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Abstract

In this paper, we present a model for an exchange economy which is an extension of the classical model as introduced by Arrow and Debreu. In the classical model, there is a finite number of commodities and a finite number of consumers. The commodities are treated separately, and so a commodity bundle is an element of the positive orthant of the Euclidean space $\mathbb{R}^l$, where l is the number of commodities. A closer look at Arrow and Debreu's model shows that this Euclidean structure is used only indirectly.

Instead of using the Euclidean structure, we allow for just the existence of commodity bundles, and do not take into consideration individual commodities. More specifically, we model the set of all possible commodity bundles in the exchange economy under consideration, by a pointed convex cone in a finite-dimensional vector space. This vector space is used only to define the suitable topological concepts in the cone, and therefore is not part of the model.

Since we do not consider separate commodities, we do not introduce prices of individual commodities. Instead, we consider price systems, which attach a positive value to every commodity bundle. These price systems are modelled by the linear functionals on the vector space that are positive on the cone of commodity bundles. The set of price systems is a cone with similar properties as the commodity cone. More precisely, the price cone is the polar cone of the commodity cone.

The commodity cone introduces a partial ordering on the commodity bundles and the price systems are compatible with this ordering. If we take the positive orthant of the Euclidean space $\mathbb{R}^l$ as the pointed convex cone then the partial ordering coincides with the Euclidean order relation on $\mathbb{R}^l$ taken in the classical approach.

In this setting, given a finite number of consumers each with an initial endowment and a preference relation on the commodity cone, we prove existence of a Walrasian equilibrium under assumptions which are essentially the same as the ones in Arrow and Debreu's model. We introduce the new concept of equilibrium function on the price system cone; zeroes of an equilibrium function correspond with equilibrium price systems. So proving existence of a Walrasian equilibrium comes down to constructing an equilibrium function with zeroes.

Contents

1 Introduction

1.1 Commodity bundles, price systems

We start with the classical description of an exchange economy in which a finite number of commodities are available. In this economy, a commodity bundle is composed of these commodities only, where each commodity is present in a certain amount. In the classical model of an exchange economy with k commodities, every commodity bundle is represented by a k-tuple of non-negative numbers $(\alpha_1, \dots \alpha_k) \in (\mathbb{R}^+)^k$. In this representation, α_i denotes the quantity of units of commodity i where $i \in \{1, \dots, k\}$. Each of the bundles $e_1 = (1, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_k = (0, \dots, 0, 1)$ represents one unit of a particular commodity and the bundles together form the natural basis to describe any commodity bundle. More precisely, a commodity bundle x is described uniquely as $x = \sum_{j=1}^{k} \alpha_i e_j$ and the collection of commodity bundles can be seen as the positive orthant of the vector space $\mathbb{R}^k$ with $\{e_1, \dots, e_k\}$ as its natural basis. The set $(\mathbb{R}^+)^k$ of all commodity bundles is called the commodity set and in this set, commodity bundles can be added and be multiplied with a positive scalar, using the addition and scalar multiplication defined on $\mathbb{R}^k$.

Commodity bundles are ordered in a natural way. The bundle x is at least as large as the bundle y if x can be split into two commodity bundles, one of which equals y, in other words, if $x - y$ is also a commodity bundle. This corresponds with the natural Euclidean order relation $\geq_E$ on $\mathbb{R}^k$ defined by

$$(\alpha_1, \dots, \alpha_k) \geq_E (\beta_1, \dots, \beta_k) \text{ if } \alpha_j \geq \beta_j \text{ for all } j \in \{1, \dots, k\}.$$

The vector space $\mathbb{R}^k$ with the Euclidean order relation is a partially ordered vector space, i.e., $\geq_E$ is reflexive, transitive, anti-symmetric, and both translation- and scaling-invariant on $\mathbb{R}^k$.

The main goal of this paper is to present a model of a pure exchange economy in which we assume just the appearence of commodity bundles, and to prove existence of an equilibrium in this setting. So we leave the classical idea, as described above, that commodities occur separately. We represent the collection of all commodity bundles by a subset C of some finite-dimensional vector space V, where C can be thought of as replacing $(\mathbb{R}^+)^k$ and V as replacing $\mathbb{R}^k$. Since, in this more general vector space model, we still want to be able to add commodity bundles and multiply any commodity bundle by a non-negative scalar, the commodity set C has to be closed with respect to these operations, i.e.,

$$\begin{cases} \forall\, x, y \in C : x + y \in C \\ \forall\, x \in C\ \forall\, \alpha \in \mathbb{R}^+ : \alpha x \in C. \end{cases}$$

Furthermore, we assume that commodity bundles do not cancel out against each other, i.e.,

$$\forall\, x, y \in C : x + y = \mathbf{0} \text{ implies } x, y = \mathbf{0}.$$

Requiring these three properties means that the set C of all commodity bundles is a pointed convex cone.

For the classical situation, where $C = (\mathbb{R}^+)^k$, the Euclidean order relation $\geq_E$ on $\mathbb{R}^k$ can also be described in the following way:

$$x \geq_E y \text{ if and only if } x - y \in (\mathbb{R}^+)^k.$$

So, a way to generalise the Euclidean order relation to fit our more general pure exchange economy model is by imposing the order relation $\geq_C$ on V where $\geq_C$ is defined by

$$x \geq_C y \text{ if and only if } x - y \in C.$$

In the appendix, we present an axiomatic introduction to pointed convex cones without using the concept of vector space. We show that each pointed convex cone induces an ordered vector space in which the cone is a total set. A pointed convex cone is called finite-dimensional if the vector space induced by it, has a finite dimension. In combination with Section 2, we will conclude that the concept of ordered vector space and the concept of pointed convex cone are interchangable. So, the essential feature in our model of an exchange economy is the use of a pointed convex cone representing the set of all commodity bundles that are present in the economy. From this cone, we obtain a vector space which yields the proper topological means to prove the existence of an equilibrium price system.

In the classical model of an exchange economy with k commodities, each price system is modelled by a vector $p = (\pi_1, \ldots, \pi_k)$, where each non-negative π_j represents the price of one unit of commodity j. At a given price system $p = (\pi_1, \ldots, \pi_k)$, the value of a commodity bundle $x = (\alpha_1, \ldots, \alpha_k)$ equals $p \cdot x = \sum_{j=1}^{k} \pi_j \alpha_j$. Regarded in this way, a price system p is a non-negative linear functional on the set of commodity bundles, since it attaches a non-negative value to every $x \in (\mathbb{R}^+)^k$. In our model, we do not have separate commodities and so we cannot speak of the price of a commodity, but only of the value of a commodity bundle. We model price systems by continuous, additive functions that map the commodity bundle cone C into $\mathbb{R}^+$, where $f(x)$ is the value of the commodity bundle x at price system f.
Recall that a function $f : C \to \mathbb{R}^+$ is called additive if for all $x, y \in C$ and for all $\alpha \geq 0$:

$$f(x + y) = f(x) + f(y) \text{ and } f(\alpha x) = \alpha f(x).$$

These functions extend to linear functionals on the corresponding vector space V and establish in this way a pointed convex cone C°, being the polar cone of C. Indeed, let V^* denote the algebraic dual of V, i.e., the vector space of all linear functionals on V. As usual we identify V and its bidual V^{**}, i.e., we identify $x \in V$ with its action $f(x)$ on every $f \in V^*$. To show this duality to full advantage, instead of $f(x)$, we write $[x, f]$ for every $f \in V^*$ and $x \in V$. With this notation $C^\circ = \{f \in V^* \mid \forall\, x \in C : [x, f] \geq 0\}$.

1.2 Consumers

Consider a model of an exchange economy with a commodity set described by a pointed convex cone C and with corresponding price system set C°. The features of a consumer are a commodity bundle w, called initial endowment, and a preference relation $\succeq$ defined on C, on the basis of which the consumer is supposed to make choices. Here, $x \succeq y$ means that the consumer considers the commodity bundle x to be at least as preferable as bundle y. By $x \succ y$, we mean $x \succeq y$ and $\neg(y \succeq x)$.
The preference relation $\succeq$ on C satisfies reflexivity ($x \succeq x$), transitivity (($x \succeq y$ and $y \succeq z$) imply $x \succeq z$) and completeness ($x \succeq y$ or $y \succeq x$). At a price system $p \in C^\circ$, a consumer can determine his income, the value $[w, p]$ of his initial endowment, and therewith his budget set $\mathcal{B}(p) := \{x \in C \mid$

$[x, p] \leq [w, p]\}$ consisting of all bundles that can be afforded by his income at price system p. Thus, the budget correspondence $\mathcal{B} : C^\circ \to 2^C$ is defined by $\mathcal{B}(p) = \{x \in C \mid [x, p] \leq [w, p]\}$ for all $p \in C^\circ$. Given a preference relation $\succeq$ and a price system $p \in C^\circ$ the demand set of the consumer is the collection of all elements of $\mathcal{B}(p)$ which are maximal with respect to the preference relation $\succeq$. Strict convexity of the preference relation guarantees that the demand set consists of at most one element. If, in addition, the preference relation is monotone with respect to the order relation $\geq_C$, then we can define a function $\mathcal{X}$ on a subdomain $\mathrm{Dom}(\mathcal{X})$ of C°, for which the demand set consists of precisely one element $\mathcal{X}(p)$. The function $\mathcal{X} : \mathrm{Dom}(\mathcal{X}) \to C$ thus defined is called the demand function and satisfies $[\mathcal{X}(p), p] = [w, p]$ for all $p \in \mathrm{Dom}(\mathcal{X})$. In Section 4 we shall prove that if the linear span of C equals V, $\mathrm{Dom}(\mathcal{X})$ equals the set of internal points of C°.

1.3 Equilibrium

Consider an exchange economy in which m consumers participate, with initial endowment w_i and preference relation $\succeq_i$ for every consumer $i \in \{1, \ldots, m\}$. Let $\mathcal{X}_i : \mathrm{Dom}(\mathcal{X}_i) \to C$ be the demand function of consumer i, where $i \in \{1, \ldots, m\}$. Suppose $\bigcap_{i=1}^m \mathrm{Dom}(\mathcal{X}_i) \neq \emptyset$, and define the total initial endowment w_{total} by $w_{\text{total}} := \sum_{i=1}^m w_i$. For every $p \in \bigcap_{i=1}^m \mathrm{Dom}(\mathcal{X}_i)$, the total demand $\mathcal{X}_{\text{total}}(p)$ at price system p is defined by

$$\mathcal{X}_{\text{total}}(p) := \sum_{i=1}^m \mathcal{X}_i(p).$$

So, in accordance with the foregoing, if the preference relations are strictly convex and monotone, $\mathcal{X}_{\text{total}}$ is a function on $\mathrm{Dom}(\mathcal{X}_{\text{total}}) = \bigcap_{i=1}^m \mathrm{Dom}(\mathcal{X}_i)$. The function $\mathcal{X}_{\text{total}}$ with domain $\mathrm{Dom}(\mathcal{X}_{\text{total}})$, is said to satisfy Walras' Law if

$$[\mathcal{X}_{\text{total}}(p), p] = [w_{\text{total}}, p] \text{ for all } p \in \mathrm{Dom}(\mathcal{X}_{\text{total}}).$$

Preference relations which are stricly convex and monotone guarantee that $\mathcal{X}_{\text{total}}$ satisfies Walras' Law. The main question in an exchange economy is the existence of a so called equilibrium price system, being a price system $p_{\text{eq}} \neq \mathbf{0}$ in $\mathrm{Dom}(\mathcal{X}_{\text{total}})$ such that $\mathcal{X}_{\text{total}}(p_{\text{eq}}) = w_{\text{total}}$.
In the following definition we introduce our concept of equilibrium function, on the basis of which, in Section 4, we prove existence of an equilibrium price system.

