\pi_S(i)} = \sum_{i\in S} -u_{i\tau_S(i)},$$

it holds that

$$\sum_{i\in S} y_i = \sum_{i\in S} (-u_{i\sigma(i)} - x_{\sigma(i)} + x_i) = \sum_{i\in S} (-u_{i\sigma(i)} - x_{\sigma(i)} + x_{\tau_S(i)}) \leq \sum_{i\in S} -u_{i\tau_S(i)} = c(S),$$

where the inequality follows from the envy-freeness of (σ, x). This proves the theorem. □

[3] A cooperative TU-game (N,c) with costs c is totally balanced if every sub-game of (N, c) has a non-empty core, i.e. for every $S \in 2^N \backslash \emptyset$ there is a vector $x_S \in \mathbb{R}^S$ such that $\sum_{i\in T} x_{Si} \leq c(T)$ for all $T \in 2^S \backslash \emptyset$ and $\sum_{i\in S} x_{Si} = c(S)$.

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