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Abstract

This paper shows how problems in 'non life'-insurance and 'non life'-reinsurance can be modelled simultaneously as cooperative games with stochastic payoffs. Pareto optimal allocations of the risks faced by the insurers and the insureds are determined. It is shown that the core of the corresponding insurance games is nonempty. Moreover, it is shown that specific core allocations are obtained when the zero utility principle is used for calculating premiums. Finally, game theory is used to give a justification for subadditive premiums.

KEYWORDS: (re)insurance, zero utility principle, cooperative game theory, Pareto optimality, core.

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1 Introduction

Classical actuarial theory has mainly focused on insurance problems from the insurer's point of view. Most of the attention is dedicated to the determination of an appropriate premium for the insured risk. Obviously, the nature of the risk is a substantial factor in this process. In this respect, there is an important difference whether the risk arises from the 'life' or the 'non life' sector. For the first, there is a profusion of statistical data on the expected remaining life available, which makes the calculation of an appropriate premium relatively easy. For the latter, however, things are a bit more complicated. In 'non life' insurance the risk is not always easy to capture in a statistical framework. Therefore, several premium calculation principles have been developed to serve this purpose, see for instance *Goovaerts, De Vylder* and *Haezendonck (1984)*.

These calculation principles, however, only take into account a part of the insurer's side of the deal. More precisely, they consider whether the premium is high enough to cover the risk. Competition arising from the presence of other insurers on the one hand, and the interests of the insured, on the other hand, are at least to a large extent ignored. It is, of course, better to consider all these aspects in an insurance deal, since the premium should not only be high enough to compensate the insurer for bearing the individual's risk, it should also be low enough so that an individual is willing to insure his risk (or a part of it) for this premium. The economic models for (re)insurance markets, which were developed from the 1960's on (cf. *Borch (1962a)* and *Bühlmann (1980), (1984)*), consider indeed the interests of both the insurers and the insureds. These models incorporate the possibility to study problems concerning fairness, Pareto optimality and market equilibrium. *Bühlmann (1980)*, for example, shows that the Esscher calculation principle results in a Pareto optimal outcome.

More recently, also game theory is used to model the interests of all parties in an insurance problem. Cooperative game theory focuses on the gains arising in multi person interactive decision situations when a part of the population decides to cooperate. The primary concerns are which coalitions ultimately form and how to divide the gains among the members of each coalition formed. This theory finds many applications in, for example, cost allocation (cf. *Moulin (1988)*) and combinatorial optimization problems (cf *Tijs (1992)*). Cost allocation problems arise when several groups of people with possibly different interests are involved in a joint project. The problem is then who pays what part of the total costs. A well known example of such a problem was faced by the Tennessee Valley Authority in the US during the 1930's. Briefly spoken, this project was designed to control the course of the Tennessee river. The problem was how the costs should be allocated to the groups that benefited from the project (see *Ransmeier (1942)*). In combinatorial optimization one can think of the construction of a network to connect households to an electrical powerplant such that everybody is supplied with electricity. Problems that arise here are which network to construct and how to allocate the costs of it.

Besides the applications just mentioned, cooperative game theory has been applied in insurance problems. Especially when insurance companies incorporate subadditive premiums, individuals can save on the premium if they decide to take a collective insurance instead of an individual one. This situation is discussed in *Alegre* and *Mercè Claramunt (1995)*. Other applications of cooperative game theory in insurance can be found in *Borch (1962b)* and *Lemaire (1991)*.

Cooperative game theory, however, still has to establish itself as an appropriate tool for exploring insurance problems. A reason for this is due to the inability of traditional cooperative game theory to incorporate the uncertainties, which play such an important role in insurance. Indeed, in classical theory the gains coalitions can obtain by cooperating are assumed to be known with certainty. Recently, however, *Suijs, Borm, De Waegenaere* and *Tijs (1995)* introduced a model, which overcomes this problem. They introduced cooperative games, which allow that the gains coalitions can make are random variables.