Definition of an equilibrium function
Let C° *be the price system cone,* $\mathcal{X}_{\text{total}}$ *the total demand function defined on* $Dom(\mathcal{X}_{\text{total}})$*, and* w_{total} *the total initial endowment for an economy with commodity cone* C*. A function* $\mathcal{E} : Dom(\mathcal{X}_{\text{total}}) \to C^\circ$ *is an equilibrium function if for every* $p \in Dom(\mathcal{X}_{\text{total}}) \backslash \{0\}$ *:*

$$\mathcal{E}(p) = 0 \textit{ if and only if } \mathcal{X}_{\text{total}}(p) = w_{\text{total}}.$$

The problem of proving existence of an equilibrium price, which we shall tackle in Section 4, can now be replaced by the problem of finding an equilibrium function with zeroes in $\text{Dom}(\mathcal{X}_{\text{total}})$.

So the main theorem of this paper is:

Existence theorem
Let there be given a model of an exchange economy with the following structure: the commodity bundle set is modelled by a finite dimensional pointed convex cone C *with corresponding order relation* $\geq_C$ *on the vector space* $V = span(C)$*; the price system set is modelled by the polar cone of* C*,* $C^\circ = \{p \in V^* \mid \forall\ x \in C : [x, p] \geq 0\}$*;* m *consumers participate in this exchange economy, each characterised by an initial endowment* $w_i \in C$ *and a continuous, monotone, strictly convex preference relation* $\succeq_i$ *on* C*,* $i \in \{1, \ldots, m\}$*. If* $C^{\circ\circ} = C$ *and if the total endowment is an interior point of* C*, then this model of an exchange economy admits an equilibrium price system.*

2 Pointed convex cones

2.1 Definition and properties

Although the use of a vector space is not an essential feature in the model of a pure exchange economy, we assume from the beginning that cones are subsets of a real vector space V. As mentioned in Section 1, an axiomatic introduction to pointed convex cones, without the concept of vector space, is presented in the appendix, at the end of this report. It turns out that a pointed convex cone can always be regarded as a total subset of a real vector space V.

Definition 2.1.1 *Let S be a subset of V. Let $n \in I\!N$, $x_1, \ldots, x_n \in S$, and $\tau_1, \ldots, \tau_n \in [0,1]$ such that $\sum_{i=1}^{n} \tau_i = 1$. Then $\sum_{i=1}^{n} \tau_i x_i$ is called a convex combination of $x_1, \ldots, x_n$. The set of all convex combinations of elements of S is the convex hull of S, denoted by $co(S)$. A set $K \subset V$ is convex if $K = co(K)$.*

Notice that the convex hull $\mathrm{co}(S)$ of S is the intersection of all convex sets that contain S.

Definition 2.1.2 *A convex subset $K \subset V$ is solid if its linear span, denoted by $span(K)$, equals V.*

Definition 2.1.3 *A non-empty subset D of V is called a cone if $\forall\ x \in D\ \forall\ \alpha \geq 0 : \alpha x \in D$. A cone D is pointed if $x \in D$ and $-x \in D$ imply $x = \mathbf{0}$.*

Note that the intersection of an arbitrary number of cones in V, is a cone in V.
A pointed convex cone D is said to be finite-dimensional if the dimension of $\mathrm{span}(D)$ is finite.

Definition 2.1.4 *A cone D_1 in V is a subcone of D if $D_1 \subset D$. For $x \in D$, $x \neq \mathbf{0}$, the pointed convex subcone $R = \{\alpha x \mid \alpha \geq 0\}$ of D is called the ray of D, generated by x.*

Proposition 2.1.5 *A subset D_1 of a pointed convex cone D is a pointed convex subcone of D if and only if D_1 is closed under addition and scalar multiplication over $\mathbb{R}^+$.*

Definition 2.1.6 *For any subset S of V the convex cone span of S, denoted by ccs(S), is the intersection of all convex cones of V, that contain S. An element of ccs(S) is called a cone combination of S.*

Note that by definition $\text{ccs}(\emptyset) = \{\mathbf{0}\}$.

Proposition 2.1.7 *Let $S \subset V$, then*

$$ccs(S) = \{x \in V \mid \exists\, n \in \mathbb{N}\ \exists\, x_1, \ldots, x_n \in S\ \exists\, \gamma_1, \ldots, \gamma_n \in \mathbb{R}^+ : x = \sum_{i=1}^{n} \gamma_i x_i\}.$$

For every $x \in D$ the set $\text{ccs}(\{x\}) = \{\alpha x \mid \alpha \geq 0\}$ is called the ray generated by x.

Definition 2.1.8 *A subset S of V is cone dependent if there is an $x \in S$ such that $x \in ccs(S \setminus \{x\})$. A set S is cone independent if S is not cone dependent.*

Definition 2.1.9 *A cone basis for a convex cone D is a cone independent set $B \subset D$ such that $ccs(B) = D$. A cone is called finitely generated if it has a finite cone basis.*

Example 1
Consider the following pointed convex cone in $\mathbb{R}^3$:

$$\begin{aligned} D_1 \ := \ & \{x \in R^3 \mid \exists\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \geq 0 : \\ & x = (\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4, \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)\}. \end{aligned}$$

This cone is finitely generated by the vectors $(1,1,1), (1,-1,1), (-1,1,1)$, and $(-1,-1,1)$.

Example 2
Define the pointed convex cone D_2 in $\mathbb{R}^3$ by:

$$D_2 := \{(\alpha_1, \alpha_2, \alpha_3) \in R^3 \mid (\alpha_1)^2 + (\alpha_2)^2 \leq (\alpha_3)^2 \text{ and } \alpha_3 \geq 0\}.$$

This cone is not finitely generated, but, as will become clear from the proof of Theorem 3.2.15, a cone basis for D_2 is $\{(\alpha_1, \alpha_2, \alpha_3) \in R^3 \mid (\alpha_1)^2 + (\alpha_2)^2 = 1 \text{ and } \alpha_3 = 1\}$.

Lemma 2.1.10 *Let B be a cone basis for a pointed convex cone D, let $b_0 \in B$, and let $\alpha : B \to \mathbb{R}^+$ be a non-negative function on B for which the set $\{b \in B \mid \alpha(b) > 0\}$ is finite. If $b_0 = \sum_{b \in B} \alpha(b) b$ then*

$$\alpha(b) = \begin{cases} 0 & \text{if } b \neq b_0 \\ 1 & \text{if } b = b_0. \end{cases}$$

Proof
Suppose $\alpha(b_0) > 1$ then $(1 - \alpha(b_0))b_0 = \sum_{b \in B \setminus \{b_0\}} \alpha(b) b \in D$ and $(\alpha(b_0) - 1)b_0 \in D$ because $\alpha(b_0) - 1 > 0$. The cone D is pointed so $(1 - \alpha(b_0))b_0 = \mathbf{0}$ which is impossible because $\alpha(b_0) \neq 1$ and $b_0 \neq \mathbf{0}$.
Suppose $\alpha(b_0) < 1$, then $b_0 = \sum_{b \in B \setminus \{b_0\}} \frac{\alpha(b) b}{1 - \alpha(b_0)}$. This is impossible since B is cone independent. We have proved that $\alpha(b_0) = 1$, so we find $\mathbf{0} = b_0 - \alpha(b_0) b_0 = \sum_{b \in B \setminus \{b_0\}} \alpha(b) b$. Now $\alpha(b)$ must be equal to zero for all $b \in B \setminus \{b_0\}$ because D is pointed. □

Note that a cone basis for a pointed convex cone and a linear basis for a vector space are two quite different concepts. For $\{e_1, \ldots, e_n\}$ being a linear basis for a vector space V, each $x \in V$ can be uniquely written as $x = \sum_{i=1}^{n} \gamma_i e_i$ with $\gamma_i \in \mathbb{R}$. However, if B is a cone basis for a pointed convex cone D each $x \in D$ can be written as $x = \sum_{b \in B} \alpha(b) b$ with $\alpha : B \to \mathbb{R}^+$ a non-negative mapping on B for which the set $\{b \in B \mid \alpha(b) > 0\}$ is finite. This representation is, in general, not unique. In fact, we have the following lemma.

Lemma 2.1.11 *If the number of elements of a cone basis B of some pointed convex cone D is higher than the dimension of span(D), then there exists an element in D which can be described as a positive combination of elements of B in at least two different ways.*

Proof
Let D be a pointed convex cone and let W be the subspace spanned by D. Let $\dim(W) = n$ and assume B is a cone basis of D with more than n elements. The set B contains a linear basis $\{b_1, \ldots, b_n\}$ of W. So there is $b_{n+1} \in B$ with $b_{n+1} \notin \{b_1, \ldots, b_n\}$ which can be uniquely described as $b_{n+1} = \sum_{i=1}^{n} \alpha_i b_i$ where at least one $\alpha_i < 0$. Define $b := \sum_{i=1}^{n} b_i$ and consider the convex combination $\tau b_{n+1} + (1 - \tau) b$ with a fixed $\tau \in (0, 1)$ so small

that $\tau\alpha_i + (1-\tau) > 0$ for all $i \in \{1, \dots, m\}$. We find $\tau b_{n+1} + (1-\tau)b = \sum_{i=1}^n (1-\tau)b_i + \tau b_{n+1}$ and $\tau b_{n+1} + (1-\tau)b = \sum_{i=1}^n (\tau\alpha_i + (1-\tau))b_i$. □

2.2 Extreme sets

In this subsection, we introduce the concept of extreme rays of a pointed convex cone. We show that a cone basis, whenever it exists, generates all extreme rays.

Definition 2.2.1 *Let K be a convex subset of V. A subset E of K is extreme in K if for all $x_1, x_2 \in K$ and $\tau \in [0,1]$ satisfying $\tau x_1 + (1-\tau)x_2 \in E$ it follows that $x_1, x_2 \in E$. A point $e \in K$ for which $\{e\}$ is extreme is an extreme point of K.*

Notice that $E = \emptyset$ and $E = K$ are both extreme sets of K. From the definition of extreme set we can deduce that the intersection and the union of a collection of extreme sets of K is again an extreme set of K.

Proposition 2.2.2 *Let E be an extreme set of a convex set K_1 and let K_2 be a convex subset of K_1, then $E \cap K_2$ is an extreme set of K_2.*

Proof
Let x be an element of $E \cap K_2$ and assume $x = \tau x_1 + (1-\tau)x_2$ for certain x_1 and x_2 in K_2, and $\tau \in [0,1]$. Since E is an extreme set of K_1 both x_1 and x_2 belong to E. We conclude that $E \cap K_2$ is an extreme set of K_2. □

In the following, we investigate the properties of extreme sets of a pointed convex cone D in V.

