Equilibria and Approximate Equilibria in Infinite Potential Games

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Abstract: Finite potential games have Nash equilibria in pure strategies. This note provides some results on the existence of equilibria or approximate equilibria if some players have infinite sets of strategies.

1 Introduction

Potential games incorporate information about Nash equilibria in a single real-valued function, called a potential, on the strategy space. These games were introduced in Monderer and Shapley (1996). After recalling the definitions of exact, ordinal and generalized ordinal potential games in section 2, it will follow easily that maxima of a potential function with respect to unilateral deviations are Nash equilibria of the game. Since a potential function of a finite potential game always has a maximum, such games have at least one Nash equilibrium in pure strategies. This need no longer be the case if infinite games are considered. If a Nash equilibrium does not exist, there may be instances of the game in which players either receive a large payoff that satisfies them or cannot gain too much from deviating. Such an instance is an approximate equilibrium. The definition of approximate equilibria is also recalled in section 2.

The main purpose of this paper is to provide some results on the existence of Nash equilibria or approximate equilibria in infinite potential games.

In section 3 we look at approximate equilibria. We show that generalized ordinal potential games in which at most one player has an infinite set of strategies always has approximate equilibria. This generalizes a theorem from Norde and Tijs (1996) on exact potential games to ordinal and generalized ordinal potential games.

An open problem from Peleg, Potters, and Tijs (1996) is solved in section 4 by showing that an ordinal potential game where all players have compact strategy sets and continuous payoff functions may not have a continuous ordinal potential function.

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2 Definitions and preliminary results

A *strategic game* is a tuple $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$, where $N = \{1,\ldots,n\}, n \in \mathbb{N}$ is the player set, for each $i \in N$ the set of player i's strategies is X^i, and $u^i : \prod_{i\in N} X^i \to \mathbb{R}$ is player i's payoff function.

For brevity, we define $X = \prod_{i\in N} X^i$ and for $i \in N$: $X^{-i} = \prod_{j\in N\setminus\{i\}} X^j$. Let $x = (x^1,\ldots,x^n) \in X$ and $i \in N$. Then $x^{-i} \in X^{-i}$ is the vector $(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)$. With a slight abuse of notation, we denote $x = (x^i, x^{-i})$.

The following definitions are due to Monderer and Shapley (1996). A strategic game $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ is

- an *exact potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x^{-i} \in X^{-i}$ and all $y^i, z^i \in X^i$:

$$u^i(y^i, x^{-i}) - u^i(z^i, x^{-i}) = P(y^i, x^{-i}) - P(z^i, x^{-i}).$$

- an *ordinal potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x^{-i} \in X^{-i}$ and all $y^i, z^i \in X^i$:

$$u^i(y^i, x^{-i}) - u^i(z^i, x^{-i}) > 0 \Leftrightarrow P(y^i, x^{-i}) - P(z^i, x^{-i}) > 0.$$

- a *generalized ordinal potential game* if there exists a function $P : X \to \mathbb{R}$ such that for all $i \in N$, for all $x^{-i} \in X^{-i}$ and all $y^i, z^i \in X^i$:

$$u^i(y^i, x^{-i}) - u^i(z^i, x^{-i}) > 0 \Rightarrow P(y^i, x^{-i}) - P(z^i, x^{-i}) > 0.$$

Such a function P is called an (exact, ordinal or generalized) *potential* of the game G. Clearly, an exact potential game is an ordinal potential game, which in its turn is a generalized ordinal potential game. In exact potential games the difference in the value of the potential equal the difference in the payoff to the deviating player. In ordinal potential games only the signs of the differences match. Generalized ordinal potential games allow for freedom in the potential if a deviating player's payoff does not change.

Economic applications of potential games include oligopolies (Slade, 1994) and congestion situations (Rosenthal, 1973).

The *potential maximizer* of a generalized ordinal potential game $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ is the set of strategy combinations $x \in X$ for which some potential P achieves a maximum. The following proposition follows immediately from these definitions.