This paper shows how the abovementioned game theoretical model can be applied to examine problems in insurance. The model we introduce incorporates insurance of personal losses as well as reinsurance of the portfolios of insurance companies. By cooperating with insurance companies individual persons are able to transfer their future random losses to the cooperating insurance companies. Thus, in doing so, individual persons conclude an insurance deal. Similarly, by cooperating with other insurers an insurance company can transfer (parts of) her insurance portfolio to the other insurers. So, the insurance company concludes a reinsurance deal.

In this model our attention is focused on Pareto optimal allocations of the risks, and on the question which premiums are fair to charge for these risk exchanges. A Pareto optimal allocation is such that there exists no other allocation which is better for all persons and insurers taking part in the game. We show that there is essentially a unique Pareto optimal allocation of risk. It will appear that this Pareto optimal allocation of the risk is independent of the insurance premiums that are paid for these risk exchanges. For determining fair premiums, we look at the 'core' of the reinsurance game. A core allocation divides the gains of cooperation in such a way that no subcoalition has an incentive to split off. We show that the core is nonempty for insurance games. Moreover, we show that the zero utility principle for calculating premiums (see *Goovaerts, De Vylder* and *Haezendonck (1984)*) results in a core allocation.

The paper is organized as follows. In Section 2 we introduce insurance games. We indicate which allocations are Pareto optimal and show that the core of reinsurance games is nonempty. In particular, we prove that the zero utility principle for calculating premiums results in a core allocation. In Section 3 we use game theory to explain why subadditive premiums are attractive for insurance companies. Finally, Section 4 provides some concluding remarks.

2 Insurance games

This section models insurance problems as cooperative games with stochastic payoffs as introduced in *Suijs et al. (1995)*. We show that by cooperating, individuals and insurers can redistribute their risks and, consequently, improve their welfare. First, we need to specify the players of the game. A player can be one of two types. A player is either an individual person or an insurer. The set of individual persons is denoted by N_P and the set of insurers is denoted by N_I. Hence, the players of the game are denoted by the set $N_I \cup N_P$.

Next, all players are assumed to be risk averse expected utility maximizers. This means that a player prefers one risk to another if the expected utility of the first exceeds the expected utility of the latter. Note that insurers are also assumed to be risk averse. Furthermore, we assume that the utility function for each player $i \in N_I \cup N_P$ can be described by $u_i(t) = \beta_i e^{-\alpha_i t}$, $(t \in \mathbb{R})$, with $\beta_i < 0$, $\alpha_i > 0$. Since $\beta_i < 0$ and $\alpha_i > 0$ imply concavity for the utility functions u_i, we have that each player is risk averse. So, for each random loss X this means that a player prefers receiving the expected loss $E(X)$ with certainty to receiving the random loss X. Moreover, the absolute measure of risk aversion for player i is constant and equals α_i. Hence, player i is more risk averse than player j if $\alpha_i > \alpha_j$. By changing the signs of the parameters β_i and α_i the utility function becomes convex, and, as a consequence, the player will be risk loving. Regarding the situations where one or more risk neutral/loving insurers are involved we confine ourselves to a brief discussion later on. Finally, note that since the utility functions are exponential that the expected utility of a random loss X need not always exist. In this paper, however, we implicitly assume that the risks are such that the expected utility exists.

To describe the future random losses of a player, we introduce the following notation. Let $\{Y_k \sim \text{Exp}(\mu_k) | k \in \mathcal{K}\}$ be a finite collection of independent exponentially distributed random variables. These variables can be interpreted as describing the random losses that could occur to individuals. They describe, for example, the monetary damages caused by cars, bikes, fires, or other people. The loss X_i for player i then equals

$$X_i \;=\; \sum_{k \in \mathcal{K}} f_{ik} Y_k, \tag{1}$$

where $0 \leq f_{ik} \leq 1$ for all $k \in \mathcal{K}$. In particular we define $\mathcal{K}_j = \{k \in \mathcal{K} | f_{jk} \neq 0\}$ for all $j \in N_I \cup N_P$. So, if player i is an insurer the loss X_i represents the loss of insurer i's portfolio. Moreover, the insurance portfolio X_i can be a combination of many random losses. In fact, they are the fractions f_{ik} of the losses that individuals have insured at this particular insurer. If player i is an individual person then X_i represents the random loss this individual might want to insure. Note that the portfolios of different players may be stochastically dependent, albeit in a very specific way. Indeed, an individual can insure part of his loss at insurer i and another part of the same loss at insurer j.

