Tilburg University

Order Based Cost Allocation Rules

Lind, M.; van Megen, F.J.C.

Publication date:
1996

Link to publication in Tilburg University Research Portal

Citation for published version (APA):
Lind, M., & van Megen, F. J. C. (1996). *Order Based Cost Allocation Rules*. (CentER Discussion Paper; Vol. 1996-56). Operations research.

Order Based Cost Allocation Rules

Mikael Lind [1]
Freek van Megen

Department of Econometrics
and
CentER for Economic Research
Tilburg University
PO Box 90153
NL - 5000 LE Tilburg
The Netherlands

e-mail: michaell@mi.aau.dk and fmegen@kub.nl

[1]Department of Operations Research, University of Aarhus, Building 530, Ny Munkegade, DK-8000 Aarhus C, Denmark. This paper was written while Mikael Lind was visiting the Department of Econometrics, Tilburg University, Tilburg, The Netherlands. Financial support from SNF and SSF is gratefully acknowledged.

Abstract

Cooperative aspects of multiple criteria decision making with respect to cost allocation in a network will be studied. The vector valued costs of a graph that connects a group of geographically scattered users to a common source have to be distributed among the users. Here a specific class of cost allocation rules, the so-called Bird rules, are studied. Bird allocations can be determined by means of a greedy algorithm. This algorithm is based on an order on the edge set. Three different orders and their Bird rules are studied. Two of the three associated Bird allocation rules will be characterized. Those two constitute a refinement of the set of stable cost allocations of the multiple criteria minimum cost spanning tree problem.

Keywords: Multi Criteria Analysis, Cooperative Game Theory, Cost Allocation, Spanning Trees, Stability.

1 Introduction

This paper studies cooperative aspects of multiple criteria decision making (MCDM) with respect to cost allocation in a network.

MCDM has gained broad interest and has been extensively described in the literature on operations research and decision theory over the last two decades. MCDM is concerned with solving problems for which the final decisions have to be acceptable with regard to several possibly conflicting objectives. Various types of decision problems have characteristics which imply that a number of objectives should be taken into consideration. Moreover, by doing so, a more detailed description of the underlying problem is obtained, thereby hopefully giving better decisions. Two good reference books on MCDM are Steuer (1986) and Vincke (1992).

Consider a network in which a number of geographically scattered users have to be connected to a common source. We assume that the costs for each link between two users or between a user and the source are known and non-negative. Furthermore, each link is assumed to carry different types of costs; for example, the construction costs and the maintenance costs of the link. We assume that the users dislike any distribution system that has costs dominated by the costs of another available distribution system. Thus, acceptable distribution systems can be found among the set of spanning trees in the network. For approximations of the set of cost efficient spanning trees see Andersen, Jörnsten and Lind (1995) and Hamacher and Ruhe (1994).

Next, the problem arises how to allocate the different types of costs of a spanning tree to the users. We investigate three cost allocation rules, each based on an ordering of the network's edge set. We will refer to the problem of finding an acceptable distribution system and allocating its costs to the users as *the cost efficient spanning tree (cest) problem.*

For the less general case, the situation in which each link is given a one-dimensional cost, the corresponding problem, a so-called minimum cost spanning tree (mcst) problem, was first studied by Claus and Kleitman (1973) and later, among others, by Bird (1976), Granot and Huberman (1981), and Feltkamp, Tijs, and Muto (1994a). In a mcst problem all minimum cost spanning trees are readily found by means of a greedy algorithm á la Prim (1957), Dijkstra (1959) or Kruskal (1956). The basic idea behind the algorithms is to add the cheapest available edges, one after another to the already generated set of edges, without constructing a cycle.

Furthermore, an allocation of the costs of a spanning tree was proposed by Bird (1976): each agent is attributed the costs of the last edge in the unique path from the source to the node that represents the agent. This allocation can be implemented easily into the greedy algorithm approach of Prim (1957) and Dijkstra (1959); see Feltkamp et al. (1994a). We will refer to this type of cost allocations as *Bird allocations.*

The basis of the greedy algorithms is an order of the edge set. For mcst problems a natural order is provided by simply arranging the edges in non-decreasing costs. In the multiple criteria case the users might reach consensus upon a common preference relation that will be used to order the edges in the network at hand. The more properties the users request for a preference relation, the fewer orders are applicable. Social choice theory might suggest a description of suitable preference relations and thereby orders.

We study three specific orders: unanimity, also called the strong version of the Pareto principle, and two versions of utilitarianism, the classical utilitarian principle and the Nash principle. The greedy algorithm is then applied to each of them, so the associated order based Bird-rules are easily deduced.

Several properties of allocation rules are described and used to characterize the Bird-rules based on unanimity and on the classical utilitarian principle, respectively. The relationship between properties on the orders and properties of their corresponding Bird-rules is investigated through a consistency property. A Bird-rule is said to be *consistent* if the conceptual meanings of the properties of respectively the Bird-rule and of the order used to define it do not contradict each other. The two characterized Bird-rules satisfy the consistency property; however, the Bird-rule based on the Nash principle does not. For a characterization of the Bird-rule defined with respect to mcst problems see Feltkamp et al. (1994a). It is shown that the Bird-rule based on the classical utilitarian principle constitutes a refinement of the Bird-rule based on unanimity. Moreover, the two consistent Bird-rules provide stable cost allocations. The stability concept is derived from a core concept for related games.

The paper is organized as follows. In section 2 the cost efficient spanning tree problem is presented. The notion of a solution concept is introduced for this setting and a short review is given of Bird allocations and Bird-rules in the mcst setting. Moreover, the order based Bird-rule is formally introduced. Three orders as well as their associated Bird-rules are described in section 3. Properties, both of orders and of solution concepts, are given. The section's main part deals with characterizations of two Bird-rules. Two stability concepts will be revealed in section 4, and the cost allocations derived from the Bird-rule based on unanimity are shown to be stable. Section 5 gives conclusions.

2 Cost Efficient Spanning Tree Problems

In a multiple criteria cost efficient spanning tree problem $M = \langle N \cup \{*\}, E, w \rangle$ a finite group of agents $N = \{1, \ldots, n\}$ all have to be connected to a common source, here denoted by $*$, via a subset of the set of links E among the agents or between an agent and the source. Each link $e \in E$ is assumed to carry m different non-negative costs, which is captured in the model by the cost function

$w : E \to \mathbb{R}^m_+$.

The costs of the distribution system have to be allocated among the agents. This implies that the agents have an incentive to cooperate since there might exist a more cost efficient distribution system than the one where each agent connects himself directly to the source. Further, any distribution system that connects all the agents to the source and which includes a cycle will be at least as expensive as the distribution system where a link in the cycle is removed. Thus, distribution systems for which the costs are not dominated by the costs of other distribution systems, and hence cost efficient ones, can be found among the set of spanning trees within the network. This explains why the problem is called a multiple criteria cost efficient spanning tree problem. We will refer to such problem as a cest problem.

Let $\mathcal{M}$ denote the class of multiple criteria cest problems. A *solution concept*, σ, for cest problems is a correspondence that assigns to every $M \in \mathcal{M}$, with $M = \langle N \cup \{*\}, E, w \rangle$, a subset of $(\mathbb{R}^m_+)^N$. Each $x \in (\mathbb{R}^m_+)^N$ is seen as the allocation of the costs $\sum_{i=1}^n x_i \in \mathbb{R}^m_+$ to the users 1 up to n. Agent $i \in N$ is allocated the costs $x_i = (x_i^1, \ldots, x_i^m) \in \mathbb{R}^m_+$.