Lemma 2.2.3 *Every extreme set E of a convex cone D is closed under multiplication by non-negative scalars, i.e. for all $x \in E$, the ray $R := \{\alpha x \mid \alpha \geq 0\}$ generated by x is a subset of E.*

Proof
Let E be an extreme set of D, let $\alpha \geq 0$, and let x be an element of E. If $x = \mathbf{0}$ there is nothing to prove, so assume $x \neq \mathbf{0}$. If $\alpha = 0$ then $\alpha x = \mathbf{0}$

and $x = \frac{1}{2}\mathbf{0} + \frac{1}{2}2x$, so $\mathbf{0}$ (and $2x$) $\in E$. Now assume $\alpha > 0$. Note that x is a convex combination of αx and $\frac{1}{\alpha}x$:

$$x = \frac{1}{1+\alpha}(\alpha x) + \frac{\alpha}{1+\alpha}(\frac{1}{\alpha}x).$$

Since E is an extreme set of the cone D, both αx and $\frac{1}{\alpha}x \in E$. □

Corollary 2.2.4 *Let D be a convex cone. Then D is pointed if and only if the zero-vector $\mathbf{0}$ is the only extreme point of D.*

Proof
If e is an extreme point of D, then $\{e\}$ is an extreme set of D. By the preceding lemma, the set $\{e\}$ has to be closed under multiplication by positive scalars. Only if $e = \mathbf{0}$, this condition is satisfied. The set $\{\mathbf{0}\}$ is indeed an extreme set of D because D is pointed and so $x + y = \mathbf{0}$ implies $x = y = \mathbf{0}$ for all $x, y \in D$. □

Definition 2.2.5 *Let D be a pointed convex cone. An extreme set E of D is an extreme ray if $E = ccs(\{x_E\})$ for some $x_E \neq \mathbf{0}$.*

Theorem 2.2.6 *Let D be a pointed convex cone. A non-empty convex extreme set E of D is a convex subcone of D.*

Proof
In the Lemma 2.2.3 we have already seen that E is closed under non-negative scalar multiplication. So, in order to prove the theorem we have only to prove that E is closed under addition. Let $x, y \in E$. The extreme set E is convex, so $\frac{1}{2}x + \frac{1}{2}y \in E$. The set E is also closed under scalar multiplication over $I\!R^+$, so $2(\frac{1}{2}x + \frac{1}{2}y) = x + y \in E$. □

We conclude this subsection by showing the relationship between a cone basis and extreme rays.

Theorem 2.2.7 *Suppose a pointed convex cone D has a cone basis B.*

1. *Let $b_0 \in B$. Then the ray generated by b_0 is an extreme ray of D;*

2. *Let x generate an extreme ray of D. Then there exists $\lambda > 0$ such that $\lambda x \in B$.*

Proof
Let $\alpha : B \to \mathbb{R}^+$ and $\beta : B \to \mathbb{R}^+$ be two non-negative functions on B with $\{b \in B \mid \alpha(b) > 0\}$ and $\{b \in B \mid \beta(b) > 0\}$ finite.

1. Let $x, y \in D$ satisfy $x = \sum_{b \in B} \alpha(b)b$ and $y = \sum_{b \in B} \beta(b)b$. Suppose $\tau x + (1-\tau)y \in \mathrm{ccs}(\{b_0\})$ for certain $\tau \in (0, 1)$. Then $b_0 = \lambda\tau x + \lambda(1-\tau)y = \sum_{b \in B} \lambda(\tau\alpha(b) + (1-\tau)\beta(b))b$ for certain $\lambda > 0$. By Lemma 2.1.10,

$$\lambda(\tau\alpha(b) + (1-\tau)\beta(b)) = \begin{cases} 0 & \text{if } b \neq b_0 \\ 1 & \text{if } b = b_0, \end{cases}$$

   so $\alpha(b) = \beta(b) = 0$ for all $b \neq b_0$. In other words $x, y \in \mathrm{ccs}(\{b_0\})$.

2. Let x be a generator of an extreme ray E satisfying $x = \sum_{b \in B} \alpha(b)b$. Since $x \neq \mathbf{0}$, we find $\sum_{b \in B} \alpha(b) > 0$. Since the vector x is a generator of E, for all $b \in B$ we have $\alpha(b) = 0$ or $b \in \mathrm{ccs}(\{x\})$. Because B is a cone basis, at most one element of B can be in $\mathrm{ccs}(\{x\})$. Since $\sum_{b \in B} \alpha(b) > 0$ there is precisely one element $b_0 \in B$ with $\alpha(b_0) > 0$, and for all $b \neq b_0$ we have $\alpha(b) = 0$. Hence $\lambda x \in B$ with $\lambda = \frac{1}{\alpha(b_0)}$.

□

Corollary 2.2.8 *Let D be a pointed convex cone and let the set of extreme rays $\{R_i \mid i \in I\}$ of D be non-empty. If in each extreme ray R_i a generator x_i is chosen, then the set $\{x_i \mid i \in I\}$ is cone independent.*

Proof
Assume there is an extreme ray generator x, satisfying $x = \sum_{i \in I} \alpha(i)x_i$, where $\alpha : I \to \mathbb{R}^+$ and $\{i \in I \mid \alpha(i) > 0\}$ is finite. Without loss of generality we may assume $\sum_{i \in I} \alpha(i) = 1$. The ray generated by x is extreme in D, so for all $i \in I$ satisfying $\alpha(i) > 0$ we find $x_i = x$. Hence, $\{x_i \mid i \in I\}$ is cone independent. □

The question whether a cone basis for some pointed convex cone D actually exists will be answered in Section 3.1, where we shall use a version of the Krein-Milman theorem in the setting of a finite-dimensional vector space (cf. [Pani93, p.191]).

3 Partially ordered vector spaces

3.1 Algebraic considerations

In this subsection we show that each pointed convex cone in a vector space V induces a partially ordered vector space.

Definition 3.1.1 *[ABB89] A partially ordered vector space is a pair $(X, \geq)$, where X is a vector space over $\mathbb{R}$ and $\geq$ is a relation on X, satisfying $\forall\, x, y, z \in X\ \forall\, \alpha \geq 0$:*

- *reflexivity: $x \geq x$,*
- *transitivity: if $x \geq y$ and $y \geq z$, then $x \geq z$,*
- *anti-symmetry: if $x \geq y$ and $y \geq x$, then $x = y$,*
- *translation-invariance: if $x \geq y$ and $z \in X$, then $x + z \geq y + z$,*
- *scaling-invariance: if $x \geq y$, then $\alpha x \geq \alpha y$.*

Lemma 3.1.2 *Let D be a cone in V. Define the order relation $\geq_D$ by*

$$x \geq_D y :\Longleftrightarrow x - y \in D,$$

then $\geq_D$ is reflexive, transitive, and anti-symmetric if and only if D is non-empty, convex, and pointed, respectively.

Proof
Suppose $\geq_D$ is reflexive, then $\forall\, x \in V : x \geq_D x$ or $\mathbf{0} = x - x \in D$. So D is non-empty.
Suppose D is non-empty, then $\mathbf{0} \in D$ because D is closed under multiplication over $\mathbb{R}^+$. Let $x \in V$, then $x \geq_D x$ because $x - x = \mathbf{0} \in D$.

Suppose $\geq_D$ is transitive. Let $x, y \in D$ and $\tau \in (0, 1)$. D is a cone so $\tau x \in D$ and $(1 - \tau)y \in D$, i.e., $\tau x \geq_D \mathbf{0}$ and $\mathbf{0} \geq_D (\tau - 1)y$. The order relation $\geq_D$ is transitive, so $\tau x \geq_D (\tau - 1)y$ and hence $\tau x + (1 - \tau)y \in D$.
Suppose D is convex and suppose $x \geq_D y$ and $y \geq_D z$ for some $x, y, z \in V$. From $x-y \in D$ and $y-z \in D$ we conclude $\frac{1}{2}(x-y)+\frac{1}{2}(y-z) = \frac{1}{2}(x-z) \in D$.

So $x \geq_D z$.

Suppose $\geq_D$ is anti-symmetric and $x \in V$ satisfies $x \in D$ and $-x \in D$. So $x \geq_D \mathbf{0}$ and $-x \geq_D \mathbf{0}$, i.e., $x = \mathbf{0}$. We conclude that D is pointed.
Suppose D is pointed and $x \geq_D y$ and $y \geq_D x$ for some $x, y \in V$. Then $x - y \in D$ and $y - x \in D$. The cone D is pointed so $x - y = \mathbf{0}$ or $x = y$. So $\geq_D$ is anti-symmetric. $\square$

Lemma 3.1.3 *Let V be a vector space and D a pointed convex cone in V, then $(V, \geq_D)$ is a partially ordered vector space. Let $(X, \geq)$ be a partially ordered vector space, then $D := \{x \in X \mid x \geq \mathbf{0}\}$ is a pointed convex cone.*

Definition 3.1.4 *An element u of a pointed convex cone D is called an order unit for D if*

$$\forall\, y \in D\ \exists\, \lambda(y), \mu(y) \geq 0 : \mu(y)u \leq_D y \leq_D \lambda(y)u.$$

Example
Let V be the vector space of all polynomials on $[0, 1]$ with degree at most equal to 5. Let $k > 5$ and let $x_i \in [0, 1]$ be given for every $i \in \{1, \ldots, k\}$. The set $D := \{p \in V \mid \forall\, i \in \{1, \ldots, k\} : p(x_i) \geq 0\}$ is a pointed convex cone in V.
The order relation $\geq_D$ on V is defined by

$$p \geq_D q :\Longleftrightarrow \forall\, i \in \{1, \ldots, k\} : p(x_i) - q(x_i) \geq 0.$$

Let $p_0 \in V$ satisfy $p_0(x_i) > 0$ for all $i \in \{1, \ldots, k\}$, and for every $p \in D$ define

$$\begin{aligned} \lambda(p) :=&\ \max_i \tfrac{p(x_i)}{p_0(x_i)} \\ \mu(p) :=&\ \min_i \tfrac{p(x_i)}{p_0(x_i)}. \end{aligned}$$

Then, for every $p \in D$, $\mu(p)p_0 \leq_D p \leq_D \lambda(p)p_0$. Hence, p_0 is an order unit for D.

Order units are used in proving the continuity of the total demand function (cf. Section 1). Existence of order units in a pointed convex cone D satisfying $\mathrm{span}(D) = V$, will be proved in Lemma 3.2.12.

To conclude this subsection on some algebraic features of partially ordered vector spaces, we remark that the model of an exchange economy which was introduced in Section 1 (and which will be described more precisely in Section 4.1) is not a special case of the model presented in [ABB89], and therefore their equilibrium theorem is, in general, not applicable.

In fact, in [ABB89] the commodity-price duality is described by a Riesz dual system $\langle E, E' \rangle$, where the Riesz space E is the commodity space and the Riesz space E' is the price space. A Riesz space (or a vector lattice) is a partially ordered vector space X, which is a lattice at the same time. That is, a partially ordered vector space $(X, \geq)$ is said to be a Riesz space whenever for every pair of vectors x and y the supremum (least upper bound) and infimum (greatest lower bound) exist in X. Using standard lattice theory notation, the supremum and infimum of the set $\{x, y\}$ will be denoted by $x \vee y$ and $x \wedge y$, respectively. Recall that in a partially ordered vector space X an element $z \in X$ is said to be the supremum of a non-empty subset A of X whenever

1. $a \leq z$ holds for each $a \in A$;

2. if $a \leq b$ holds for all $a \in A$ and some $b \in X$, then $z \leq b$.

The condition that the commodity space is a Riesz-space is rather restrictive, although satisfied by the classical situation where X equals $I\!R^n$ with the Euclidean structure.
In our model of an exchange economy, the commodity bundle set is modelled by the positive cone D of a partially ordered vector space $(V, \geq_D)$. As the following example will clarify, a partial order relation $\geq_D$ induced by a pointed convex cone D, does not necessarily induce a lattice structure.