Now, let us focus on the possibilities that occur when players decide to cooperate.

Therefore, consider a coalition S of players. If the members of S decide to cooperate, the total loss $X_S \in L^1(\mathbb{R})$ of the coalition equals the sum of the individual losses of the members of S, i.e., $X_S = \sum_{i \in S} X_i$. Subsequently, the loss X_S has to be allocated to the members of S.

In allocating the loss X_S we distinguish the following three cases. In the first case, coalition S consists of insurers only. So, $S \subset N_I$. Such a coalition is assumed to allocate the loss X_S in the following way. First, a coalition S allocates a fraction $r_{ij} \in [0,1]$ of the loss X_j of insurer $j \in S$ to insurer $i \in S$. So, insurer i bears a total loss of $\sum_{j \in S} r_{ij} X_j$, where $r_{ij} \in [0,1]$ and $\sum_{k \in S} r_{kj} = 1$. This is called proportional reinsurance. This part of the allocation of X_S for coalition S can be described by a matrix $R \in \mathbb{R}_+^{S \times S}$, where r_{ij} represents the fraction insurer i bears of insurer j's loss X_j. Second, the insurers are allowed to make deterministic transfer payments. This means that each insurance company $i \in S$ also receives an amount $d_i \in \mathbb{R}$ such that $\sum_{j \in S} d_j = 0$. These transfer payments can be interpreted as the aggregate premium insurers have to pay for the actual risk exchanges.

In the second case, coalition S consists of individual persons only. So, $S \subset N_P$. Then the gains of cooperation are assumed to be nil. That is, we do not allow any risk exchanges between the persons themselves. For, that is what the insurers are for in the first place. As a result, the only allocations (d, R) of X_S which are allowed are of the form $r_{ii} = 1$ for all $i \in S$ and $r_{ij} = 0$ for all $i, j \in S$ with $i \neq j$. If, however, one wants to allow risk exchanges by the individual persons then the resulting situation is similar to the case where only insurance companies cooperate. Consequently, allocations can be described in the same way.

In the third and last case, coalition S consists of both insurers and individual persons. So, $S \subset N_I \cup N_P$. Now cooperation can take place in two different ways. First, insurers are allowed to exchange (parts of) their portfolios with other insurers. Second, individual persons may transfer (parts of) their risks to insurers. Again, individual persons are not allowed to exchange risks with each other. Moreover, we assume that insurers cannot transfer (parts of) their portfolios to individuals.

Summarizing we can say that there a several restrictions on allocations. To be more precise, denote by S_I the set of insurers of coalition S, i.e., $S_I = S \cap N_I$, and by S_P the set of individuals of coalition S, i.e., $S_P = S \cap N_P$. Then an allocation $(d, R) \in \mathbb{R}^S \times \mathbb{R}_+^{S \times S}$ is feasible for the coalition S if for all $i \in S_P$ and all $j \in S$ with $i \neq j$ it holds that $r_{ij} = 0$ and $\sum_{i \in S} r_{ij} = 1$ for all $j \in S$. Finally, we denote by $Z(S)$ the set of all feasible allocations for S.