Proposition 2.1 *Let $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ be a generalized ordinal potential game and P a potential for G. If $x \in X$ is a Nash equilibrium of $\langle N, \{x^i\}_{i\in N}, \{P\}_{i\in N}\rangle$, i.e., of the game with all payoff functions replaced by P, then x is a Nash equilibrium of G. In particular, every finite generalized ordinal potential game has at least one Nash equilibrium, since the potential maximizer is nonempty.*

If G is an exact or ordinal potential game and x is a Nash equilibrium of G, then x is also a Nash equilibrium of $\langle N, \{x^i\}_{i\in N}, \{P\}_{i\in N}\rangle$. This is not necessarily true for generalized ordinal potential games.

Example 2.1 Consider a two-player game with strategy space $X = X^1 \times X^2$ and $u^1(x) = u^2(x) = 0$ for all $x \in X$. Then any function $P : X \to \mathbb{R}$ is a generalized ordinal potential function, so the maxima of P w.r.t. unilateral deviations not necessarily pick out all pure Nash equilibria of the game.

An *improvement path* is a sequence $(x_1, x_2, \ldots)$ of elements $x_k \in X$ such that for all $k \in \mathbb{N}$ the strategy combinations x_k and x_{k+1} differ in exactly one, say the $i(k)$-th, coordinate and $u^{i(k)}(x_k) < u^{i(k)}(x_{k+1})$. A finite improvement path $(x_1, \ldots, x_k)$, with $k \geq 3$, is an *improvement cycle* if $x_1 = x_k$.

Lemma 2.1 *Let $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ be a generalized ordinal potential game. Then G contains no improvement cycles.*

Proof. Suppose, to the contrary, that $(x_1, \ldots, x_k)$ is an improvement cycle of G. Let P be a potential for G. Since for all $l = 1, ..., k-1 : u^{i(l)}(x_l) < u^{i(l)}(x_{l+1})$, we have $P(x_1) < \cdots < P(x_k) = P(x_1)$, a contradiction. □

Let $\epsilon > 0, k \in \mathbb{R}$. A strategy $x^i \in X^i$ of player i is called an *ϵ-best response* to $x^{-i} \in X^{-i}$ if

$$u^i(x^i, x^{-i}) \geq \sup_{y^i \in X^i} u^i(y^i, x^{-i}) - \epsilon$$

and a *k-guaranteeing response* to $x^{-i} \in X^{-i}$ if

$$u^i(x^i, x^{-i}) \geq k.$$

If x^i is either an ϵ-best or k-guaranteeing response (or both) to x^{-i}, it is called an *(ϵ, k)-best response*. Notice that an (ϵ, k)-best response to x^{-i} always exists. A strategy combination $(x^1, \ldots, x^n)$ is called an *ϵ-equilibrium* of the game G if for each $i \in N$, x^i is an ϵ-best response to x^{-i}. It is called an *(ϵ, k)-equilibrium* if x^i is an (ϵ, k)-best response to x^{-i} for all $i \in N$. In such an equilibrium, each player can gain at most ϵ from deviating or receives at least a utility of k.

A game is called *weakly determined* if it has an (ϵ, k)-equilibrium for every $\epsilon > 0$ and every $k \in \mathbb{R}$.

We conclude this section with some examples to illustrate these definitions. Notice that a one-person game is trivially a potential game.

Example 2.2 Consider a one-person game with the player having strategy space $\mathbb{Z}$ and $u(x) = x$ for all $x \in \mathbb{Z}$. This game has no Nash equilibria, but is weakly determined, since for every $k \in \mathbb{R}$, $x = \lfloor k+1 \rfloor$ is a k-guaranteeing response, where for $r \in \mathbb{N}$, $\lfloor r \rfloor$ is the largest integer not exceeding r.

Example 2.3 Consider a one-person game with the player having strategy space $(0, \infty)$ and $u(x) = -\frac{1}{x}$ for all $x \in (0, \infty)$. This game has no Nash equilibria, but for every $\epsilon > 0$, $x > \frac{1}{\epsilon}$ is an ϵ-equilibrium.

The following example from Norde and Tijs (1996) shows that infinite potential games may not be weakly determined.

Example 2.4 Consider the $\infty \times \infty$-bimatrix game with payoff functions $u^1(i,j) = i - j$ and $u^2(i,j) = j - i$, where $i, j \in \mathbb{N}$. This is an exact potential game, with a potential $P(i,j) = i + j$ for all $i, j \in \mathbb{N}$. Clearly, this game does not have (ϵ, k)-equilibria whenever $k > 0$.