Example
Consider the pointed convex cone D_1 defined in Example 1 of Section 2.1:

$$\begin{aligned} D_1 := \quad & \{x \in I\!R^3 \mid \exists\, \alpha_1, \ldots, \alpha_4 \geq 0 : \\ & x = \alpha_1(1,1,1) + \alpha_2(1,-1,1) + \alpha_3(-1,1,1) + \alpha_4(-1,-1,1)\}. \end{aligned}$$

The ordering corresponding with the cone D_1 is defined by:

$$x \geq_{D_1} y \Longleftrightarrow x - y \in D_1.$$

Let $x = (0, -1, 1) \in D_1$ and $y = (0, 1, 1) \in D_1$. The collection of all upper bounds of the set $\{x, y\}$ is equal to $(x + D_1) \cap (y + D_1)$. Since both $(1, 0, 2)$ and $(-1, 0, 2)$ are elements of this intersection, they are both upper bounds of $\{x, y\}$. After tedious, but elementary calculations one can show that there is no $z \in (x+D_1)\cap(y+D_1)$ such that both $z \leq_{D_1} (1, 0, 2)$ and $z \leq_{D_1} (-1, 0, 2)$, i.e., there is no least upper bound of $\{x, y\}$. We conclude that $(\mathbb{R}^3, \geq_{D_1})$ is not a vector lattice.

3.2 Topological considerations

Let V be a finite dimensional real vector space. If, for a basis $\{e_1, \ldots, e_n\}$ and for every element $x = \sum_{i=1}^n \alpha_i e_i$ in V, we define $\| x \| := \sum_{i=1}^n | \alpha_i |$, then $\| \,.\, \|$ defines a norm on V. Denote the corresponding linear topology on V by $T(V)$. It is well-known that $T(V)$ is the only topology on V which is linear. As a consequence all norms in V are equivalent, i.e., they generate the topology $T(V)$. Therefore, in the sequel we shall not specify a norm on V.

We recall some elementary notions.

Let $S \subset V$ and $x, y \in V$. The complement of S is the set $V \setminus S$.
An element x of S is an interior point of S if there is $O \in T(V)$ satisfying $x \in O$ and $O \subset S$. The set of all interior points of S is called the interior of S and is denoted by $\text{int}(S)$. Note that $\text{int}(S) \in T(V)$.
An element x of V is an adherent point of S if every $O \in T(V)$ satisfying $x \in O$ contains an element of S. The closure of S, denoted by $\text{cl}(S)$, is the set of all adherent points of S. If $S = \text{cl}(S)$, then S is closed.
The set $\text{cl}(S) \cap \text{cl}(V \setminus S)$ is called the boundary of S and is denoted by ∂S.
The set S is bounded if $\forall\, O \in T(V)\ \exists\, \lambda > 0 : S \subset \lambda O$, where $\lambda O := \{x \in V \mid \exists\, y \in O : x = \lambda y\}$. By the Heine-Borel theorem any closed and bounded set in V is compact.
For any norm $\| \,.\, \|$ in V the dual norm $\| \,.\, \|_*$ is, for every $f \in V^*$, defined by

$$\| f \|_* := \max\{|[x, f]| : \| x \| = 1\}.$$

Recall from Section 1 that for every $f \in V^*$ and $x \in V$, by $[x, f]$ we denote $f(x)$.

Identifying V and its bidual space $(V^*)^*$, i.e., identifying each $x \in V$ with its action $f \to [x, f]$ on V^*, we have $\| x \| = \| x \|_{**}$ where $\| \,.\, \|_{**}$ is, for every $x \in V$, defined by

$$\| x \|_{**} := \max\{|[x, f]| : \| f \|_* = 1\}.$$

Definition 3.2.1 *Let S be a subset of V. The polar set S° of S is given by $S^\circ = \{f \in V^* \mid \forall\, x \in S : [x, f] \geq 0\}$. The bipolar set $S^{\circ\circ}$ of a set $S \subset V$ is given by $S^{\circ\circ} = (S^\circ)^\circ = \{x \in V \mid \forall\, f \in S^\circ : [x, f] \geq 0\}$.*

Note that for every $S \subset V$, the set S° is closed in V. Since we identify V and V^{**}, the set $S^{\circ\circ}$ is closed in V.

Also, note that our definition of polar set coincides with the one in [Pani93, p.83] and not with the one in [Conw90, p.126].

Lemma 3.2.2 *For every $S \subset V$, $S \neq \{\mathbf{0}\} : S^{\circ\circ} = cl(ccs(S))$.*

Proof
Clearly, $\text{ccs}(S) \subset \{x \in V \mid \forall\; f \in S^\circ : [x, f] \geq 0\} = S^{\circ\circ}$. The set $S^{\circ\circ}$ is closed in V so $\text{cl}(\text{ccs}(S)) \subset S^{\circ\circ}$. Suppose $x \notin \text{cl}(\text{ccs}(S))$, we shall show that $x \notin S^{\circ\circ}$. Because $\text{cl}(\text{ccs}(S))$ is closed and convex, by the Hahn-Banach Theorem [Conw90, p.78], there is a functional $f_0 \in V^*$ such that $[x, f_0] < 0$ and $[y, f_0] > 0$ for all $y \in \text{cl}(\text{ccs}(S))$. This functional f_0 is an element of S° and so $x \notin S^{\circ\circ}$. □

If S is a finite set, then $S^{\circ\circ} = \text{ccs}(S)$ (cf. Appendix A.2).
By Lemma 3.2.2, a convex cone D is closed if and only if $D^{\circ\circ} = D$.

Corollary 3.2.3 *Let D be a closed convex cone in V. Then for all $x, y \in V$:*

$$x \geq_D y \text{ if and only if } \forall\; f \in D^\circ : [x, f] \geq [y, f].$$

Lemma 3.2.4 *Let D be a solid closed convex cone and let $x_0 \in D$. Then $x_0 \in int(D)$ if and only if $\forall\; f \in D^\circ \setminus \{\mathbf{0}\} : [x_0, f] > 0$.*

Proof
Suppose $x_0 \in \text{int}(D)$ and suppose there exists $f \in D^\circ$ such that $[x_0, f] = 0$. Since $x_0 \in \text{int}(D)$ there is an $O \in T(V)$ satisfying $x_0 + O \subset D$. For all $y \in O : [y, f] = [x_0 + y, f] \geq 0$, from which we conclude $f = \mathbf{0}$.
For the converse, suppose $x_0 \in \partial D$. Since D is convex, $\text{int}(D)$ is convex, so by the Weak Separation Theorem of Minkowski ([Pani93, p.60])

$$\exists\, f_0 \in V^* \; \exists\, \alpha \in I\!\!R : \begin{cases} \forall\, \lambda \geq 0 : & [\lambda x_0, f_0] \leq \alpha \\ \forall\, x \in \text{int}(D) : & [x, f_0] \geq \alpha. \end{cases}$$

On the one hand we can choose λ equal to 0, and on the other hand $\text{int}(D)$ contains a sequence of elements converging to $\mathbf{0}$. So, we find $\alpha = 0$, and as a consequence $f_0 \in D^\circ$. By choosing λ equal to 1, we find $[x_0, f_0] \leq 0$ and this is a contradiction. □

Lemma 3.2.5 *Let D be a solid closed convex cone in V. Then D is pointed if and only if D° has an interior point, i.e.,*

$$\exists\, f_0 \in D^\circ \; \forall\, x \in D \setminus \{\mathbf{0}\} : [x, f_0] > 0.$$

Proof
Since $\mathbf{0}$ is an extreme point of D, the set $D \setminus \{\mathbf{0}\}$ is convex. The intersection of $D \setminus \{\mathbf{0}\}$ and $(-D) \setminus \{\mathbf{0}\}$ is empty because D is pointed. By the Weak Separation Theorem of Minkowski ([Pani93, p.60]) there is $f_0 \in V^*$ satisfying $\forall\, x \in D \setminus \{\mathbf{0}\} : [x, f_0] > 0$. Conversely, suppose there is an $x \neq \mathbf{0}$ in D satisfying $-x \in D$. By Corollary 3.2.3, $[x, f] = 0$ for all $f \in D^\circ$, so D° does not have an interior point. □

Lemma 3.2.6 *Let D be a convex cone in V. Then D is solid if and only if int$(D) \neq \emptyset$.*

Proof
Evidently $\text{int}(D) \neq \emptyset$ implies that D is solid.
Since the linear span of D equals V, D contains a linear basis $\{e_1, \ldots, e_n\}$ of V. Define $x_0 := \sum_{i=1}^n e_i$, then x_0 is an interior point of D because for every $f \in D^\circ$, $f \neq \mathbf{0} : [x_0, f] > 0$. □

Next, we give a summary of the propositions proved above, and their dual versions.

Corollary 3.2.7 *Let D (and therefore also D°) be a convex cone. Then*

$$\begin{aligned} int(D) \neq \emptyset &\iff D \text{ is solid},\\ int(D^\circ) \neq \emptyset &\iff D^\circ \text{ is solid}. \end{aligned}$$

If, in addition, D is closed then

$$\begin{aligned} x \geq_D y &\iff \forall\, f \in D^\circ : [x, f] \geq [y, f],\\ f \geq_{D^\circ} g &\iff \forall\, x \in D : [x, f] \geq [x, g]. \end{aligned}$$

If D is closed and also solid then

$$\begin{aligned} D \text{ is pointed} &\iff int(D^\circ) \neq \emptyset,\\ x_0 \in int(D) &\iff \forall\, f \in D^\circ : [x_0, f] > 0. \end{aligned}$$

If D° is (closed and) solid then

$$\begin{aligned} D^\circ \text{ is pointed} &\iff int(D) \neq \emptyset,\\ f_0 \in int(D^\circ) &\iff \forall\, x \in D : [x, f_0] > 0. \end{aligned}$$

Theorem 3.2.8 *Let D be a closed pointed convex cone in V. For all $f \in int(D^\circ)$, the set $\{x \in D \mid [x, f] = 1\}$ is compact.*

Proof

The set $\{x \in D \mid [x, f] = 1\}$ is closed. We shall prove that it is bounded. Consider a norm $\| \, . \, \|$ in V. Suppose there is a sequence $(x_n)_{n \in I\!N}$ in $\{x \in D \mid [x, f] = 1\}$ such that $\| x_n \| \to \infty$. The vector space V being finite-dimensional, the sequence $\left(\frac{x_n}{\|x_n\|}\right)_{n \in I\!N}$ has a convergent subsequence $\left(\frac{x_{nk}}{\|x_{nk}\|}\right)_{k \in I\!N}$ with certain limit y, $\| y \| = 1$. Moreover, $[\frac{x_n}{\|x_n\|}, f] \to 0$. We conclude that $[y, f] = 0$. Since $f \in \text{int}(D^\circ)$ we find $y = \mathbf{0}$. This is a contradiction. □

Corollary 3.2.9 *Let D be a pointed convex cone in V. The set $\{f \in D^\circ \mid [x, f] = 1\}$ is compact for all $x \in int(D)$.*

Theorem 3.2.10 *Let S be a subset of a closed pointed convex cone D, let $p_0 \in int(D^\circ)$. Then S is bounded if and only if* $\sup\{s \in S \mid [s, p_0]\} < \infty$.

Proof
We only prove " $\Longleftarrow$ ". Define $\alpha := \sup\{s \in S \mid [s, p_0]\}$, then the set S is a subset of the compact set $\{x \in D \mid [x, p_0] \leq \alpha\}$ and therefore bounded. $\square$

Brouwer's Fixed Point Theorem [Conw90, p.149]
Let K be a non-empty compact convex subset of a finite-dimensional normed vector space X and let $\mathcal{F} : K \to K$ be a continuous function, then there exists $x \in K$ such that $\mathcal{F}(x) = x$, i.e., $\mathcal{F}$ has a fixed point in K.

Brouwer's Fixed Point Theorem has the following consequence for continuous functions on pointed convex cones.