2.1 Cost Allocation Through a Greedy Algorithm

The two issues a cest problem is concerned with, the one of determining an acceptable distribution system and the related one of allocating the costs of the system, can be solved in an integrated approach. A greedy algorithm is used to determine distribution systems and to define a solution concept that describes the allocation of the costs of each system.

The greedy algorithm introduced by Prim (1957) and Dijkstra (1959) is a classical method to determine a minimum cost spanning tree. This approach for the single criterion mcst problem boils down to adding nodes and edges one after another to a subgraph connected to the source. In each iteration one of the cheapest edges is added without constructing a cycle. A minimum cost spanning tree results. A straightforward adaptation of this approach to multiple criteria cest problems, by adding non-dominated edges in each step, does not in general lead to an efficient spanning tree as is seen in the next example.

Example 1.
Let $N = \{1, 2, 3\}$ and let the costs of the edges be as in the following figure.

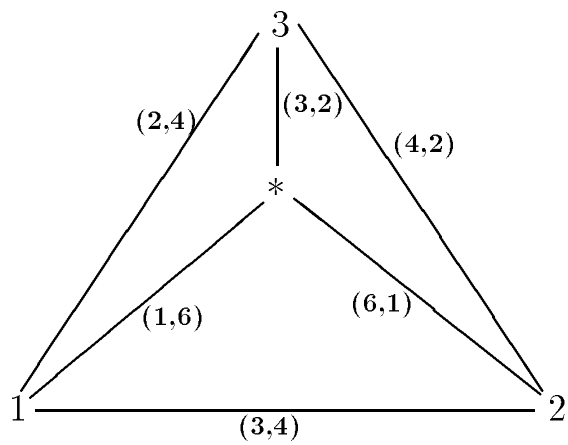

Figure 1.

Here, by taking non-dominated edges in each iteration, one might select the following sequence of edges: $(*, 1), (*, 3), (*, 2)$. This leads to a spanning tree T which costs $(10, 9)$. However, T is not efficient since its cost is dominated by the tree $\langle N \cup \{*\}, \{(*, 3), (1, 3), (2, 3)\}\rangle$ which costs $(9, 8)$. Notice on the other hand that the set of spanning trees which can be constructed by adding non-dominated edges in each iteration contains all efficient spanning trees.

For mcst problems a rule for the allocation of the cost of a spanning tree between its users is suggested by Bird (1976). Each user is allocated the cost of the first edge on the unique path from his node to the source. The implementation of the Bird-rule into the Prim-Dijkstra algorithm is described in Feltkamp et al. (1994a). In each step the cost of the new edge is allocated to the agent represented by the newly added node. The algorithmic method constitutes a solution concept that yields for every mcst problem the set of Bird allocations associated to the set of minimum cost spanning trees.

It will be clear that for a cest problem Bird allocations for a given spanning tree can be defined in analogy to the mcst case. Also, a greedy algorithm that produces spanning trees in a similar way as above can be extended with the cost allocation step.

The greedy algorithm applied in mcst problems uses a natural order on the edge set; edge e is preferred to edge f if the cost of e is less than the cost of f. In multiple criteria cest problems, however, such natural order is not explicitly available. Instead we assume that the agents somehow reach agreement on an order that will be used in the greedy algorithm.

Now, consider a cest problem $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$. By $R(w)$ we denote a partial order on the edge set (i.e. $R(w)$ is *reflexive*: $eR(w)e$ for all $e \in E$ and *transitive*: $eR(w)f$ and $fR(w)g$ implies $eR(w)g$ for all $e, f, g \in E$) defined with respect to the cost function. Let $e, f \in E$ be such that $eR(w)f$. This means that the agents as a group believe that it is as least as good to use the edge e in the distribution system as edge f given that they have to choose among the edges in order to construct the distribution system. Notice that there might exist edges which are not comparable, i.e. $e, f \in E$ and neither $eR(w)f$ nor $fR(w)e$. We write R instead of $R(w)$ if no ambiguity can occur.

The order based greedy algorithm builds a spanning tree and allocates the cost of the newly added edge to the agent represented by the newly added node. Denote for each set of edges D the set of nodes incident with D by $\nu(D)$.

The order based greedy algorithm
Given: $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and R a partial order on the edge set E.

1. Set $T = \emptyset$.
2. Take an edge $e \in E \setminus T$ incident to $\{*\} \cup \nu(T)$ such that the subgraph spanned by $\{e\} \cup T$ does not contain a cycle and such that no $f \in E \setminus T$ incident to $\{*\} \cup \nu(T)$ exists for which the subgraph spanned by $\{f\} \cup T$ does not contain a cycle and fRe and $\neg(eRf)$.
3. Assign cost $w(e)$ to the agent represented by the unique node that has been connected to the source in the previous step.
4. $T := T \cup \{e\}$.
5. Continue with step 2 unless all nodes are connected to the source.

A spanning tree is constructed and an element in the solution concept σ_R is found. Denote the collection of edge sets that the algorithm based on R for the problem $M \in \mathcal{M}$ can give as output by $\mathcal{T}_R(M)$. Clearly, $\mathcal{T}_R(M)$ is finite for each M. The solution concept σ_R assigns to every cest problem M, with ordering R on the edge set, the set of cost allocations which are provided by the algorithm. Thus, for each $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and each order R we have

$$\sigma_R(M) = \{(w((p_T(1), 1)), \ldots, w((p_T(n), n))) \mid T \in \mathcal{T}_R(M)\}$$

where for each spanning tree $\langle N \cup \{*\}, T\rangle$ and for each $i \in N$ the node which is the immediate predecessor of i in the unique path from i to the source is denoted by $p_T(i)$.

2.2 The Order Based Bird-rule

An order R on the edge set is said to satisfy *unanimity* if for all $e, f \in E$ the following holds:

$$\begin{aligned} w(e) \leq w(f) \quad &\Rightarrow eRf. \\ w(e) \leq w(f), w(e) \neq w(f) \quad &\Rightarrow eRf \text{ and } \neg fRe. \end{aligned}$$

Unanimity, sometimes referred to as the strong version of the Pareto principle, cf. d'Aspremont and Gevers (1977), implies that the agents prefer to pay as little as possible of the costs of the different criteria. In the single criterion case this means that the edges will be ordered in non-decreasing costs. Throughout the paper we assume that the agents agree upon an order which satisfies unanimity. This is in accordance with the assumption that the agents dislike any distribution system which costs are dominated by the costs of another available distribution system. However, the order based greedy algorithm might still construct a non-efficient spanning tree as it is seen in example 1.

To take care of this problem it seems reasonable to restrict the solution to the cost allocations in $\sigma_R(M)$ which are non-dominated with respect to any other cost allocation in $\sigma_R(M)$. The restrictive step added to the greedy algorithm leads to a subset of $\sigma_R(M)$ for each $M \in \mathcal{M}$. The solution concept thus defined will be called the *Bird-rule with respect to R*. We denote it B_R and it is for each $M \in \mathcal{M}$ given formally by

$$B_R(M) = \{x \in \sigma_R(M) \mid \nexists y \in \sigma_R(M) : \sum_{i \in N} y_i \leq \sum_{i \in N} x_i \text{ and } \sum_{i \in N} y_i \neq \sum_{i \in N} x_i\}.$$

3 Characterizations of Order Based Bird-rules

For a multiple criteria cest problem $M = \langle N \cup \{*\}, E, w \rangle \in \mathcal{M}$ the cost allocations derived by the Bird-rule, $B_R(M)$, depend directly on the order R.