Example 2.1 Let $N_I = \{1, 2\}$, $N_P = \{4, 5\}$ and $\mathcal{K} = \{1, 2, 3, 4, 5\}$. So, there are five independent exponentially distributed risks. Next, suppose that $X_1 = \frac{1}{3}Y_1 + Y_2$, $X_2 = \frac{1}{3}Y_1 + Y_3$, $X_4 = Y_4$ and $X_5 = Y_5$. Consider the coalition $S = \{1, 4, 5\}$. Then $X_S = X_1 + X_4 + X_5 = \frac{1}{3}Y_1 + Y_2 + Y_4 + Y_5$. A feasible allocation for S is the following. Let $d = (3, -2, -1)$ and $r_{11} = 1$, $r_{14} = \frac{1}{2}$, $r_{44} = \frac{1}{2}$, $r_{15} = \frac{1}{5}$ and $r_{55} = \frac{4}{5}$. Then insurer 1 receives

$$(d, R)_1 = 3 - (X_1 + \tfrac{1}{2}X_4 + \tfrac{1}{5}X_5) = 3 - (\tfrac{1}{3}Y_1 + Y_2 + \tfrac{1}{2}Y_4 + \tfrac{4}{5}Y_5),$$

individual 4 receives

$$(d, R)_4 = -2 - \tfrac{1}{2}X_4 = -2 - \tfrac{1}{2}Y_4,$$

and individual 5 receives

$$(d, R)_5 = -1 - \tfrac{4}{5}X_4 = -1 - \tfrac{4}{5}Y_5.$$

So, individuals 4 and 5 pay a premium of 2 and 1, respectively, to insurer 1 for the insurance of their losses.

In conclusion, an insurance game Γ with player set $N_I \cup N_P$ is described by the tuple $(N_I \cup N_P, (X_S)_{S \subset N_I \cup N_P}, (u_i)_{i \in N_I \cup N_P})$, where N_I is the set of insurers, N_P the set of individuals, $X_S \in L^1(\mathbb{R})$ the random loss for coalition S, and u_i the utility function for player $i \in N_I \cup N_P$. Recall that $X_S = \sum_{i \in S} X_i$ for all $S \subset N_I \cup N_P$ and that all players are constant absolute risk averse expected utility maximizers. The class of all such insurance games with insurers N_I and individuals N_P is denoted by $IG(N_I, N_P)$.

2.1 Pareto optimal distributions of risk

Since the preferences of both an individual and an insurer are described by means of a utility function we can look at the deterministic (or certainty) equivalent of random payoffs for each of them. The deterministic equivalent of a random payoff is the amount of money for which a player is indifferent between receiving the random payoff and receiving this amount of money with certainty. For the utility functions considered in our model, we can define the deterministic equivalent of a random payoff X by $m_i(X) = u_i^{-1}(E(u_i(X)))$ provided that the expected utility exists. Then for all these random payoffs X it holds that $E(u_i(m_i(X)) = u_i(m_i(X)) = E(u_i(X))$. Since the expected utilities equal each other, player i is indifferent between the random payoff X and the deterministic payoff $m_i(X)$. Moreover, for the insurance games introduced in this section the deterministic equivalent is such that the results stated in *Suijs* and *Borm (1996)* can be applied. One of their results concerns the Pareto optimality[1] of an allocation. For insurance games this result reads as follows.

Proposition 2.2 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d, R) \in Z(S)$ is Pareto optimal for coalition S if and only if

$$\sum_{i \in S} m_i((d,R)_i) = \max_{(\tilde{d},\tilde{R}) \in Z(S)} \sum_{i \in S} m_i((\tilde{d},\tilde{R})_i). \tag{2}$$

So, an allocation is Pareto optimal for coalition S if and only if this allocation maximizes the sum of the deterministic equivalents. To determine these allocations, we first need to calculate the deterministic equivalent of an allocation (d, R) for S for player $i \in S$. Therefore, let $S \subset N_I \cup N_P$ and $(d, R) \in Z(S)$. The random loss coalition S has to allocate equals $X_S = \sum_{i \in S} X_i$. Given a feasible allocation $(d, R) \in Z(S)$, the random payoff to player $i \in S$ equals

$$(d,R)_i = d_i - \sum_{j \in S} r_{ij} X_j$$

if $i \in S_I$ and

$$(d,R)_i = d_i - r_{ii} X_i$$

[1] An allocation (d, R) of the loss X_S is Pareto optimal for coalition S if there exists no feasible allocation $(\tilde{d}, \tilde{R})$ of X_S such that each member of S is better off, i.e., $E(u_i((\tilde{d},\tilde{R})_i)) > E(u_i((d,R)_i))$ for all $i \in S$.