Theorem 3.2.11 *Let D be a closed pointed convex cone in a finite dimensional vector space and let $\mathcal{G} : D \setminus \{\mathbf{0}\} \to D$ be a continuous function. Then there is $x \in D \setminus \{\mathbf{0}\}$ such that $\mathcal{G}(x) = \alpha x$ for some $\alpha \geq 0$. In fact, for all $f_0 \in int(D^\circ)$ there is $x \in D$ such that $\mathcal{G}(x) = [\mathcal{G}(x), f_0]x$.*

Proof
Let $f_0 \in \text{int}(D^\circ)$. The set $K := \{x \in D \mid [x, f_0] = 1\}$ is non-empty, convex and compact by Theorem 3.2.8. Define $\mathcal{F}(x) := \frac{x+\mathcal{G}(x)}{1+[\mathcal{G}(x), f_0]}$ for all $x \in K$, then $\mathcal{F} : K \to K$ is a continuous function. By the preceding theorem the function $\mathcal{F}$ has a fixed point x in K, so

$$x = \mathcal{F}(x) = \frac{x + \mathcal{G}(x)}{1 + [\mathcal{G}(x), f_0]},$$

hence $\mathcal{G}(x) = [\mathcal{G}(x), f_0]x$. $\square$

As mentioned in Section 2, we use the concept of order unit to prove that the demand functions are continuous.

Theorem 3.2.12 *Let D be a solid closed pointed convex cone in V. Then every $x_0 \in int(D)$ is an order unit for D.*

Proof
By Corollary 3.2.7, $\text{int}(D) \neq \emptyset$. Let $x_0 \in \text{int}(D)$ and define $K := \{f \in D^\circ \mid [x_0, f] = 1\}$. Then $D^\circ = \{\alpha f \mid f \in K \text{ and } \alpha \geq 0\}$, and by Corollary 3.2.9, K is compact. From Definition 3.1.4 and Corollary 3.2.3 we conclude that x_0 is an order unit if and only if

$$\forall\, x \in D\, \exists\, \lambda(x), \mu(x) \geq 0\, \forall\, f \in K : \mu(x)[x_0, f] \leq [x, f] \leq \lambda(x)[x_0, f].$$

For every $x \in D$ define

$$\begin{array}{ll} \lambda(x) & := \max\{[x, f] \mid f \in K\} \\ \mu(x) & := \min\{[x, f] \mid f \in K\}. \end{array}$$

Then $\mu(x)[x_0, f] \leq [x, f] \leq \lambda(x)[x_0, f]$ for all $f \in K$. □

Observe that from the proof it follows that $\mu(x) > 0$ if and only if $x \in \text{int}(D)$.

Corollary 3.2.13 *Let D be a solid closed pointed convex cone in V, let $x_0 \in int(D)$, and let $(x_n)_{n \in I\!N}$ be a sequence in $int(D)$, with limit x_0. Then there are sequences $(\lambda_n)_{n \in I\!N}$ and $(\mu_n)_{n \in I\!N}$ such that*

$$\mu_n x_0 \leq_D x_n \leq_D \lambda_n x_0 \text{ and } \lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \lambda_n = 1.$$

Proof
Take $\lambda_n = \lambda(x_n)$ and $\mu_n = \mu(x_n)$ as in the proof of Theorem 3.2.12. □

In Section 2.2 we introduced the concept of cone basis, and we left open the question whether a cone basis actually exists. We shall now prove that any closed pointed convex cone has a cone basis. In the appendix we shall pay attention to pointed convex cones with a finite cone basis.

Theorem 3.2.14 (Krein-Milman) *(cf. [Pani93, p.191]) Each non-empty compact convex set K in a finite-dimensional vector space is the convex hull of its set of extreme points, i.e., $K = co(ext(K))$.*

Theorem 3.2.15 *If D is a closed pointed convex cone then D has a cone basis.*

Proof
By Lemma 3.2.5, $\text{int}(D^\circ) \neq \emptyset$. Let $f_0 \in \text{int}(D^\circ)$, then by Theorem 3.2.8 the set $K := \{x \in D \mid [x, f_0] = 1\}$ is compact. The set K is also convex, so by the Krein-Milman Theorem $K = \text{co}(\text{ext}(K))$, and therefore $D = \text{ccs}(K) = \text{ccs}(\text{ext}(K))$. We shall prove that $\text{ext}(K)$ is a cone independent set.

> **Claim:** Every e in $\text{ext}(K)$ generates an extreme ray of D.
> **Proof:** Let $\mu \geq 0$. Take $\mu e = \tau x_1 + (1-\tau)x_2$ for some $x_1, x_2 \in D$ and $\tau \in (0,1)$. If $\mu = 0$ then $x_1 = x_2 = \mathbf{0}$ because $\mathbf{0}$ is an extreme point of D. So assume $\mu > 0$. If $x_1 = \mathbf{0}$ or $x_2 = \mathbf{0}$ there is nothing to prove, so we assume $x_1 \neq \mathbf{0} \neq x_2$. Now, $e = \tau\frac{x_1}{\mu} + (1-\tau)\frac{x_2}{\mu} = \tau y_1 + (1-\tau)y_2$ where $y_1, y_2 \in D$. We shall prove that y_1 and y_2 (and therefore also x_1 and x_2) are elements of the ray $\{\alpha e \mid \alpha \geq 0\}$ generated by e.
>
> Since $1 = [e, f_0] = \tau[y_1, f_0] + (1-\tau)[y_2, f_0]$, we can write $e = \tau[y_1, f_0]\frac{y_1}{[y_1, f_0]} + (1-\tau)[y_2, f_0]\frac{y_2}{[y_2, f_0]}$ which is a convex combination of $\frac{y_1}{[y_1, f_0]}$ and $\frac{y_2}{[y_2, f_0]}$, both elements of K.

Conversely, by Proposition 2.2.2, every extreme ray $\{\alpha x \mid \alpha \geq 0\}$ corresponds with an extreme point $\alpha_0 x$ of K, where $\alpha_0 \in \mathbb{R}^+$ is such that $[\alpha_0 x, f_0] = 1$. By Corollary 2.2.8 the set $\text{ext}(K)$ is cone independent. Since $K = \text{co}(\text{ext}(K))$ and $D = \{\alpha a \mid \alpha \geq 0 \text{ and } a \in K\}$, $\text{ext}(K)$ is a cone basis in D. $\square$

4 A model for a pure exchange economy

4.1 First analysis of the model

In Section 1 we suggested the following model of an exchange economy:

- The commodity bundle set is modelled by a finite-dimensional pointed convex cone C with corresponding order relation $\geq_C$ on the vector space $V = \mathrm{span}(C)$,
- The price system set is modelled by the polar cone of C,

$$C^\circ := \{p \in V^* \mid \forall\, x \in C : [x, p] \geq 0\},$$

- m consumers participate in this exchange economy, each characterised by an initial endowment $w_i \in C$ and a preference relation $\succeq_i$ on C, $i \in \{1, \ldots, m\}$.

The main theorem of this paper, stating existence of an equilibrium price system, is now presented.

Existence Theorem
Consider the model of an exchange economy, described above. Under the assumptions that

- $C^{\circ\circ} = C$,
- $\forall\, p \in C^\circ \setminus \{\mathbf{0}\} : [w_{\text{total}}, p] > 0$,
- *each preference relation* $\succeq_i$ *is*
  - *continuous:* $\forall\, y \in C$ *the sets* $\{x \in C \mid x \succeq_i y\}$ *and* $\{x \in C \mid y \succeq_i x\}$ *are closed in* V *with respect to the linear toplogy on* V*,*
  - *monotone:* $\forall\, x, y \in C : x \geq_C y$ *implies* $x \succeq_i y$*,*
  - *strictly convex:* $\forall\, x, y \in C, \tau \in (0, 1) : x \succeq_i y$ *and* $x \neq y$ *imply* $\tau x + (1 - \tau)y \succ_i y$*,*

this exchange economy model admits an equilibrium price system.

The rest of this section is devoted to a proof of this theorem, using the mathematical results obtained in Sections 2 and 3.
For the definition of the budget correspondence $\mathcal{B}_i$, and the demand function $\mathcal{X}_i$ for every $i \in \{1, \ldots, m\}$, we refer to Section 1.

Lemma 4.1.1 *Let $i \in \{1, \ldots, m\}$, $p \in C^\circ$, and $x \in \mathcal{B}_i(p)$. Assume $[w_i, p] > 0$. If $x \succeq_i y$ holds for all $y \in C$ satisfying $[y, p] < [w_i, p]$, then $x \succeq_i y$ holds for all $y \in C$ satisfying $[y, p] \leq [w_i, p]$.*

Proof
Assume $x \succeq_i y$ holds for all $y \in C$ satisfying $[y, p] < [w_i, p]$. Let $y \in \mathcal{B}_i(p)$ satisfy $[y, p] = [w_i, p]$. For all $\lambda \in [0, 1)$ we have $[\lambda y, p] < [w_i, p]$ and thus $x \succeq_i \lambda y$. The preference relation $\succeq_i$ being continuous we conclude $x \succeq_i y$. □

Next we prove that each demand set contains at most one element and we show that Walras' Law is valid.

Lemma 4.1.2 *Let $p \in C^\circ$ and $i \in \{1, \ldots, m\}$. Then the demand set contains at most one element.*

Proof
Let $p \in C^\circ$ and suppose x and y are both elements of the demand set. On the one hand, the preference relation $\succeq_i$ is strictly convex so if $x \neq y$ then $\tau x + (1 - \tau) y \succ_i x$ for all $\tau \in (0, 1)$. On the other hand, the budget set $\mathcal{B}_i(p) := \{z \in C \mid [z, p] \leq [w_i, p]\}$ is convex, whence $\tau x + (1 - \tau) y \in \mathcal{B}_i(p)$. □

Corollary 4.1.3 *Let $y \in C$ and $p \in Dom(\mathcal{X}_i)$. If $y >_C \mathcal{X}_i(p)$ then $y \notin \mathcal{B}_i(p)$.*

Proof
Due to the monotony of the preference relation, $y >_C \mathcal{X}_i(p)$ implies $y \succeq_i \mathcal{X}_i(p)$. Since the demand set consists of precisely one element, $y \succeq_i \mathcal{X}_i(p)$ and $y \neq \mathcal{X}_i(p)$ imply $y \notin \mathcal{B}_i(p)$. □

Lemma 4.1.4 *Let $i \in \{1, \ldots, m\}$. If $p \in Dom(\mathcal{X}_i)$, then $[\mathcal{X}_i(p), p] = [w_i, p]$.*

Proof
Let $x_0 \in \text{int}(C)$, $p \in \text{Dom}(\mathcal{X}_i)$, and suppose $[\mathcal{X}_i(p), p] < [w_i, p]$. Choose $\lambda > 0$ such that

1. $\lambda x_0 >_C \mathcal{X}_i(p)$
2. $[\lambda x_0, p] > [w_i, p]$.

Consider the convex combination $\tau\lambda x_0 + (1-\tau)\mathcal{X}_i(p)$ with $\tau \in (0,1)$ so small that $[\tau\lambda x_0 + (1-\tau)\mathcal{X}_i(p), p] \leq [w_i, p]$. Then $\tau\lambda x_0 + (1-\tau)\mathcal{X}_i(p) \in \mathcal{B}_i(p)$ and $\tau\lambda x_0 + (1-\tau)\mathcal{X}_i(p) >_C \mathcal{X}_i(p)$. By Corollary 4.1.3, we come to a contradiction. □

Recall that on $\bigcap_{i=1}^m \text{Dom}(\mathcal{X}_i)$ the total demand function $\mathcal{X}_{\text{total}}$ is defined by $\mathcal{X}_{\text{total}}(p) := \sum_{i=1}^m \mathcal{X}_i(p)$. In Lemma 4.2.2 we shall prove that for all $i \in \{1, \ldots, m\}$ it holds that $\text{Dom}(\mathcal{X}_i) = \text{Dom}(\mathcal{X}_{\text{total}}) = \text{int}(C^\circ)$.