We assumed that R satisfies unanimity and hence in the single criterion case the edges are simply ordered in non-decreasing costs. However, in the m-dimensional setting several orders suggest themselves as being reasonable. To choose an order on the edge set the agents might select a mapping from the set of cost-functions to the set of orderings on the edge set and then use the derived order. Moreover, they might agree upon some properties which the mapping has to satisfy and use these to define an acceptable order. So, to determine which order to use, it is helpful to take also other properties than unanimity into consideration.

Properties for orderings on a finite set of alternatives are studied within the context of social choice theory. Several of such properties can be interpreted into the framework of cest problems. The following list of possible properties for an order $R(\cdot)$, regarded as a mapping from the set of cost-functions to the set of orderings of the edge set, is considered.

Anonymity.
For all $w : E \rightarrow \mathbb{R}^m_+$, for every permutation μ on E and for all $e, f \in E$

$$\mu^{-1}(e) R(w) \mu^{-1}(f) \Leftrightarrow e R(w_\mu) f$$

where $w_\mu(e) = w(\mu(e))$ for all $e \in E$.

Neutrality.
For all $w : E \to \mathbb{R}^m_+$, for every permutation π on $\{1,\ldots,m\}$ and for all $e,f \in E$

$$eR(w)f \Leftrightarrow eR(w^\pi)f$$

where $w^\pi_t(e) = w_{\pi(t)}(e)$ for $t = 1,\ldots,m$ for all $e \in E$.

Scale-independence.
For all $w : E \to \mathbb{R}^m_+$, $w > 0$, for all $a \in \mathbb{R}^m_{++}$ and for all $e,f \in E$

$$eR(w)f \Leftrightarrow eR(a * w)f$$

where $(a * w)(e) = (a_1 w_1(e), \ldots, a_m w_m(e))$ for all $e \in E$.

Zero-independence.
For all $w : E \to \mathbb{R}^m_+$, for all $b \in \mathbb{R}^m$ for which $(w+b)(e) \geq 0$ for all $e \in E$ and for all $e,f \in E$

$$eR(w)f \Leftrightarrow eR(w+b)f$$

where $(w+b)(e) = (w_1(e)+b_1, \ldots, w_m(e)+b_m)$ for all $e \in E$.

Independence.
For every pair of cost functions, $w, \overline{w} : E \to \mathbb{R}^m_+$, and for all $e,f \in E$ such that $w(e) = \overline{w}(e)$ and $w(f) = \overline{w}(f)$

$$eR(w)f \Leftrightarrow eR(\overline{w})f.$$

If $R(\cdot)$ satisfies anonymity and neutrality it follows that the deduced order on the edge set is not influenced by the names or the indexation of neither the nodes nor the criteria. Moreover, if $R(\cdot)$ satisfies scale- and zero-independence the scaling of the different criteria does not affect the derived order. The different criteria can be measured in any terms that we like, as long as only affine transformations of the measurements are made. Finally, independence says that if two edges are comparable with respect to the derived order R then the order of the two edges should not depend of the cost of the other edges.

Now, to examine the relationship between the properties of an order and the properties of the order based Bird-rule and to provide characterizations of various Bird-rules, we present a list of properties for a solution concept σ on $\mathcal{M}$. Some more notation is introduced first. For all $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ let the collection of edge sets associated with spanning trees on $\langle N \cup \{*\}, E\rangle$ be denoted by $\mathcal{T}_N$. Denote for each $D \subseteq E$ the corresponding cost by $w(D)$, i.e. $w(D) = \sum_{e \in D} w(e)$ and denote the set of costs associated with the set of efficient spanning trees by $Eff(M)$, i.e. let

$$Eff(M) = \{w(T) \mid T \in \mathcal{T}_N \text{ and } \nexists \overline{T} \in \mathcal{T}_N : w(\overline{T}) \leq w(T), w(\overline{T}) \neq w(T)\}.$$

We state the following possible properties for a solution concept σ on $\mathcal{M}$.

Non-emptiness.
$\sigma(M) \neq \emptyset$, $\forall M \in \mathcal{M}$.

Cost Efficiency.
$\{\sum_{i \in N} x_i \mid x \in \sigma(M)\} \subseteq Eff(M)$, $\forall M \in \mathcal{M}$.

Anonymity.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and for every permutation μ on N

$$\{x_{\mu(i)} \in \mathbb{R}_+^m \mid x \in \sigma(N \cup \{*\}, E, w)\} = \{x_i \in \mathbb{R}_+^m \mid x \in \sigma(N \cup \{*\}, E, w_\mu)\} \ \forall i \in N$$

where $w_\mu((\mu(i), \mu(j))) = w((i,j))$ for all $i, j \in N, i \neq j$.

Neutrality.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and for every permutation π on $\{1, \dots, m\}$

$$x \in \sigma(N \cup \{*\}, E, w) \Leftrightarrow (x_1^\pi, \dots, x_n^\pi) \in \sigma(N \cup \{*\}, E, w^\pi)$$

where $y_{\pi(t)}^\pi = y_t$ for all $t \in \{1, \dots, m\}$ for all $y \in \mathbb{R}^m$.

Scale-independence.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and for all $a \in \mathbb{R}_{++}^m$,

$$\sigma(N \cup \{*\}, E, a * w) = \{(a * x_1, \dots, a * x_n) \mid x \in \sigma(N \cup \{*\}, E, w)\}$$

where $(a * y)_t = a_t y_t$ for all $t \in \{1, \dots, m\}$ for all $y \in \mathbb{R}^m$.

Zero-independence.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and for all $b \in \mathbb{R}^m$ for which $(w + b)(e) \geq 0$ for all $e \in E$

$$\sigma(N \cup \{*\}, E, w + b) = \{(x_1 + b, \dots, x_n + b) \mid x \in \sigma(N \cup \{*\}, E, w)\}$$

where $(y + b)_t = y_t + b_t$ for all $t \in \{1, \dots, m\}$ for all $y \in \mathbb{R}^m$.

The last two properties of the list take leafs of spanning trees into account. We call $i \in N$ a *leaf* of the spanning tree $\langle N \cup \{*\}, T\rangle$ if no $j \in N$ exists such that i is the predecessor of j in the unique path from $*$ to j within the tree.

Leaf-consistency.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and for all $x \in \sigma(N \cup \{*\}, E, w)$, if $i \in N$ is a leaf in a spanning tree $\langle N \cup \{*\}, T\rangle$ for which $\sum_{i \in N} x_i = w(T)$ then

$$x^{-i} \in \sigma((N \setminus \{i\}) \cup \{*\}, E^{-i}, w) \text{ and } \sum_{j \in N \setminus \{i\}} x_j^{-i} = w(T \cap E^{-i})$$

where $x^{-i} = (x_j)_{j \in N \setminus \{i\}}$, $E^{-i} = \{(l,k) \in E \mid l \neq i, k \neq i\}$ and w should be restricted to the domain E^{-i}.

Converse leaf-consistency.
For all $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ with $|N| \geq 2$ and for all $x \in (\mathbb{R}^m_+)^N$, if $\sum_{i \in N} x_i = w(T)$ for an efficient spanning tree $\langle N \cup \{*\}, T\rangle$ and $x^{-i} \in \sigma((N \setminus \{i\}) \cup \{*\}, E^{-i}, w)$ for all $i \in N$ where i is a leaf in $\langle N \cup \{*\}, T\rangle$, then

$$x \in \sigma(N, E, w).$$

For a solution concept σ on $\mathcal{M}$ non-emptiness says that a cost allocation will be attributed to each problem. The solution concept satisfies cost efficiency if each cost allocation derived by σ exactly covers the costs of an efficient spanning tree. The interpretation of the next four properties: anonymity, neutrality, scale- and zero-independence is analogical to the interpretation given for the same properties with respect to an order. Further, leaf-consistency says that the costs allocated to the other agents should not be affected if an agent situated in a leaf of the distribution system pays his share and leaves it. Converse leaf-consistency, on the other hand, says that if a cost allocation covers the costs of an efficient spanning tree and is consistent with the problems ignoring any agent represented by a leaf of the tree then the allocation is an element of the set of cost allocations generated by the solution concept.