if $i \in S_P$. Consequently, we have that the deterministic equivalent of $(d,R)_i$ equals[2]

$$m_i((d,R)_i) = \begin{cases} d_i + \sum_{k\in\mathcal{K}} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k}\alpha_i r_{ii} f_{ik}\right), & \text{if } i \in S_P, \\ d_i + \sum_{j\in S}\sum_{k\in\mathcal{K}} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k}\alpha_i r_{ij} f_{jk}\right), & \text{if } i \in S_I. \end{cases} \quad (3)$$

The sum of deterministic equivalences then equals

$$\begin{aligned} \sum_{i\in S} m_i((d,R)_i) &= \sum_{i\in S_P}\sum_{k\in\mathcal{K}} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k}\alpha_i r_{ii} f_{ik}\right) \\ &\quad + \sum_{i\in S_I}\sum_{j\in S}\sum_{k\in\mathcal{K}} \frac{1}{\alpha_i} \log\left(1 - \frac{1}{\mu_k}\alpha_i r_{ij} f_{jk}\right). \end{aligned} \quad (4)$$

Hence the sum of deterministic equivalents is independent of the vector of transfer payments d. Intuitively, this is quite clear. Indeed, an increase in d_i for player i implies that d_j decreases for at least one other player j since $\sum_{h\in S} d_h = 0$. Consequently, Pareto optimality is solely determined by the choice of the allocation risk exchange matrix R of the random losses. In fact, the next theorem shows that there is a unique allocation risk exchange matrix R^* inducing Pareto optimality.

Theorem 2.3 Let $\Gamma \in IG(N_I, N_P)$ and $S \subset N_I \cup N_P$. An allocation $(d, R^*) \in Z(S)$ is Pareto optimal for S if and only if

$$r^*_{ij} = \begin{cases} \dfrac{\frac{1}{\alpha_i}}{\sum_{h\in S_I}\frac{1}{\alpha_h}} & \text{, if } i,j \in S_I, \\ \dfrac{\frac{1}{\alpha_i}}{\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

PROOF: We have to show that R^* is the unique solution of

$$\begin{aligned} \max \sum_{i\in S_P}\sum_{k\in\mathcal{K}} \tfrac{1}{\alpha_i} \log\left(1 - \tfrac{1}{\mu_k}\alpha_i r_{ii} f_{ik}\right) &+ \sum_{i\in S_I}\sum_{j\in S}\sum_{k\in\mathcal{K}} \tfrac{1}{\alpha_i} \log\left(1 - \tfrac{1}{\mu_k}\alpha_i r_{ij} f_{jk}\right) \\ \text{s.t.:}\quad r_{jj} + \textstyle\sum_{i\in S_I} r_{ij} &= 1, \quad \text{for all } j \in S_P, \\ \textstyle\sum_{i\in S_I} r_{ij} &= 1, \quad \text{for all } j \in S_I, \\ r_{ii} &\geq 0, \quad \text{if } i \in S_P, \\ r_{ij} &\geq 0, \quad \text{if } i \in S_I \text{ and } j \in S. \end{aligned}$$

[2]The proof is stated in the Appendix.