3 Equilibria and approximate equilibria in infinite games

Norde and Tijs (1996) provide theorems on equilibrium existence for several classes of games with an exact potential. Their proofs are largely based either on the equality sign in the definition of an exact potential or on a characterization of exact potential games in terms of coordination and dummy games (See Facchini et al. (1995); a similar characterization for Cournot games was given in Slade (1994)). As a consequence, their proofs do not carry over to ordinal or generalized ordinal potential games. Theorem 3.1 generalizes one of their results. If at most one player in a generalized ordinal potential game has an infinite set of strategies, the game has (ϵ, k)-equilibria for all $\epsilon > 0, k \in \mathbb{R}$.

Theorem 3.1 *Let $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ be a generalized ordinal potential game. If $X^1, \ldots, X^{n-1}$ are finite sets, then G is weakly determined.*

Proof. Let P be a potential for G. For each $x^n \in X^n$ fix $\phi(x^n) \in \arg\max_{x^{-n}\in X^{-n}} P(x^n, x^{-n})$. Let $\epsilon > 0, k \in \mathbb{R}$. Construct a sequence $\gamma = (x_1, x_2, \ldots)$ in X as follows: Take $x^n \in X^n$, define $x_1 = (x^n, \phi(x^n))$. Let $m \in \mathbb{N}$. Suppose x_m is defined. If m is odd, and

- x_m^n is not an (ϵ, k)-best response to x_m^{-n}, take $x_{m+1} = (x^n, x_m^{-n})$ with x^n an (ϵ, k)-best response to x_m^{-n};
- otherwise, stop.

If m is even, and

- $x_m^{-n} \notin \arg\max_{x^{-n}\in X^{-n}} P(x_m^n, x^{-n})$, take $x_{m+1} = (x_m^n, \phi(x_m^n))$;
- otherwise, stop.

If the sequence γ is finite, the terminal point is clearly an (ϵ, k)-equilibrium. So now assume this sequence is infinite.

Since the sets $X^1, \ldots, X^{n-1}$ are finite, there exist $l, m \in \mathbb{N}$ such that l is even, m is odd, $l < m$, and $x_l^{-n} = x_m^{-n}$. By construction, $P(x_l) < P(x_m)$, which implies $u^n(x_l) \leq u^n(x_m)$. But x_l^n is an (ϵ, k)-best response to $x_l^{-n} = x_m^{-n}$, so x_m^n is an (ϵ, k)-best response to x_m^{-n}. Since $x_m^{-n} = \phi(x_m^n)$, the other players cannot improve at all. Hence x_m is an (ϵ, k)-equilibrium. □

Example 2.4 indicates that this result cannot be extended to include two or more players with an infinite strategy set.

Under different assumptions we can also establish existence, like in the following theorem. Recall that a real-valued function f on a topological space T is called *upper semi-continuous (u.s.c.)* if for each $c \in \mathbb{R}$ the set $\{x \in T | f(x) \geq c\}$ is closed.

Theorem 3.2 *Let $G = \langle N, \{X^i\}_{i\in N}, \{u^i\}_{i\in N}\rangle$ be a generalized ordinal potential game. If $X^1, \ldots, X^{n-1}$ are finite, X^n is a compact topological space and u^n is u.s.c. in the n-th coordinate, then G has a Nash equilibrium.*

Proof. Fix for each $x^{-n} \in X^{-n}$ an element $\phi(x^{-n}) \in \Phi(x^{-n}) = \arg\max_{z\in X^n} u^n(z, x^{-n})$, which is possible by the upper semi-continuity and compactness conditions.

Suppose that G is not determined. Let $x^{-n} \in X^{-n}$. Take $x_1 = (\phi(x^{-n}), x^{-n})$. Then there exists an infinite improvement path $(x_1, x_2, \ldots)$ such that for each $k \in \mathbb{N}$, if $x_k^n \notin \Phi(x_k^{-n})$, then $x_{k+1} = (\phi(x_k^{-n}), x_k^{-n})$, and otherwise $x_{k+1} = (z, x_k^{-i})$ for some $i \in N$ and $z \in X^i$ satisfying $u^i(z, x_k^{-i}) > u^i(x_k)$.