We have stated two lists. A list of properties defined for orderings on the edge set and a list of properties associated with solution concepts on $\mathcal{M}$. Now, recall that each Bird-rule is defined with respect to an order on the edge set. Such an order might imply or can be defined by a set of properties in which the agents believe. Thus, it would only make sense to regard the associated Bird-rule as a reasonable solution concept if the conceptual meaning of the properties for the Bird-rule and the properties of the order do correspond. If an order satisfies neutrality and the rule does too, we say that the rule is consistent with respect to neutrality. Thus, consistency with respect to anonymity, neutrality, scale- and zero-independence can be defined. The correspondence between unanimity and cost efficiency will be given by consistency with respect to unanimity. In this paper each order is assumed to satisfy unanimity and hence the associated Bird-rules all have to satisfy cost efficiency to be consistent w.r.t. unanimity. Further, we will call a Bird-rule *consistent* if the rule is consistent w.r.t. the properties used to define the underlying order of the rule.

3.1 Bird Allocations Based on Unanimity

A prerequisite for the definition of the Bird-rule with respect to an order is that the order satisfies unanimity. This seems quite natural since we assume that the agents confronted with two distribution systems, prefer the cheapest. Consider now for each multiple criteria cest problem $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ the partial order $\succeq$ defined by

$$\forall e, f \in E: \quad e \succeq f \Leftrightarrow w(e) \leq w(f).$$

It is straightforward to show that the related mapping $\succeq(\cdot)$ satisfies unanimity, anonymity, neutrality, scale- and zero-independence and independence. Moreover, $\succeq$ provides the constraints an arbitrary order on the edge set has to fulfill to satisfy unanimity.

The next proposition shows that the corresponding Bird-rule, $B_{\succeq}$, satisfies all the properties for a solution concept on $\mathcal{M}$ we have listed so far.

Proposition 1 *$B_{\succeq}$ satisfies non-emptiness, cost efficiency, anonymity, neutrality, scale- and zero-independence, leaf-consistency and converse leaf-consistency.*

Proof. Let $M \in \mathcal{M}$. It follows from Serafini (1986) and is easy to prove that $Eff(M) \subseteq \{\sum_{i \in N} x_i \mid x \in \sigma_{\succeq}(M)\}$ and thus by the definition of $B_{\succeq}(M)$ we have $Eff(M) = \{\sum_{i \in N} x_i \mid x \in B_{\succeq}(M)\}$. Hence, $B_{\succeq}$ satisfies non-emptiness and cost efficiency. It is straightforward to show that $B_{\succeq}$ satisfies the other properties too. □

An immediate consequence of Proposition 1 is that $B_{\succeq}$ is consistent w.r.t. anonymity, neutrality, zero- and scale-independence and more important w.r.t. unanimity. Thus, $B_{\succeq}$ is a consistent solution concept. Furthermore, note that $B_{\succeq}(M)$ supports the costs of all efficient spanning trees for M. To provide a characterization of $B_{\succeq}$ we use the following lemma which is an adaptation of lemma 3.6 in Feltkamp et al. (1994a).

Lemma 2 *Let φ be a solution concept on $\mathcal{M}$ which satisfies cost efficiency and leaf-consistency and let ψ be a solution concept on $\mathcal{M}$ which satisfies non-emptiness, efficiency and converse leaf-consistency. Then*

$$\varphi(M) \subseteq \psi(M) \qquad \forall M \in \mathcal{M}.$$

Proof. The proof uses induction in the number of agents. First, let $\langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ be such that $|N| = 1$. Then, by non-emptiness and cost efficiency of ψ and cost efficiency of φ we have $\varphi(N \cup \{*\}, E, w) \subseteq \psi(N \cup \{*\}, E, w)$. Now, let $k \in \mathbb{N}$, $k \geq 2$. Assume $\varphi(M) \subseteq \psi(M)$ for all $M \in \mathcal{M}$ with $|N| = k-1$ and let $M \in \mathcal{M}$ be such that $|N| = k$. Assume $\varphi(M) \neq \emptyset$ and let $x \in \varphi(M)$. By cost efficiency there exists an efficient spanning tree $\langle N \cup \{*\}, T\rangle$ such that $\sum_{i \in N} x_i = w(T)$. By leaf-consistency of φ and the induction hypothesis it follows that

$$x^{-i} \in \varphi((N \setminus \{i\}) \cup \{*\}, E^{-i}, w) \subseteq \psi((N \setminus \{i\}) \cup \{*\}, E^{-i}, w)$$

for all $i \in N$ which are leafs in $\langle N \cup \{*\}, T\rangle$. So, by converse leaf-consistency of ψ we conclude that $x \in \psi(M)$. □

Theorem 3 *$B_{\succeq}$ is the unique solution concept on $\mathcal{M}$ which satisfies non-emptiness, cost efficiency, leaf-consistency and converse leaf-consistency.*

Proof. By Proposition 1 it follows that $B_{\succeq}$ satisfies the properties. Next, let σ denote a solution concept which satisfies non-emptiness, cost efficiency, leaf-consistency and converse leaf-consistency. Let $M \in \mathcal{M}$. Then by Lemma 2 we have $\sigma(M) \subseteq B_{\succeq}(M)$ and $B_{\succeq}(M) \subseteq \sigma(M)$. □

Moreover, the four properties: non-emptiness, cost efficiency, leaf-consistency and converse leaf-consistency are logically independent. First, consider the solution concept given by $\sigma(M) = \emptyset$ for all $M \in \mathcal{M}$. Apart from non-emptiness it satisfies the properties. Second, notice that the solution concept defined by $\sigma(M) = (w(E), \dots, w(E))$ for all $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ satisfies non-emptiness, leaf-consistency and converse leaf-consistency but not cost efficiency. Third, let

$$\sigma(M) = \{x \in (\mathbb{R}^m_+)^N \mid \sum_{i \in N} x_i \in Eff(M)\} \qquad \forall M \in \mathcal{M}.$$

$\sigma(M)$ satisfies non-emptiness, cost efficiency and converse leaf-consistency but is not leaf-consistent. Finally, we consider the solution concept defined for all $M \in \mathcal{M}$ by

$$\sigma(M) = \{x \in B_{\succeq}(M) \mid \sum_{i \in N} x_i \leq_L \sum_{i \in N} y_i, \forall y \in B_{\succeq}(M)\} \tag{1}$$

where $\leq_L$ denotes the lexicographic order on $\mathbb{R}^m$. (Let $x, y \in \mathbb{R}^m$. Then $x \leq_L y$ if and only if $x = y$ or there exists a $k \in \{1, \dots, m\}$ such that $x_t = y_t$ for $t = 1, \dots, k-1$ and $x_k < y_k$.) Notice that σ satisfies non-emptiness, cost efficiency and leaf-consistency but that it is easy to find a cest problem $M = \langle N \cup \{*\}, E, w\rangle$ for which $\sigma(M)$ does not coincide with $B_{\succeq}(M)$. Hence, by Theorem 3 , σ does not satisfy converse leaf-consistency.