Corollary 4.1.5 (Walras' Law) $\forall\, p \in Dom(\mathcal{X}_{\text{total}}) : [\mathcal{X}_{\text{total}}(p), p] = [w_{\text{total}}, p]$.

Proof
Cf. Lemma 4.1.4. □

Lemma 4.1.6 *Let $i \in \{1, \ldots m\}$ and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in $Dom(\mathcal{X}_i)$ with limit p. If $[w_i, p] > 0$ and the sequence $(\mathcal{X}_i(p_n))_{n\in\mathbb{N}}$ is bounded, then $p \in Dom(\mathcal{X}_i)$.*

Proof
Let $(\mathcal{X}_i(p_n))_{n\in\mathbb{N}}$ be bounded. Then we may assume that the sequence $\mathcal{X}_i(p_n)$ is convergent to some $x \in C$. From $\mathcal{X}_i(p_n) \to x$ and $p_n \to p$, we conclude that $x \in \mathcal{B}_i(p)$. We show that x is the optimal element of $\mathcal{B}_i(p)$. Let $y \in \mathcal{B}_i(p)$. By Lemma 4.1.1 we may assume that $[y, p] < [w_i, p]$. The sequence $(p_n)_{n\in\mathbb{N}}$ converging to p, implies $\exists\, N \in \mathbb{N}\ \forall\, n > N : [y, p_n] < [w_i, p_n]$, i.e., $y \in \mathcal{B}_i(p_n)$ and $[y, p_n] \neq [w_i, p_n]$. Lemma 4.1.4 states that y cannot be an optimal element of $\mathcal{B}_i(p_n)$, so $\mathcal{X}_i(p_n) \succeq_i y$, for all $n > N$. The sequence $(\mathcal{X}_i(p_n))_{n\in\mathbb{N}}$ converging to x and $\succeq_i$ being continuous, we conclude that $x \succeq_i y$. □

Lemma 4.1.7 *Let $p \in Dom(\mathcal{X}_{\text{total}})$ satisfy $\mathcal{X}_{\text{total}}(p) \leq_C w_{\text{total}}$. Then $\mathcal{X}_{\text{total}}(p) = w_{\text{total}}$, i.e. p is an equilibrium price system.*

Proof
Define $y := w_{\text{total}} - \mathcal{X}_{\text{total}}(p)$, then $y \in C$ and by Walras' law, $[y, p] = 0$. Define $z_i := \mathcal{X}_i(p) + \frac{1}{m}y$, then $z_i \in \mathcal{B}_i(p)$, for all $i = \{1, \ldots, m\}$. If y were non-zero then $z_i >_C \mathcal{X}_i(p)$ and so by Corollary 4.1.3 we arrive at a contradiction. □

As mentioned before, the proof of the existence of an equilibrium price system can be split into two subproblems (for the concept of equilibrium function we refer to Section 1):

- Find an equilibrium function $\mathcal{E}$.
- Given this equilibrium function $\mathcal{E}$, find a $p \in \text{Dom}(\mathcal{X}_{\text{total}})$ such that $\mathcal{E}(p) = 0$.

For the second part we have presented Theorem 3.2.11. Therefore we shall prove there is an equilibrium function $\mathcal{E} : C^\circ \setminus \{\mathbf{0}\} \to C^\circ$, and then apply the mentioned theorem for $D = C^\circ$.

4.2 Existence of an equilibrium

Let $x_0 \in \text{int}(C)$. Consider the hyperplane $H := \{p \in V^* \mid [x_0, p] = 1\}$ of the dual space V^*. Let $\Phi : I\!R^{n-1} \to H$ be an affine parametrisation of H. The set $\Phi^{\leftarrow}(H \cap C^\circ)$ is compact and convex in $I\!R^{n-1}$. Let λ be the standard Lebesque measure on $I\!R^{n-1}$, and let μ be the measure on H induced by Φ and λ. In particular, for every subset A of H we have $\mu(A) = \lambda(\Phi^{\leftarrow}(A))$, and for a vector-valued function f on (a subset of) H, for which $f \circ \Phi$ is continuous:

$$\int_A f d\mu = \int_{\Phi^{\leftarrow}(A)} (f \circ \Phi) d\lambda.$$

Define the function $\mathcal{S}_0$ on $\text{Dom}(\mathcal{X}_{\text{total}})$ by

$$\begin{aligned} \mathcal{S}_0(p) &:= \int_{H \cap C^\circ} \max\{0, [\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q]\} q d\mu(q) \\ &= \int_L ([\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q]) \, q d\mu(q), \end{aligned}$$

where $L := \{p \in H \cap C^\circ \mid [\mathcal{X}_{\text{total}}(p), q] > [w_{\text{total}}, q]\}$. Observe that for each $p \in \text{Dom}(\mathcal{X}_{\text{total}})$, the function $q \mapsto \max\{0, [\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q]\}q$ is continuous and C°-valued. The set L is open in $H \cap C^\circ$ and therefore measurable. From this we conclude that $\mathcal{S}_0(p)$ is properly defined for every $p \in \text{Dom}(\mathcal{X}_{\text{total}})$.

Lemma 4.2.1 *The function $\mathcal{S}_0$ is an equilibrium function.*

Proof
Clearly, if p is an equilibrium price system then $\mathcal{S}_0(p) = 0$. Now, assume for some $p \in \text{Dom}(\mathcal{X}_{\text{total}})$ we have $\mathcal{S}_0(p) = 0$. Then

$$\begin{aligned} 0 &= [\mathcal{X}_{\text{total}}(p), \mathcal{S}_0(p)] - [w_{\text{total}}, \mathcal{S}_0(p)] \\ &= \int_L ([\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q])^2 d\mu(q). \end{aligned}$$

It follows that $\mu(L) = 0$ and therefore

$$[\mathcal{X}_{\text{total}}(p), q] \leq [w_{\text{total}}, q] \;\; \mu\text{-almost everywhere on } H \cap C^\circ.$$

Define f on H by $f(q) := [\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q]$ then $f \circ \Phi$ is continuous, and $f \circ \Phi \leq 0$ on $\Phi^{\leftarrow}((H \cap C^\circ) \setminus L)$ where $\lambda(\Phi^{\leftarrow}(L)) = 0$. We conclude that the continuous function $f \circ \Phi \leq 0$ on $\Phi^{\leftarrow}(H \cap C^\circ)$ and therefore $f \leq 0$ on $H \cap C^\circ$. Hence, for all $q \in H \cap C^\circ$ it holds that $[\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q] \leq 0$ and so $\mathcal{X}_{\text{total}}(p) \leq_C w_{\text{total}}$ (cf. Corollary 3.2.3). □

Observe that by the preceding proof, Walras' Law has the following consequence. Let $\alpha \geq 0$ and $p \in \text{Dom}(\mathcal{X}_{\text{total}})$, then

$$\begin{aligned} \mathcal{S}_0(p) = \alpha p &\implies [\mathcal{X}_{\text{total}}(p), \mathcal{S}_0(p)] - [w_{\text{total}}, \mathcal{S}_0(p)] = 0 \\ &\iff \mathcal{S}_0(p) = 0. \end{aligned}$$

We want to use Theorem 3.2.11 to prove existence of a $p \in \text{Dom}(\mathcal{X}_{\text{total}})$ satisfying $\mathcal{S}_0(p) = \alpha p$, for some $\alpha \geq 0$. This is why we first determine the domain of the total demand function $\mathcal{X}_{\text{total}}$. It turns out that the function $\mathcal{S}_0$ is not defined on the whole of the pointed convex cone C°. Secondly, we replace $\mathcal{S}_0$ by a related equilibrium function $\mathcal{S}$, defined on the whole of C°, and thirdly, we prove continuity of this function $\mathcal{S}$ on $C^\circ \setminus \{\mathbf{0}\}$.

Lemma 4.2.2 $\forall\, i \in \{1, \ldots, m\}$: *$Dom(\mathcal{X}_i) = int(C^\circ)$.*

Proof
Let $i \in \{1, \dots, m\}$. We have to prove that precisely the elements p of $\mathrm{int}(C^\circ)$ have the property that the demand set at price system p consists of exactly one element.

> **Claim 1:** $\mathrm{Dom}(\mathcal{X}_i) \supseteq \mathrm{int}(C^\circ)$.
> **Proof:** In Section 3 we showed that the budget set $\mathcal{B}_i(p)$ is compact for all $p \in \mathrm{int}(C^\circ)$. Suppose $\mathcal{B}_i(p)$ contains no optimal element, i.e. $\forall\, y \in \mathcal{B}_i(p)\ \exists\, z \in \mathcal{B}_i(p) : z \succ_i y$. Define for all $z \in \mathcal{B}_i(p)$ the set $G(z) := \{y \in \mathcal{B}_i(p) \mid z \succ_i y\}$. The preference relation $\succeq_i$ is continuous, so every $G(z)$ is an open set in $\mathcal{B}_i(p)$ with repsect to the relative topology on $\mathcal{B}_i(p)$. Since $\mathcal{X}_i(p) = \emptyset$, every $x \in \mathcal{B}_i(p)$ is an element of at least one $G(z)$. The collection $\{G(z) \mid z \in \mathcal{B}_i(p)\}$ is an open cover of $\mathcal{B}_i(p)$ and because $\mathcal{B}_i(p)$ is compact, there is a finite subset Z of $\mathcal{B}_i(p)$ such that $\mathcal{B}_i(p) = \bigcup_{z \in Z} G(z)$. The preference relation $\succeq_i$ being transitive, Z has an optimal element $z_0 \in \mathcal{B}_i(p)$. So there is a $z_1 \in Z$ such that $z_0 \in G(z_1)$ and this is in contradiction with the optimality of z_0. Hence, $\mathcal{X}_i(p) \neq \emptyset$.
>
> **Claim 2:** $\forall\, p \in \partial C^\circ : \mathcal{X}_i(p) = \emptyset$.
> **Proof:** Let $p \in \partial C^\circ$ and suppose y is a maximal element of $\mathcal{B}_i(p)$. Because $p \in \partial C^\circ$, by Lemma 3.2.4 there is $x >_C \mathbf{0}$ such that $[x, p] = 0$. Clearly, $x + y \in \mathcal{B}_i(p)$ and $x + y >_C y$, but this is impossible by Corollary 4.1.3. Hence, $\mathcal{B}_i(p)$ has no maximal element.

□

Corollary 4.2.3 *$Dom(\mathcal{S}_0) = int(C^\circ)$.*

Because we want to apply Corollary 3.2.11, we need the function $\mathcal{S}_0$ to be defined not only on $\mathrm{int}(C^\circ)$, but also on ∂C°. To solve this problem, we are going to adjust $\mathcal{S}_0$ and extend its adjustment to a function $\mathcal{S}$ on C°. We shall do this in such a way that the function $\mathcal{S} : C^\circ \to C^\circ$ is an equilibrium function.

Firstly, we define the function $\eta : I\!R \to [0,1]$ by

$$\eta(\alpha) := \begin{cases} 0 & \text{if } \alpha \leq 0 \\ \alpha & \text{if } 0 < \alpha < 1 \\ 1 & \text{if } 1 \leq \alpha, \end{cases}$$

and we choose a fixed $p_0 \in \text{int}(C^\circ)$, see Lemma 3.2.5. With this function η we can now define the new function $\mathcal{S}$ (here we feel inspired by [ArHa71]).