Since X^{-n} is finite and player n uses only strategies from $\{\phi(x^{-n}) | x^{-n} \in X^{-n}\}$, there exist $k, l \in \mathbb{N}, k < l$, such that $x_k = x_l$. Hence $(x_k, x_{k+1}, \ldots, x_l)$ is an improvement cycle. However, Lemma 2.1 shows that the absence of improvement cycles is necessary for the existence of a potential function, which yields the desired contradiction. □

A similar result for a different class of potential games is given in Voorneveld (1996).

4 Continuity of potential functions

Peleg, Potters, and Tijs (1996) study properties of the potential maximizer. It was left as an open problem in their paper whether ordinal potential games on a compact strategy space with payoff functions u^i which are continuous in the i-th coordinate have a non-empty potential maximizer or, even stronger, whether all such ordinal potential games possess a continuous potential. The result from this section indicates that this is not the case, even if payoff functions are continuous in each coordinate.

Theorem 4.1 *There exists an ordinal potential game with compact strategy spaces and continuous payoff functions for which no potential achieves a maximum and which consequently has no continuous ordinal potential function.*

Proof. Consider the game with $N = \{1, 2\}$, $X^1 = X^2 = [0, 1]$, and payoff functions defined as

$$u^1(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy^6}{(x^2+y^2)^3} & \text{otherwise} \end{cases}$$

and

$$u^2(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{x^6 y}{(x^2+y^2)^3} & \text{otherwise} \end{cases} .$$

Clearly, these payoff functions are continuous. Moreover,

$$P(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ \frac{xy}{(x^2+y^2)^3} & \text{otherwise} \end{cases}$$

is a non-continuous (consider the image of the sequence $\{(\frac{1}{n}, \frac{1}{n})\}_{n=1}^{\infty}$) ordinal potential for the game. This follows easily from $u^1(x, y) = y^5 P(x, y)$ and $u^2(x, y) = x^5 P(x, y)$.

Now consider any ordinal potential Q for this game and the path C in the strategy space from $(1, 1)$ to $(\frac{1}{2}, 1)$ to $(\frac{1}{2}, \frac{1}{2}) \ldots (\frac{1}{2^n}, \frac{1}{2^n})$ to $(\frac{1}{2^{n+1}}, \frac{1}{2^n}) \ldots$ This path is depicted in Figure 1.

For $n \in \mathbb{N}_0$ and $y = \frac{1}{2^n}$ the functions $u^1(\cdot, y)$ and (hence) $Q(\cdot, y)$ are strictly decreasing on $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. We will work out this case and leave other similar cases to the reader. The partial derivative of u^1 with respect to x equals

$$\frac{\partial u^1(x,y)}{\partial x} = y^5 \frac{\partial P(x,y)}{\partial x} = \frac{y^6(y^2 - 5x^2)}{(x^2+y^2)^4}.$$