3.2 Bird Allocations Based on Utilitarianism

Consider a multiple criteria cest problem $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$. Since $\{\sum_{i \in N} x_i \mid x \in B_{\succeq}(M)\} = Eff(M)$ we have that $B_{\succeq}(M)$ generally contains several elements. By using a *complete* order R on the edge set (i.e. eRf or fRe for all $e, f \in E$ and R is reflexive and transitive) which satisfies unanimity we get $\sigma_R(M) \subseteq \sigma_{\succeq}(M)$ for all $M \in \mathcal{M}$. Moreover, it will be possible to find a cest problem M for which $\sigma_R(M) \subset \sigma_{\succeq}(M)$. Thus, σ_R is a *refinement* of $\sigma_{\succeq}$. Further, if B_R satisfies cost efficiency, B_R constitutes a refinement of $B_{\succeq}$.

We will consider two complete orders on the edge set which both satisfy unanimity. They are well-studied within social choice theory and are based on respectively the classical utilitarian principle and the Nash principle.

For each cest problem $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ the order based on the classical utilitarian principle, $\Sigma(w)$, is defined by

$$\forall e, f \in E: \qquad e\Sigma(w)f \Leftrightarrow \sum_{t=1}^{m} w_t(e) \leq \sum_{t=1}^{m} w_t(f).$$

The order based on the Nash principle, $\Pi(w)$, is defined only if $w(e) > 0$ for all $e \in E$. However, if this is the case then $\Pi(w)$ is given by

$$\forall e, f \in E: \qquad e\Pi(w)f \Leftrightarrow \prod_{t=1}^{m} w_t(e) \leq \prod_{t=1}^{m} w_t(f).$$

Notice that a well-defined order derived from the Nash principle is equivalent to the order obtained from the classical utilitarian principle where a logarithmic transformation is performed on each cost component.

The next theorem provides characterizations of both orders discussed above. The theorem is stated in the framework of multiple criteria cest problems. A proof of the first statement has been given by d'Aspremont and Gevers (1977) in the setting of social choice theory. For the last part see Moulin (1988), Theorem 2.3.

Theorem 4 *Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ with $|N| \geq 2$. Let R denote a complete order on the edge set E. Then*

- *R satisfies unanimity, neutrality, zero-independence and independence if and only if*
$$\forall e, f \in E: \qquad eRf \Leftrightarrow e\Sigma f.$$
- *Assume $w(e) > 0$ for all $e \in E$. Then, R satisfies unanimity, neutrality, scale-independence and independence if and only if*
$$\forall e, f \in E: \qquad eRf \Leftrightarrow e\Pi f.$$

It is an immediate consequence of Theorem 4 that there does not exist a complete order on the edge set which satisfies unanimity, neutrality, scale- and zero-independence and independence.

A Bird-rule can be defined with respect to each of the orders, Σ and Π, since they both satisfy unanimity. We will denote these two Bird-rules by respectively B_Σ and B_Π. The following lemma gives an equivalent definition of B_Σ.

Lemma 5 *Let $M \in \mathcal{M}$. Then*

$$\begin{aligned} B_\Sigma(M) &= \sigma_\Sigma(M) \\ &= \{(w((p_T(1),1)),\ldots,w((p_T(n),n))) \mid T \in argmin_{T \in \mathcal{T}_N} \sum_{t=1}^{m} w_t(T)\}. \end{aligned}$$

Proof. Let $\overline{T} \in \text{argmin}_{T \in \mathcal{T}_N} \sum_{t=1}^{m} w_t(T)$ and notice that $\overline{T}$ can be found by a greedy algorithm applied to the mcst problem $\langle N \cup \{*\}, E, \overline{w} \rangle$ where the cost for each edge $e \in E$ is defined by $\overline{w}(e) = \sum_{t=1}^{m} w_t(e)$. It follows from the definition of Σ and $\sigma_\Sigma(M)$ that

$$\sigma_\Sigma(M) = \{(w((p_T(1), 1)), \ldots, w((p_T(n), n))) \mid T \in \text{argmin}_{T \in \mathcal{T}_N} \sum_{t=1}^{m} w_t(T)\}.$$

By the definition of B_Σ we have $B_\Sigma(M) \subseteq \sigma_\Sigma(M)$. Furthermore, for each $T \in \mathcal{T}_N$ with $\sum_{t=1}^{m} w_t(T)$ minimal it holds that there does not exist any element in $\mathcal{T}_N$ which costs dominate $w(T)$. Therefore, $B_\Sigma(M) = \sigma_\Sigma(M)$. □

For σ_Π and B_Π it can be shown in a similar way that $B_\Pi(M) \subseteq \sigma_\Pi(M) = \{(w((p_T(1), 1)), \ldots, w((p_T(n), n))) \mid T \in \text{argmin}_{T \in \mathcal{T}_N} \prod_{t=1}^{m} w_t(T)\}$ for each $M \in \mathcal{M}$ where $w(e) > 0$ for all $e \in E$.

In the following we will concentrate on B_Σ. To achieve a characterization of B_Σ based on the introduced properties we first show that B_Σ does not satisfy scale-independence and converse leaf-consistency.

Example 1, continued.
Let M denote the cest problem corresponding to figure 1. Then $B_\Sigma(M) = \{((2,4),(4,2),(3,2))\}$.

Now, let $a = (1,3)$. Then, $B_\Sigma(N \cup \{*\}, E, a * w) = \{((2,12),(6,3),(3,6))\} \neq \{(a * (2,4), a * (4,2), a * (3,2))\}$. We conclude that B_Σ does not satisfy scale-independence.

Next, consider the tree $\langle N \cup \{*\}, T \rangle = \langle N \cup \{*\}, \{(*,3),(1,3),(1,2)\} \rangle$. It is easy to see that $\langle N \cup \{*\}, T \rangle$ is efficient. Let $x = ((2,4),(3,4),(3,2))$ and hence $\sum_{i \in N} x_i = w(T)$. The only leaf in $\langle N \cup \{*\}, T \rangle$ is node 2 and $B_\Sigma(\{1,3,*\}, E^{-2}, w) = \{((2,4),(3,2))\} = \{(x_1, x_3)\}$. It follows that B_Σ does not satisfy converse leaf-consistency since $x \notin B_\Sigma(M)$.

However, B_Σ satisfies most of the listed properties for a solution concept on $\mathcal{M}$ as it is stated in the following proposition. A proof can be given by means of Lemma 5.

Proposition 6 *B_Σ satisfies non-emptiness, cost efficiency, anonymity, neutrality, zero-independence and leaf-consistency.*

It follows from Theorem 4 and Proposition 6 that B_Σ is consistent with respect to neutrality, zero-independence and unanimity. Furthermore, it is easy to see that B_Σ is consistent w.r.t. anonymity. Thus, B_Σ is a consistent solution concept.

Next, since B_Σ is leaf-consistent and cost efficient and $B_\succeq$ satisfies non-emptiness, cost efficiency and converse leaf-consistency (Propositions 1 and 6) Lemma 2 shows that $B_\Sigma(M) \subseteq B_\succeq(M)$ for all $M \in \mathcal{M}$. Furthermore, it is easy to find a cest problem M for which $B_\Sigma(M) \subset B_\succeq(M)$. Thus, B_Σ is a refinement of $B_{\succeq}$. To give a characterization of B_Σ we describe the notion of maximality and the so-called tree-additivity property. A solution concept, σ, on $\mathcal{M}$ is called *maximal* with respect to a list of properties, if σ satisfies the various properties and if the cost allocations for every solution concept on $\mathcal{M}$ that satisfies the same properties are included in the cost allocations obtained by σ. Secondly, a solution concept, σ, on $\mathcal{M}$ is said to satisfy *tree-additivity* if the following holds for all pairs $M = \langle N \cup \{*\}, E, w\rangle$, $\overline{M} = \langle N \cup \{*\}, E, \overline{w}\rangle \in \mathcal{M}$.