Definition 4.2.4 *Let $p_0 \in int(C^\circ)$. The function $\mathcal{S} : C^\circ \to C^\circ$ is given by*

$$\mathcal{S}(p) := \begin{cases} (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{S}_0(p) + \eta(\mathcal{Z}(p, p_0))p_0 & p \in int(C^\circ) \\ p_0 & p \in \partial C^\circ, \end{cases}$$

where $\mathcal{Z}(p, q) = [\mathcal{X}_{\text{total}}(p), q] - [w_{\text{total}}, q]$.

Theorem 4.2.5 *The function $\mathcal{S}$ is an equilibrium function. Further, if $p \in C^\circ$ satisfies $\mathcal{S}(p) = 0$ then $p \in int(C^\circ)$.*

Proof
Suppose $\mathcal{S}(p) = 0$ for some $p \in C^\circ$, then from the definition of $\mathcal{S}$ it follows that $p \in \text{int}(C^\circ)$. Because $\mathcal{S}_0$ is an equilibrium function defined on $\text{int}(C^\circ)$, we find the following sequence of equivalences: $\forall\, p \in \text{int}(C^\circ)$:

$$\begin{aligned} \mathcal{S}(p) = 0 &\iff (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{S}_0(p) + \eta(\mathcal{Z}(p, p_0))p_0 = 0 \\ &\iff (1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{S}_0(p) = 0 \text{ and } \eta(\mathcal{Z}(p, p_0))p_0 = 0 \\ &\iff \mathcal{S}_0(p) = 0 \text{ and } \eta(\mathcal{Z}(p, p_0)) = 0 \\ &\iff p \text{ is an equilibrium price system.} \end{aligned}$$

□

Because of Theorem 3.2.11 the following lemma is interesting.

Lemma 4.2.6 *Let $p \in C^\circ$. The following two statements are equivalent:*

1. $\exists\, \alpha \geq 0 : \mathcal{S}(p) = \alpha p$
2. $\mathcal{S}(p) = 0$.

Proof
Suppose $\mathcal{S}(p) = \alpha p$ for some $\alpha \geq 0$. From the definition of $\mathcal{S}$ it immediately follows that $p \in \text{int}(C^\circ)$. Applying Walras' Law yields

$$[\mathcal{X}_{\text{total}}(p), \mathcal{S}(p)] - [w_{\text{total}}, \mathcal{S}(p)] = \alpha[\mathcal{X}_{\text{total}}(p), p] - [w_{\text{total}}, p] = 0.$$

Using the definition of $\mathcal{S}(p)$ for $p \in \text{int}(C^\circ)$ we find

$$(1 - \eta(\mathcal{Z}(p, p_0)))\mathcal{Z}(p, \mathcal{S}_0(p)) + \eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0) = 0. \qquad (*)$$

Clearly, $\mathcal{Z}(p, \mathcal{S}_0(p)) = \int_{H \cap C^\circ} \max\{0, \mathcal{Z}(p, q)\}\mathcal{Z}(p, q)d\mu(q) \geq 0$, and so the first term of $(*)$ is non-negative. We conclude that the second term of $(*)$ has to be non-positive, i.e.

$$\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0) \leq 0.$$

Now, suppose $\mathcal{Z}(p, p_0)$ were positive. Then from $\eta(\mathcal{Z}(p, p_0)) > 0$ it follows that $\eta(\mathcal{Z}(p, p_0))\mathcal{Z}(p, p_0) > 0$ which cannot be the case. Hence $\mathcal{Z}(p, p_0) \leq 0$ or $\eta(\mathcal{Z}(p, p_0)) = 0$. By equation $(*)$ this results in the conclusion that $0 = \mathcal{Z}(p, \mathcal{S}_0(p)) = [\mathcal{X}_{\text{total}}(p), \mathcal{S}_0(p)] - [w_{\text{total}}, \mathcal{S}_0(p)]$. So $\mathcal{S}_0(p) = 0$. By the definition of $\mathcal{S}$ we find $\mathcal{S}(p) = 0$. □

In order to prove that $\exists\, p \in \text{int}(C^\circ)\ \exists\, \alpha \geq 0 : \mathcal{S}(p) = \alpha p$, we prove that $\mathcal{S}$ is continuous on $C^\circ \setminus \{\mathbf{0}\}$ and then apply Theorem 3.2.11.
The proof that $\mathcal{S}$ is continuous on $C^\circ \setminus \{\mathbf{0}\}$ is based on the following lemmas.

Lemma 4.2.7 *Let $(p_n)_{n \in I\!N}$ be a sequence in int(C°) converging to some $p \in$ int(C°). For every $i \in \{1, \ldots, m\}$, the budget correspondence $\mathcal{B}_i$ has the following two properties:*

1. *If $x_n \in \mathcal{B}_i(p_n)$ for each n, then there is a subsequence $(x_{nk})_{k \in I\!N}$ that converges to some $x \in \mathcal{B}_i(p)$.*
2. *For each $x \in \mathcal{B}_i(p)$ there exists a sequence $(x_n)_{n \in I\!N}$ satisfying $x_n \in \mathcal{B}_i(p_n)$ and $x_n \to x \in C$.*

Proof
Let $i \in \{1, \ldots, m\}$.

1. By Lemma 3.2.12, p is an order unit for C° and since $p_n \to p$, by Corollary 3.2.13 there are sequences λ_n and μ_n, both converging to 1, that satisfy

$$\forall\, n \in I\!N\ \exists\ \lambda_n, \mu_n > 0 : \mu_n p \leq_C p_n \leq_C \lambda_n p.$$

Define $K := \max_n\{\frac{\lambda_n}{\mu_n}\}$. Assume the sequence $(x_n)_{n\in I\!N}$ satisfies $\forall\, n \in I\!N : x_n \in \mathcal{B}_i(p_n)$. Then $\mu_n[x_n, p] \leq [x_n, p_n] \leq [w_i, p_n] \leq \lambda_n[w_i, p]$ and so $[x_n, p] \leq K[w_i, p]$ for all $n \in I\!N$.

Claim: The sequence $(x_n)_{n\in I\!N}$ is bounded.
Proof: Suppose $(x_n)_{n\in I\!N}$ were unbounded, then $[\frac{x_n}{\|x_n\|}, p] \leq \frac{K}{\|x_n\|}[w_i, p] \to 0$. The sequence $\left(\frac{x_n}{\|x_n\|}\right)_{n\in I\!N}$ is bounded and therefore has a convergent subsequence with limit $y \in C$. Now we obtain a contradiction, since $[y, p] = 0$ would imply $p \notin \text{int}(C^\circ)$.

As a result from the claim, $(x_n)_{n\in I\!N}$ has a convergent subsequence $(x_{nk})_{k\in I\!N}$ with limit $x \in C$. Since $\forall\, k \in I\!N : [x_{nk}, p_{nk}] \leq [w_i, p_{nk}]$ the limit x belongs to $\mathcal{B}_i(p)$.

2. Let $x \in \mathcal{B}_i(p)$. If $[x, p] < [w_i, p]$ then $\exists\ N \in I\!N\ \forall\ n > N : [x, p_n] < [w_i, p_n]$ so for all $n > N$ choose $x_n := x$ and we are done.

   Now assume $[x, p] = [w_i, p]$. Because $p \in \text{int}(C^\circ)$ we have $[w_i, p] > 0$. So, $\exists\ N \in I\!N\ \forall\ n > N : \frac{1}{2}[w_i, p_n] < [x, p_n]$. Let $n > N$. If $x \in \mathcal{B}_i(p_n)$ define $t_n := 1$. If $x \notin \mathcal{B}_i(p_n)$, i.e., if $[x, p_n] > [w_i, p_n]$ then define $t_n := \frac{[w_i, p_n]}{2[x, p_n] - [w_i, p_n]}$ For all $n > N : t_n \in [0, 1]$ and $\lim_{n\to\infty} t_n = 1$. Now put $x_n := t_n x + (1 - t_n)\frac{1}{2}w_i$ then $[x_n, p_n] = [w_i, p_n]$, and $x_n \to x$ as $n \to \infty$.

□

Lemma 4.2.7 expresses the type of continuity of the budget correspondences that we need in order to prove the continuity of the individual demand functions $\mathcal{X}_i$, and they together imply the continuity of the total demand function $\mathcal{X}_{\text{total}}$.

Lemma 4.2.8 *For each $i \in \{1, \ldots, m\}$, $\mathcal{X}_i$ is a continuous function on int(C°).*

Proof
Let $i \in \{1, \ldots, m\}$ and let $(p_n)_{n \in I\!N}$ be a sequence in $\mathrm{int}(C^\circ)$ converging to some $p \in \mathrm{int}(C^\circ)$. By 1) of the preceding lemma, the sequence $\mathcal{X}_i(p_n)$ has a subsequence $\mathcal{X}_i(p_{nk})$ that converges to some $x \in \mathcal{B}_i(p)$. Let $y \in \mathcal{B}_i(p)$. By 2) of the preceding lemma, for all $k \in I\!N$ there is $y_{nk} \in \mathcal{B}_i(p_{nk})$ satisfying $y_{nk} \to y$. Since the preference relation $\succeq_i$ is continuous, $\forall\ k \in I\!N : \mathcal{X}_i(p_{nk}) \succeq_i y_{nk}$ implies $x \succeq_i y$. So, $x = \mathcal{X}_i(p)$. We conclude that any convergent subsequence $\mathcal{X}_i(p_{nk})$ of $\mathcal{X}_i(p_n)$ has $\mathcal{X}_i(p)$ as its limit. Furthermore, every subsequence of $\mathcal{X}_i(p_n)$ has a convergent subsequence. These two observations imply that $\mathcal{X}_i(p_n) \to \mathcal{X}_i(p)$: suppose $\mathcal{X}_i(p_n)$ does not converge to $\mathcal{X}_i(p)$, then there is an $\epsilon > 0$ and a subsequence $\mathcal{X}_i(p_{nk})$ such that $\| \mathcal{X}_i(p) - \mathcal{X}_i(p_{nk}) \| > \epsilon$ for some norm $\| \,.\, \|$ on V. But this subsequence has a converging subsequence with limit $\mathcal{X}_i(p)$, which is a contradiction. □

Lemma 4.2.9 *The function $\mathcal{S}_0$ is continuous on int(C°).*

Proof
Since $H \cap C^\circ$ is compact, we can define

$$M := \int_{H \cap C^\circ} \| q \|_*^2 d\mu(q).$$

Here $\| \,.\, \|_*$ denotes the norm, dual to a chosen norm $\| \,.\, \|$ on V.