Since the objective function is strictly concave in r_{ij} for all relevant combinations of $i, j \in S$, it is sufficient to prove that R^* solves this maximization problem. The Karush-Kuhn-Tucker conditions[3] tell us that this is indeed the case if there exists $\lambda_j \in \mathbb{R}$ ($j \in S$), $\nu_{jj} \geq 0$ ($j \in S_P$) and $\nu_{ij} \geq 0$ ($i \in S_I,\ j \in S$) such that

$$\begin{aligned}
\sum_{k \in \mathcal{K}_j} \frac{-1}{\frac{\mu_k}{f_{jk}} - \alpha_j r_{jj}} &= \lambda_j - \nu_{jj}, && \text{for all } j \in S_P, \\
\sum_{k \in \mathcal{K}_j} \frac{-1}{\frac{\mu_k}{f_{jk}} - \alpha_i r_{ij}} &= \lambda_j - \nu_{ij}, && \text{for all } i \in S_I \text{ and all } j \in S, \\
\nu_{ii} r_{ii} &= 0, && \text{for all } i \in S_P, \\
\nu_{ij} r_{ij} &= 0, && \text{for all } i \in S_I \text{ and all } j \in S.
\end{aligned}$$

Substituting r^*_{ij} gives $\nu_{ij} = 0$ for all relevant combinations of $i, j \in S$ and

$$\begin{aligned}
\lambda_j &= -\sum_{k \in \mathcal{K}_j} f_{jk} \left(\mu_k - \frac{f_{jk}}{\sum_{h \in S_I \cup \{j\}} \frac{1}{\alpha_h}} \right)^{-1}, && \text{for all } j \in S_P, \\
\lambda_j &= -\sum_{k \in \mathcal{K}_j} f_{jk} \left(\mu_k - \frac{f_{jk}}{\sum_{h \in S_I} \frac{1}{\alpha_h}} \right)^{-1}, && \text{for all } j \in S_I.
\end{aligned}$$

Consequently, R^* solves the maximization problem. □

So, for a Pareto optimal allocation of a loss X_j within S one has to distinguish between two cases. In the first case the index j refers to an insurer and in the second case j refers to an individual. When X_j is the loss of insurer $j \in S_I$, the loss is allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S. When X_j is the loss of individual $j \in S_P$, the loss is allocated proportionally to $\frac{1}{\alpha_i}$ among all insurers in coalition S and individual j himself. Note that by the feasibility constraints nothing is allocated to the

[3] The Karush-Kuhn-Tucker conditions read as follows:

$$\begin{aligned}
\text{If } f(x) &= \max_y f(y) \\
&\quad \text{s.t.} \quad g_k(y) \leq 0, \quad k \in K \\
&\quad\quad\quad\ \ g_l(y) = 0, \quad l \in L
\end{aligned}$$

then there exist $\nu_k \geq 0$ ($\forall k \in K$) and $\lambda_l \in \mathbb{R}$ ($\forall l \in L$) such that

$$\begin{aligned}
\nabla f(x) &= \sum_{k \in K} \nu_k \cdot \nabla g_k(x) + \sum_{l \in L} \lambda_l \cdot \nabla g_l(x) \\
\nu_k \cdot g_k(x) &= 0, \text{ for all } k \in K.
\end{aligned}$$

Moreover, if f is strictly concave and g_k ($k \in K$), g_l ($l \in L$) are convex then the reverse of the statement also holds and the maximum is unique.

other individuals. Moreover, the less risk averse a player is, the larger his share in the risk will be. Furthermore, Pareto optimality does not depend on the parameters μ_k of the losses Y_k $(k \in \mathcal{K})$. Finally, remark that if only reinsurance of the insurance portfolios is considered, that is, $N_P = \emptyset$ then the Pareto optimal allocation coincides with the Pareto optimal allocation of (re)insurance markets discussed in *Bühlmann (1980)*.

Example 2.4 In this example all monetary amounts can be assumed to be in thousands of dollars. Consider the following situation in automobile insurance with three insurance companies and two individual persons. So, $N_I = \{1, 2, 3\}$ and $N_P = \{4, 5\}$. The utility function of each player can be described by $u_i(t) = e^{-\alpha_i t}$ with $\alpha_1 = 0.33$, $\alpha_2 = 0.1$, $\alpha_3 = 0.25$, $\alpha_4 = 0.4$ and $\alpha_5 = 0.25$, respectively. So insurer 2 is the least risk averse player and individual 4 is the most risk averse player. Each insurance company bears the risk of all the cars contained in its insurance portfolio. A car can be one of two types. The first type corresponds to an average saloon car which generates relatively low losses. The second type corresponds to an exclusive sportscar generating relatively high losses. Formally, the monetary loss generated by a car is described by the exponential probability distribution Exp(5) when it is of type 1 and by Exp(0.5) when it is of type 2. Thus the expected loss of a type 1 car and a type 2 car equal \$ 0.2 and \$ 2, respectively.