If there exists an $x \in \sigma(M)$ and a $y \in \sigma(\overline{M})$ for which $\sum_{i\in N} x_i = w(T)$ and $\sum_{i\in N} y_i = \overline{w}(T)$ for a spanning tree $\langle N \cup \{*\}, T\rangle$, then there exists a $z \in (\mathbb{R}^m_+)^N$ such that

$$z \in \sigma(N \cup \{*\}, E, w + \overline{w}) \text{ and } \sum_{i\in N} z_i = (w + \overline{w})(T).$$

Tree-additivity states that if a spanning tree supports a solution for two different cost-functions then it should also support a solution for the cest problem where the two cost-functions are added together.

Theorem 7 *B_Σ is the maximal solution concept on $\mathcal{M}$ which satisfies non-emptiness, cost efficiency, neutrality, leaf-consistency and tree-additivity.*

Proof. It follows from Proposition 6 that B_Σ satisfies non-emptiness, efficiency, neutrality and leaf-consistency. In order to prove that B_Σ satisfies tree-additivity let $M = \langle N \cup \{*\}, E, w\rangle$, $\overline{M} = \langle N \cup \{*\}, E, \overline{w}\rangle \in \mathcal{M}$ be such that there exists an $x \in B_\Sigma(M)$, a $y \in B_\Sigma(\overline{M})$ and a spanning tree $\langle N \cup \{*\}, T\rangle$ for which $\sum_{i\in N} x_i = w(T)$ and $\sum_{i\in N} y_i = \overline{w}(T)$. Let $z, \overline{z} \in (\mathbb{R}^m_+)^N$ be defined by $z_i = w((p_T(i), i))$ and $\overline{z}_i = \overline{w}((p_T(i), i))$ for all $i \in N$. Then, $\sum_{i\in N}[z_i + \overline{z}_i] = w(T) + \overline{w}(T)$. Moreover, we have by Lemma 5

$$\sum_{t=1}^{m}[w_t(T) + \overline{w}_t(T)] = \min_{\tilde{T}\in\mathcal{T}_N} \sum_{t=1}^{m} w_t(\tilde{T}) + \min_{\tilde{T}\in\mathcal{T}_N} \sum_{t=1}^{m} \overline{w}_t(\tilde{T}) \leq \min_{\tilde{T}\in\mathcal{T}_N} \sum_{t=1}^{m}(w_t(\tilde{T}) + \overline{w}_t(\tilde{T}))$$

and thus $z + \overline{z} \in B_\Sigma(N \cup \{*\}, E, w + \overline{w})$.

Next, let σ denote a solution concept which satisfies the five properties. It suffices to prove that $\sigma(M) \subseteq B_\Sigma(M)$ for all $M \in \mathcal{M}$. The proof uses induction in the number of agents. First, let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ be such that $|N| = 1$. Then, by non-emptiness and cost efficiency of $\sigma(M)$ we get $\sigma(M) = \{w((*, 1))\}$. From the definition of B_Σ we have $B_\Sigma(M) = \{w((*, 1))\}$. Thus, $\sigma(M) = B_\Sigma(M)$. Secondly, let $k \in \mathbb{N}$. Assume $\sigma(M) \subseteq B_\Sigma(M)$ for all $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ with $|N| = k - 1$. Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ be such that $|N| = k$. By non-emptiness of σ we have a $x \in \sigma(M)$. We will show that $x \in B_\Sigma(M)$. There exists an efficient spanning tree $\langle N \cup \{*\}, T\rangle$ for which $\sum_{i\in N} x_i = w(T)$ since σ satisfies cost efficiency. Let $i \in N$ denote a leaf in $\langle N \cup \{*\}, T\rangle$. Then, by leaf-consistency of σ and the induction hypothesis we have

$$x^{-i} \in \sigma((N \setminus \{i\}) \cup \{*\}, E^{-i}, w) \subseteq B_\Sigma((N \setminus \{i\}) \cup \{*\}, E^{-i}, w) \qquad (2)$$

and $\sum_{j\in N\setminus\{i\}} x_j = w(T\cap E^{-i})$. Therefore, $x_i = w((p_T(i),i))$. Moreover, by (2) and the definition of B_Σ there exists a spanning tree $\langle (N\setminus\{i\})\cup\{*\},\widehat{T}\rangle$ such that $x_j = w((p_{\widehat{T}}(j),j))$ for all $j\in N\setminus\{i\}$. Now, let $\overline{T}\in\mathcal{T}_N$ be defined by $\overline{T} = \{(p_T(i),i)\}\cup\widehat{T}$. Notice that $\sum_{j\in N} x_j = w(T) = w(\overline{T})$. By Lemma 5 it suffices to show that

$$\sum_{j\in N}\sum_{t=1}^{m} w_t((p_{\overline{T}}(j),j)) = \min_{T\in\mathcal{T}_N}\sum_{t=1}^{m} w_t(T)$$

or equivalent, see Ahuja, Magnanti and Orlin (1993), Theorem 13.1, that for each edge e in the tree $\langle N\cup\{*\},\overline{T}\rangle$ no edge f exists such that $\sum_{t=1}^m w_t(f) < \sum_{t=1}^m w_t(e)$ and $\langle N\cup\{*\},(\overline{T}\setminus\{e\})\cup\{f\}\rangle$ is a spanning tree. So, assume that there exists an $e\in\overline{T}$ and an $f\in E\setminus\overline{T}$ with $\sum_{t=1}^m w_t(f) < \sum_{t=1}^m w_t(e)$ and $\{f\}\cup(\overline{T}\setminus\{e\})\in\mathcal{T}_N$. For $s=1,\ldots,m$ let the permutation $\pi[s]$ on $\{1,\ldots,m\}$ be defined by

$$\pi[s](t) = \begin{cases} t+s-1 & \text{if} \quad t+s\le m+1. \\ t+s-m-1 & \text{if} \quad t+s > m+1. \end{cases}$$

Notice that

$$\sum_{s=1}^{m} w^{\pi[s]}(e) = (\sum_{t=1}^{m} w_t(e),\ldots,\sum_{t=1}^{m} w_t(e)) > (\sum_{t=1}^{m} w_t(f),\ldots,\sum_{t=1}^{m} w_t(f)).$$

By neutrality of σ it follows $(x_1^{\pi[s]},\ldots,x_n^{\pi[s]})\in\sigma(N\cup\{*\},E,w^{\pi[s]})$ for $s = 1,\ldots,m$. Also $\sum_{i=1}^n x_i^{\pi[s]} = w^{\pi[s]}(T)$ for each $s\in\{1,\ldots,m\}$, so by tree-additivity of σ there exists a $z\in\sigma(N\cup\{*\},E,\sum_{s=1}^m w^{\pi[s]})$ for which $\sum_{j\in N} z_j = \sum_{s=1}^m w^{\pi[s]}(\overline{T})$. This leads to a contradiction with the cost efficiency of σ. □

In the following we will show that the different properties mentioned in Theorem 7 are logically independent. First, notice that the two solution concepts discussed immediately after Theorem 3 satisfy the five different properties except respectively non-emptiness and efficiency. Secondly, consider the solution concept defined for all $M = \langle N\cup\{*\},E,w\rangle\in\mathcal{M}$ by

$$\sigma(M) = \{x\in(\mathbb{R}_+^m)^N \mid \sum_{i\in N} x_i = w(T), T\in \text{argmin}_{T\in\mathcal{T}_N}\sum_{t=1}^{m} w_t(T)\}.$$

It can be shown that σ satisfies non-emptiness, efficiency, neutrality and tree-additivity. However, σ is not leaf-consistent. Thirdly, notice that the solution concept defined by (1) satisfies the five properties used in Theorem 7 except neutrality. Finally, by Proposition 1 we have that $B_\succeq$ satisfies non-emptiness, efficiency, neutrality and leaf-consistency. However, $B_\succeq$ does not satisfy tree-additivity.