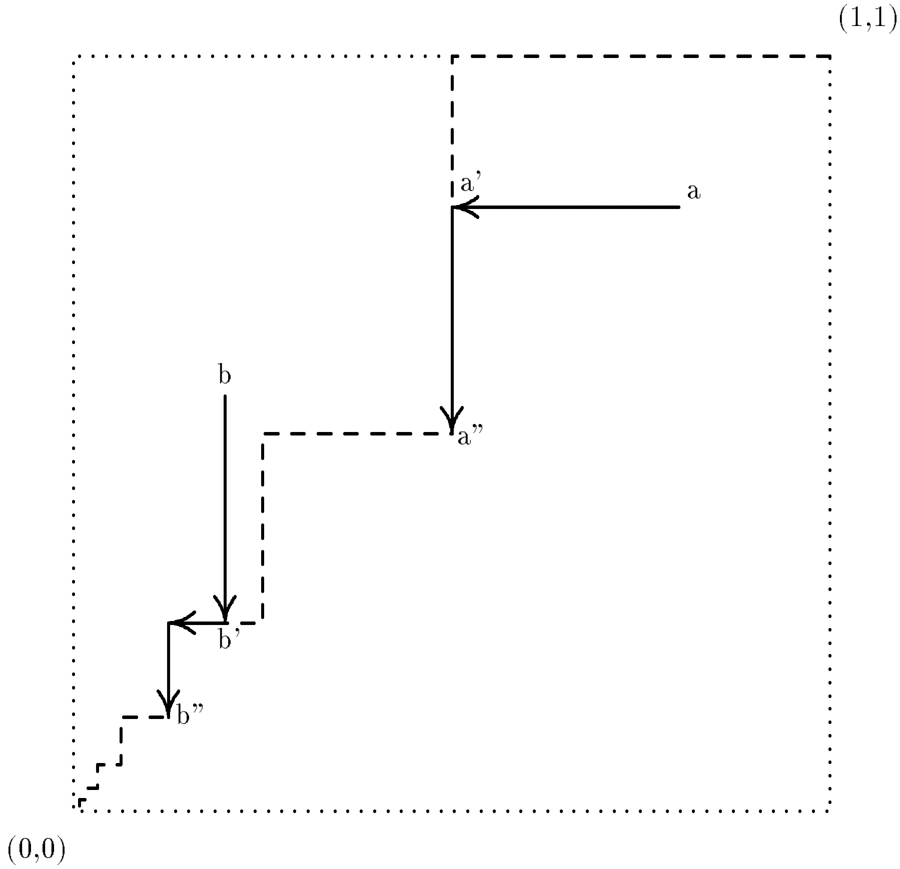

Figure 1: The strategy space and path C from Theorem 3.1

Since $\frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}$, we have that $\frac{1}{2^{2n}} - \frac{5}{2^{2n}} \leq y^2 - 5x^2 \leq \frac{1}{2^{2n}} - \frac{5}{2^{2n+2}}$, which is equivalent to $\frac{-4}{2^{2n}} \leq y^2 - 5x^2 \leq \frac{2^2}{2^{2n+2}} - \frac{5}{2^{2n+2}} = \frac{-1}{2^{2n+2}} < 0$.

Similarly, for $n \in \mathbb{N}$ and $x = \frac{1}{2^n}$ the functions $u^2(x, \cdot)$ and (hence) $Q(x, \cdot)$ are strictly decreasing on $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. This implies that Q must strictly increase along the path C from $(1,1)$ to $(0,0)$.

Also $Q(x,0) = Q(1,0) < Q(1,1)$ and $Q(0,y) = Q(0,1) < Q(1,1)$. Once again using the above, if (x,y) lies to the right of C, like the point a in Figure 1, and (x',y) is on C, like the point a', then $Q(x,y) < Q(x',y)$, since given $y \in (0,1)$, there exists a $n \in \mathbb{N}$ such that $\frac{1}{2^n} \leq y < \frac{1}{2^{n-1}}$. Then by definition $(\frac{1}{2^n}, y)$ is on C and $u^1(\cdot, y)$ is strictly decreasing on $[\frac{1}{2^n}, 1]$.

Also, if (x,y) lies to the left of C, like the point b, and (x,y') is on C, like the point b', then $Q(x,y) < Q(x,y')$, since, given $x \in (0,1)$, there exists an $n \in \mathbb{N}$ such that $\frac{1}{2^{n+1}} \leq x < \frac{1}{2^n}$. Then by definition $(x, \frac{1}{2^n})$ is on C and $u^2(x, \cdot)$ is strictly decreasing on $[\frac{1}{2^n}, 1]$.

Therefore, for any $(x,y) \in [0,1]^2$, we have $Q(x,y) < Q(\frac{1}{2^n}, \frac{1}{2^n})$ for some $n \in \mathbb{N}$. For the points a and b in Figure 1, such points are denoted by a'' and b'', respectively. Since the sequence $\{Q(\frac{1}{2^n}, \frac{1}{2^n})\}_{n=0}^{\infty}$ is strictly increasing, Q has no maximum, which is what we had to prove.

The continuity of a potential function for this game together with the compactness of the strategy space in the product topology would imply the existence of a maximum, contradicting our proof. Hence this game has no continuous potential. □

Notice that continuity, however, is too strong a requirement. Reasonable conditions may exist under which a potential turns out to be upper semi-continuous, which given the compactness of the strategy space would still result in a maximum.

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