Claim: Let $p_1, p_2 \in \mathrm{int}(H \cap C^\circ)$ and $q \in C^\circ$. Then

$$\begin{array}{l} |\max\{0, [\mathcal{X}_{\mathrm{total}}(p_1), q] - [w_{\mathrm{total}}, q]\} - \\ \max\{0, [\mathcal{X}_{\mathrm{total}}(p_2), q] - [w_{\mathrm{total}}, q]\}| \\ \leq \| \mathcal{X}_{\mathrm{total}}(p_1) - \mathcal{X}_{\mathrm{total}}(p_2) \| \, \| q \|_*. \end{array}$$

Proof: For $\alpha \in I\!R : \max\{0, \alpha\} = \frac{1}{2}(|\alpha| - \alpha)$. So, for $\alpha, \beta \in I\!R$:

$$\begin{array}{rl} & |\max\{0, \alpha\} - \max\{0, \beta\}| \\ = & \frac{1}{2} \mid |\alpha| - |\beta| - (\alpha - \beta) \mid \\ \leq & \frac{1}{2} \mid |\alpha| - |\beta| \mid + \frac{1}{2} |\alpha - \beta| \\ \leq & |\alpha - \beta| \end{array}$$

From this, we conclude:

$$
\begin{aligned}
& |\max\{0, [\mathcal{X}_{\text{total}}(p_1), q] - [w_{\text{total}}, q]\} - \\
& \max\{0, [\mathcal{X}_{\text{total}}(p_2), q] - [w_{\text{total}}, q]\}| \\
\leq \ & |[\mathcal{X}_{\text{total}}(p_1), q] - [\mathcal{X}_{\text{total}}(p_2), q]| \\
\leq \ & \| \mathcal{X}_{\text{total}}(p_1) - \mathcal{X}_{\text{total}}(p_2) \| \| q \|_*.
\end{aligned}
$$

Using the above claim, we find for $p_1, p_2 \in \text{int}(C^\circ)$:

$$
\begin{aligned}
\| \mathcal{S}_0(p_1) - \mathcal{S}_0(p_2) \|_* &\leq \int_{H \cap C^\circ} \| \mathcal{X}_{\text{total}}(p_1) - \mathcal{X}_{\text{total}}(p_2) \| \| q \|_*^2 d\mu(q) \\
&= \| \mathcal{X}_{\text{total}}(p_1) - \mathcal{X}_{\text{total}}(p_2) \| M.
\end{aligned}
$$

Since $\mathcal{X}_{\text{total}}$ is continuous on $\text{int}(C^\circ)$, it follows that $\mathcal{S}_0$ is continuous on $\text{int}(C^\circ)$. □

Theorem 4.2.10 *The function $\mathcal{S} : C^\circ \setminus \{\mathbf{0}\} \to C^\circ$ is continuous.*

Proof
Since $\mathcal{S}_0$ and $\mathcal{X}_{\text{total}}$ are continuous on $\text{int}(C^\circ)$, the function $q \mapsto \eta(\mathcal{Z}(q, p_0))$ is continuous on $\text{int}(C^\circ)$ and so $\mathcal{S}$ is continuous on $\text{int}(C^\circ)$. Remains to prove that $\mathcal{S}$ is continuous on the boundary of $C^\circ \setminus \{\mathbf{0}\}$. Let $(p_n)_{n \in \mathbb{N}}$ be a sequence in C° with limit $p \in \partial C^\circ, p \neq \mathbf{0}$, and suppose the sequence $(\mathcal{S}(p_n))_{n \in \mathbb{N}}$ does not converge to $\mathcal{S}(p) = p_0$. Then there is a subsequence $(\mathcal{S}(p_{nk}))_{k \in \mathbb{N}}$ such that for all $k \in \mathbb{N} : \mathcal{S}(p_{nk}) \neq p_0$. Notice that for all $k \in \mathbb{N} : p_{nk} \in \text{int}(C^\circ)$. Since $[w_{\text{total}}, p] > 0$, there is $i_0 \in \{1, \ldots, m\}$ satisfying $[w_{i_0}, p] > 0$. Combining Lemma 4.1.6 and Lemma 4.2.2, yields that the sequence $(\mathcal{X}_{i_0}(p_n))_{n \in \mathbb{N}}$ is unbounded, and so the sequence $(\mathcal{X}_{\text{total}}(p_n))_{n \in \mathbb{N}}$ is unbounded in C. Hence, there is $k \in \mathbb{N} : [\mathcal{X}_{\text{total}}(p_{nk}), p_0] - [w_{\text{total}}, p_0] \geq 1$ (cf. Corollary 3.2.10). So $\mathcal{S}(p_{nk}) = p_0$. This is in contradiction with the assumption that $\mathcal{S}(p_n) \not\to p_0$.
□

Finally we come to the proof of the main theorem of this paper.

Proof of Existence Theorem
Since the equilibrium function $\mathcal{S}$ is continuous on $C^\circ \setminus \{\mathbf{0}\}$, applying Theorem 3.2.11 yields that there is some $p \in C^\circ \setminus \{\mathbf{0}\}$ such that $\mathcal{S}(p) = \alpha p$ for some $\alpha \geq 0$. By Lemma 4.2.6 p is an equilibrium price system. □

A Appendix

A.1 Axiomatic introduction to pointed convex cones

In the model of a pure exchange economy that is described in Section 4.1 we introduce the commodity set as being a pointed convex cone in a vector space $V = \text{span}(C)$. The vector space V has a minor role in this model. In fact, only the notion of a pointed convex cone is needed to describe the set of commodity bundles. However, the theory of finite-dimensional vector spaces is used in the proof that the polar cone C° does not only contain the zero-functional, and is used for deriving a number of topological results for C and C°. So, the vector space V is a mathematical tool only and, in fact, not part of the model.
In this section of the appendix we show that, in modelling a pure exchange economy, the use of a vector space can be circumvented, by the following axiomatic definition of a pointed convex cone.

Definition A.1.1 *A pointed convex cone is a set C of elements with the following properties:*

- *To every pair, x and y, of elements of C there corresponds an element $x + y \in C$, called the sum of x and y, in such a way that:*

  C_1: *addition is commutative: $x + y = y + x$,*

  C_2: *addition is associative: $x + (y + z) = (x + y) + z$,*

  C_3: *there exists an element in C, called a zero-element, denoted by $\mathbf{0}$, satisfying $x + y = \mathbf{0} \Leftrightarrow (x = \mathbf{0}$ and $y = \mathbf{0})$,*

  C_4: *for every $x \in C$, the mapping $add_x : C \to C$ defined by $add_x(y) = y + x$ is injective: $x + y = x + z$ implies $y = z$.*

- *To every pair $x \in C$ and $\alpha \in I\!R^+$, there corresponds an element $\alpha x \in C$, called the product of x and α, in such a way that*

  C_5: *multiplication over $I\!R^+$ is associative: $\alpha(\beta x) = (\alpha\beta)x$,*

  C_6: $1 \cdot x = x$,

  C_7: *multiplication over $I\!R^+$ is distributive with respect to addition: $\alpha(x + y) = \alpha x + \alpha y$,*

C_8**:** *multiplication over $I\!R^+$ is distributive with respect to scalar addition:* $(\alpha + \beta)x = \alpha x + \beta x$.

The terms "pointed" and "convex" refer to two vector space properties of cones. A subset S of some vector space V is called a cone if $\alpha x \in S$ for all $x \in S$ and $\alpha \geq 0$. A cone C in a vector space V is called pointed if $x, (-x) \in C$ implies that x equals the zero-vector of V. This definition is equivalent with axiom C_3. A subset S of some vector space V is called convex if $\tau x + (1 - \tau)y \in S$ for all $x, y \in S$ and $\tau \in [0, 1]$. Thus, a cone in a vector space is convex if it is closed under addition.

Lemma A.1.2 *For every pointed convex cone C, the zero-element is unique and satisfies the following properties*

$$\begin{array}{lll} 1. & \forall\, \alpha \in I\!R^+\backslash_{\{0\}} & : \alpha\mathbf{0} = \mathbf{0} \\ 2. & \forall\, x \in C & : x + \mathbf{0} = x \\ 3. & \forall\, x \in C & : 0x = \mathbf{0}. \end{array}$$

Each pointed convex cone C corresponds with a vector space V over the real numbers $I\!R$. The construction of this vector space is similar to the one that is used to construct the set of integers from the natural numbers $I\!N$. Define the equivalence relation $\sim$ on the product set $C \times C$ in the following way:

$$(x_1, y_1) \sim (x_2, y_2) :\Longleftrightarrow x_1 + y_2 = y_1 + x_2$$

For all $(x_1, y_1) \in C \times C$, the set $[(x_1, x_2)] := \{(x, y) \in C \times C \mid (x, y) \sim (x_1, y_1)\}$ is called the equivalence class of (x_1, y_1). Let V be the collection of all equivalent classes, so $V = (C \times C)/_\sim$. We can define the following addition and scalar multiplication on V:

$$\begin{array}{rcl} [(x_1, y_1)] + [(x_2, y_2)] & := & [(x_1 + x_2, y_1 + y_2)] \\ \alpha[(x, y)] & := & \left\{ \begin{array}{l} [(\alpha x, \alpha y)] \text{ if } \alpha \geq 0 \\ [((-\alpha)y, (-\alpha)x)] \text{ if } \alpha < 0. \end{array} \right. \end{array}$$

It is not difficult to show that these definitions are independent of the choices of the representatives.
With these definitions V is the vector space generated by C.

A.2 Finitely generated cones

In this subsection of the appendix we consider a special class of convex cones: the class of finitely generated ones. We shall show that for a model of a pure exchange economy where the commodity set is described by a finitely generated pointed cone, construction of an equilibrium function is less difficult than in the general case.

Definition A.2.1 *A convex cone D is called finitely generated if its cone basis is a finite set in V.*

Since every finitely generated cone D is closed in V (cf. [Pani93]), we have $D^{\circ\circ} = D$.

Lemma A.2.2 *If S is a finite set in V, then S° is a finitely generated cone in V^*.*

Proof
For the proof of this lemma, we refer to [Tiel79] and [Scha94]. □

Corollary A.2.3 *The polar cone of a finitely generated cone is finitely generated.*

Proof
Let D be a finitely generated cone with cone basis $\{b_1, \ldots, b_k\}$, then $D^{\circ} = \{f \in V^* \mid \forall\, i \in \{1, \ldots, k\} : [b_i, f] \geq 0\} = \{b_1, \ldots, b_k\}^{\circ}$. □

Let D be a finitely generated cone with cone basis $\{b_1, \ldots, b_k\}$ and D° its polar cone with basis $\{d_1, \ldots, d_l\}$. The order relation $\geq_D$ on D is described by (cf. Corollary 3.2.7):

$$x \geq_D y \text{ if and only if } \forall\, j \in \{1, \ldots, l\} : [x, d_j] \geq [y, d_j].$$

Equivalently, the order relation $\geq_{D^{\circ}}$ is described by

$$f \geq_{D^{\circ}} g \text{ if and only if } \forall\, i \in \{1, \ldots, k\} : [b_i, f] \geq [b_i, g].$$

Furthermore, if D is solid and pointed we have

$$\begin{array}{lcl} x_0 \in \mathrm{int}(D) & \Longleftrightarrow & \forall\, j \in \{1, \ldots, l\} : [x_0, d_j] > 0 \\ f_0 \in \mathrm{int}(D^\circ) & \Longleftrightarrow & \forall\, i \in \{1, \ldots, k\} : [b_i, f_0] > 0 \end{array}$$

Recall the model of an exhange economy, described in Section 4.1 and add the extra assumption that C is finitely generated with cone basis $\{b_1, \ldots, b_k\}$. Let C° be generated by $\{d_1, \ldots, d_l\}$. Then $\mathcal{S}_0 : \mathrm{int}(C^\circ) \to C^\circ$ defined by

$$\mathcal{S}_0(p) := \sum_{j=1}^{l} \max\{0, [\mathcal{X}_{\text{total}}(p) - w_{\text{total}}, b_j]\} b_j$$

is an equilibrium function.

Proof

The cone C is pointed, so $\mathcal{S}_0(p) = 0$ implies

$$\forall\, j \in \{1, \ldots, k\} : \max\{0, [\mathcal{X}_{\text{total}}(p) - w_{\text{total}}, b_j]\} = 0.$$

So $[\mathcal{X}_{\text{total}}(p) - w_{\text{total}}, b_j] \leq 0$ for all $j \in \{1, \ldots, l\}$ and therefore $\mathcal{X}_{\text{total}}(p) \leq_C w_{\text{total}}$. By Lemma 4.1.7 p is an equilibrium price system. □

From this equilibrium function $\mathcal{S}_0 : \mathrm{int}(C^\circ) \to C^\circ$ we can construct a continuous equilibrium function $\mathcal{S}$, defined on the whole set C°, in the same way as described in Section 4.2.

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