Though B_Σ is the maximal solution concept, it is not the only solution concept that satisfies the five properties used in Theorem 7. Regard for example the following one. Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and $\sigma(M)$ be the allocation in $B_\Sigma(M)$ that is supported by the eventually unique spanning tree with the least maximal degree among the trees which edge sets occur in $\mathcal{T}_\Sigma(M)$. If more than one spanning tree has this least maximal degree $\sigma(M)$ consists of the allocations in $B_\Sigma(M)$. Formally,

$$\sigma(M) = \begin{cases} \{x \in B_\Sigma(M) \mid \sum_{i \in N} x_i = w(T)\} & \text{if } \operatorname{argmin}_{T \in \mathcal{T}_\Sigma(M)} \max_{i \in N \cup \{*\}} d_T(i) = \{T\} \\ B_\Sigma(M) \text{ otherwise} \end{cases}$$

where $d_T(i)$ denotes for each $i \in N \cup \{*\}$ the *degree* of node i in $\langle N \cup \{*\}, T\rangle$, i.e. the number of edges in the spanning tree $\langle N \cup \{*\}, T\rangle$ which are incident to node i. We have $\sigma(M) \subseteq B_\Sigma(M)$ for all $M \in \mathcal{M}$. Moreover, σ satisfies the properties listed in Theorem 7 and one can find a multiple criteria cest problem $\langle N \cup \{*\}, E, w\rangle$ for which $\sigma(N \cup \{*\}, E, w)$ is strictly included in $B_\Sigma(N \cup \{*\}, E, w)$. Hence, the five properties do not define a unique solution concept.

We will not discuss B_Π into detail. Our statement considers the discrepancy between the order Π and the related rule B_Π. By means of Example 1 it is easy to see that B_Π does not satisfy cost efficiency, though unanimity is a property of Π. Thus, B_Π is not consistent w.r.t. unanimity and by Theorem 4 we conclude that B_Π is not a consistent solution concept. On the other hand, B_Π does satisfy anonymity, neutrality and scale-independence and it is consistent w.r.t. these properties since Π satisfies the corresponding properties.

Example 1, continued.
Let M denote the multiple criteria cest problem associated with figure 1. It is easy to see that $B_\Pi(M) = \{((1,6),(6,1),(3,2))\}$. Now, let $x \in B_\Pi(M)$. Then, we have $\sum_{i \in N} x_i = (10,9) > (9,8) = w((1,3)) + w((*,3)) + w((2,3))$. We conclude that B_Π does not satisfy cost efficiency.

4 Stability of cost allocations

The aim of this section is to present two stability conditions for cost allocation rules on multiple criteria cest problems and to show that the Bird-rule based on unanimity provides stable cost allocations. A cost allocation is said to be *stable* if no coalition of agents can improve upon its payment by constructing a spanning tree on its own. Several stability conditions can be defined since one can have various interpretations of an improvement within the multiple criteria framework. A set of cost allocations which satisfy a given collection of stability conditions is called a *core*.

We will consider two core concepts: the Cartesian product core (CPC) and the stable outcome core (SOC).

4.1 The Cartesian Product Core

The Cartesian product core of a cest problem $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ is generated by the cores of the related single criterion mcst problems. For each criterion $t \in \{1, \ldots, m\}$ the related mcst problem Mt is defined as the problem $\langle N \cup \{*\}, E, w^t\rangle$ where $w^t(e) = (w(e))_t$ for all $e \in E$. It represents the problem in which all the attention is focussed on the t'th criterion. With regard to the t'th criterion the cost by which a non-empty coalition of agents, $S \subseteq N$, can build its own spanning tree is given by
$c_{Mt}(S) = \min\{w^t(T) \mid \langle S \cup \{*\}, T\rangle$ is a spanning tree$\}$. It has been proved by Granot and Huberman (1981) that each mcst problem has a non-empty core. The core of Mt is defined by

$$Core(Mt) = \{y \in \mathbb{R}^N_+ \mid \sum_{i \in N} y_i = c_{Mt}(N), \ \sum_{i \in S} y_i \leq c_{Mt}(S) \quad \forall S \subseteq N, S \neq \emptyset\}.$$

$Core(Mt)$ is the set of cost allocations for the t'th criterion by which no coalition can improve upon its payment of the criterion by constructing its own network.

The *Cartesian product core* for M, denoted $CPC(M)$, is defined by

$$\begin{aligned} CPC(M) &= \textstyle\prod_{t=1}^m Core(Mt) \\ &= \{x \in (\mathbb{R}^m_+)^N \mid (x^t_i)_{i \in N} \in Core(Mt) \text{ for all } t \in \{1, \ldots, m\}\}. \end{aligned}$$

Clearly, $CPC(M)$ is non-empty for each $M \in \mathcal{M}$. For cost allocations which belong to the Cartesian product core no coalition can improve upon its payment in any of the criteria. Hence, no coalition has an incentive to split and strive for its own spanning tree. Moreover, the Cartesian product core is related to the so-called multicriteria core defined for multicriteria n-person games in characteristic function form by Bergstresser and Yu (1977).

Next, assume $m \geq 2$ and consider two different criteria, $s, t \in \{1, \ldots, m\}$. Since the set of efficient spanning trees by which the t'th criterion is minimized might differ considerable from the set by which the s'th criterion is minimized, there may not exists a spanning tree with an edge set T such that $w^t(T) = c_{Mt}(N)$ and $w^s(T) = c_{Ms}(N)$. In particular a Cartesian product core allocation may not cover the cost of any spanning tree in the network. We have, however,

Proposition 8 *Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and $x \in CPC(M)$. Then, x covers the costs of a connected and spanning subnetwork on $N \cup \{*\}$ if and only if there exists a spanning tree $\langle N \cup \{*\}, T\rangle$ with an edge set $T \in \mathcal{T}_N$ by which all the criteria are minimized, simultaneously.*

Proof. Let $T \in \mathcal{T}_N$ be such that $w(T) \leq w(\overline{T})$ for all $\overline{T} \in \mathcal{T}_N$. Thus, $w^t(T) \leq w^t(\overline{T})$ for all $\overline{T} \in \mathcal{T}_N$ and for all criteria $t = 1, \ldots, m$. So $w^t(T) = c_{Mt}(N) = \sum_{i \in N} x^t_i$ for $t = 1, \ldots, m$. □

The Cartesian product core is a natural solution concept for multiple criteria problems but Proposition 8 reflects its major drawback. For multiple criteria cest problems it only provides cost allocations which cover the costs of a connected and spanning subnetwork if the problem has non-conflicting criteria.

4.2 The Stable Outcome Core

The Cartesian product core takes the valuation of the different criteria into account separately. The stable outcome core on the other hand is a solution concept based on stability for each coalition of agents where all the criteria are taken into account. Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and assume $m \geq 2$. The stable outcome core, $SOC(M)$, is defined as the set of allocations $x \in (\mathrm{IR}^m_+)^N$ for which

- There exists an efficient spanning tree with edge set $T \in \mathcal{T}_N$ such that $\sum_{i \in N} x_i = w(T)$.
- For all non-empty coalitions $S \subseteq N$ there does not exist a spanning tree $\langle S \cup \{*\}, T\rangle$ such that $w(T) \leq \sum_{i \in S} x_i$ and $w(T) \neq \sum_{i \in S} x_i$.

A related version of the stable outcome core can be found in the literature on multi-commodity games, see Derks and Tijs (1986) and Nouweland, Aarts and Borm (1989).

A stable outcome core allocation covers the costs of a spanning and connected subnetwork with node set $N \cup \{*\}$. Furthermore, the last statement in the definition implies that an arbitrary coalition cannot improve upon its cost allocation in a criterion by constructing a network of its own without increasing its payment of at least one other criterion.

In order to show that the Bird-rule based on unanimity $B_{\succeq}$ provides stable outcome core allocations for each problem $M \in \mathcal{M}$ we need the following lemma. A proof can be found in Granot and Huberman (1981).

Lemma 9 *For all $T \in \mathcal{T}_N$ and each spanning tree $\langle S \cup \{*\}, D\rangle$, where $S \subseteq N$, $S \neq \emptyset$: the graph $\langle N \cup \{*\}, D \cup T \setminus \{(p_T(i), i) \mid i \in S\}\rangle$ is a spanning tree.*

Replacing, in a given tree with edge set $T \in \mathcal{T}_N$, for all agents $i \in S$ the edges to their predecessors by the edges of a spanning tree $\langle S \cup \{*\}, D\rangle$ yields a spanning tree with edge set in $\mathcal{T}_N$.

The main result of this section is the following theorem.

Theorem 10 *$B_{\succeq}(M) \subseteq SOC(M)$ for each $M \in \mathcal{M}$.*

Proof. Let $M = \langle N \cup \{*\}, E, w\rangle \in \mathcal{M}$ and let $x \in B_{\succeq}(M)$. By the definition of $B_{\succeq}$ and Proposition 1 there exists an efficient spanning tree with edge set $T \in \mathcal{T}_N$ such that $x_i = w((p_T(i), i))$ for all $i \in N$. So, $\sum_{i \in N} x_i = w(T)$.

Suppose that there exists a non-empty coalition $S \subseteq N$ such that $\sum_{i \in S} x_i$ is dominated by the cost of a spanning tree $\langle S \cup \{*\}, D\rangle$. Hence, $w(D) \leq \sum_{i \in S} x_i$ and $w(D) \neq \sum_{i \in S} x_i$. By Lemma 9 it follows $\overline{T} = D \cup T \setminus \{(p_T(i), i) \mid i \in S\} \in \mathcal{T}_N$. Moreover

$$w(T) = \sum_{i \in N} x_i = \sum_{i \in N \setminus S} w((p_T(i), i)) + \sum_{i \in S} x_i \geq \sum_{i \in N \setminus S} w((p_T(i), i)) + w(D) = w(\overline{T})$$

and in a similar way it follows $w(T) \neq w(\overline{T})$. This contradicts the cost efficiency of T. □

An immediate consequence of Theorem 10 is the non-emptiness of the stable outcome core. Moreover, if there for a cest problem $M \in \mathcal{M}$ exists a spanning tree with node set $N \cup \{*\}$ by which all the criteria are minimized, it can be proved in a similar way that $B_{\succeq}(M) \subseteq CPC(M)$. In such a case we also have that $CPC(M) \subseteq SOC(M)$.

5 Conclusion

In this paper we have studied the cost allocation problem which arises when a group of users, which are concerned with the allocation of multiple costs, have to be connected to a common source. The reason for doing so was at least twofold. First of all, by taking a number of various objectives into consideration a more detailed description of the cost allocation problem is obtained. Secondly, it was shown that findings from social choice theory and cooperative game theory can be used to analyze a multiple criteria combinatorial optimization problem. The relationship between the properties of, respectively, orders on the edge set and of cost allocations was studied by means of characterizations of social welfare functions based on utilitarianism. The stability of cost allocations was investigated using core concepts from cooperative game theory.

An implementation of a cost allocation rule into Kruskal's algorithm (Kruskal, 1956), that consist of a method of adding 'cheap' edges, not constructing cycles, without the obligation to have a connected set of edges in each step, requires assumptions on the division of the cost of the added edge among the agents in the two joined components. Feltkamp (1994b) presented some ideas in this setting for real-valued cost functions. This approach and the possibility to start with other orders, how to choose these and to characterize the resulting rules, as well as a study of strategic properties of cost allocation rules for cest problems are topics for further research.

Acknowledgements

We would like to thank Peter Borm and Stef Tijs for several stimulating discussions and helpful comments.

References

Ahuja, R.K., Magnanti, T.L., and Orlin, J.B. (1993), *Network Flows*, Prentice Hall, Englewood Cliffs, New Jersey.

Andersen, K.A., Jörnsten, K., and Lind, M. (1995), "On bicriterion minimal spanning trees: an approximation", Publication no: 95/13, Department of Operations Research, University of Aarhus, Denmark. To appear in *Computers & Operations Research.*

d'Aspremont, C., and Gevers, L. (1977), "Equity and the informational basis of collective choice", *Review of Economic Studies* 44, 199-209.

Bergstresser, K., and Yu, P.L. (1977), "Domination structures and multicriteria problems in n-person games", *Theory and Decision* 8, 5-48.

Bird, C.G. (1976), "On cost allocation for a spanning tree: a game theoretic approach", *Networks* 6, 335-350.

Claus, A., and Kleitman, D.J. (1973), "Cost allocation for a spanning tree", *Networks* 3, 289-304.

Derks, J.M.J., and Tijs, S.H. (1986), "Totally balanced multi-commodity games and flow games", *Methods of Operations Research* 54, 335-347.

Dijkstra, E.W. (1959), "A note on two problems in connection with graphs", *Numerische Mathematik* 1, 269-271.

Feltkamp, V., Tijs, S.H., and Muto, S. (1994a), "Bird's tree allocation revisited", Discussion paper No. 9435, CentER for Economic Research, Tilburg University, The Netherlands.

Feltkamp, V., Tijs, S.H., and Muto, S. (1994b), "On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems", Discussion paper No. 9410, CentER for Economic Research, Tilburg University, The Netherlands.

Granot, D., and Huberman, G. (1981), "Minimum cost spanning tree games", *Mathematical Programming* 21, 1-18.

Hamacher, H.W., and Ruhe, G. (1994), "On spanning tree problems with multiple objectives", *Annals of Operations Research* 52, 209-230.

Kruskal Jr., J.B. (1956), "On the shortest spanning subtree of a graph and the traveling salesman problem", *Proceedings of the American Mathematical Society* 72, 48-50.

Moulin, H. (1988), *Axioms of Cooperative Decision Making*, Econometric Society Monographs No. 15, Cambridge University Press, Cambridge.

Nouweland, A. van den, Aarts, H., and Borm, P. (1989), "Multi-commodity games", *Methods of Operations Research* 63, 329-338.

Prim, R.C. (1957), "Shortest connection networks and some generalizations", *The Bell System Technical Journal* 36, 1389-1401.

Serafini, P., "Some considerations about computational complexity for multi objective combinatorial problems", in: J. Jahn and W. Krabs (eds.) (1986), *Recent Advances and Historical Development of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems no. 294, Springer-Verlag, Berlin, 222-232.

Steuer, R.E. (1986), *Multiple Criteria Optimization: Theory, Computation, and Application*, John Wiley & Sons, New York

Vincke, P. (1992), *Multicriteria Decision-aid*, John Wiley & Sons, Chichester.