The insurance portfolio of insurer 1 consists of 1800 cars of type 1 and 10 cars of type 2. For insurer 2 the portfolio consists of 900 cars of type 1 and 25 cars of type 2. Finally, the portfolio of insurer 3 consists of 300 cars of type 1 and 90 cars of type 2. The expected loss for insurer 1 then equals $1800 \cdot 0.2 + 10 \cdot 2 =$ \$ 380. The expected losses for insurer 2 and 3 then equal \$ 230 and \$ 240, respectively. The two individual persons each possess one car. Player 4's car is of type 1 and player 5's car is of type 2. So, the expected losses are \$ 0.2 and \$ 2, respectively.

Next, let X_i denote the loss of player i. If all players cooperate, the Pareto optimal risk allocation matrix of the total random loss $X_1 + X_2 + X_3 + X_4 + X_5$ equals

$$R^* = \begin{bmatrix} \frac{3}{17} & \frac{3}{17} & \frac{3}{17} & \frac{6}{39} & \frac{3}{21} \\ \frac{10}{17} & \frac{10}{17} & \frac{10}{17} & \frac{20}{39} & \frac{10}{21} \\ \frac{4}{17} & \frac{4}{17} & \frac{4}{17} & \frac{8}{39} & \frac{4}{21} \\ 0 & 0 & 0 & \frac{5}{39} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{21} \end{bmatrix}.$$

Consequently, a Pareto optimal allocation (d, R^*) yields the payoffs

$$\begin{aligned} (d,R^*)_1 &= d_1 - \tfrac{3}{17}(X_1+X_2+X_3) - \tfrac{6}{39}X_4 - \tfrac{3}{21}X_5, \\ (d,R^*)_2 &= d_2 - \tfrac{10}{17}(X_1+X_2+X_3) - \tfrac{20}{39}X_4 - \tfrac{10}{21}X_5, \\ (d,R^*)_3 &= d_3 - \tfrac{4}{17}(X_1+X_2+X_3) - \tfrac{8}{39}X_4 - \tfrac{4}{21}X_5, \\ (d,R^*)_4 &= d_4 - \tfrac{5}{39}X_4, \\ (d,R^*)_5 &= d_5 - \tfrac{4}{21}X_5. \end{aligned}$$

The determination of the allocation risk exchange matrix is, of course, only one part of the allocation. We still have to determine the vector of transfer payments d, that is, the premiums that have to be paid. Although an allocation (d, R^*) may be Pareto optimal for any choice of d, not every d is satisfactory from a social point of view. An insurer will not agree with insuring the losses of other players if he is not properly compensated, that is, if he does not receive a fair premium for the insurance. Similarly, insurance companies and individuals only agree to insure their losses if the premium they have to pay is reasonable. Consequently, there is a conflict of interests; both insurance companies and individuals want to pay a low premium for insuring their own losses, while insurance companies want to receive a high premium for bearing the losses of other players. So the question remains which premiums are reasonable? This is the subject of the next subsection.

2.2 The core of insurance games

In our quest for fair premiums we look at core allocations of insurance games. The core is one of the most important solution concepts in game theory. It is generally accepted by game theorists that if the core is a nonempty set of allocations, then the allocation on which the players agree should be a core allocation. The core contains allocations that induce a form of stability for the coalition of all players involved. For a more general discussion of the core we refer to *Aumann (1961)* and *Scarf (1967)*. In the context of insurance games, an allocation is a core allocation if there is no subcoalition that wants to part company with the grand coalition $N_I \cup N_P$ because this subcoalition can acheive a better allocation on their own. Formally, this means that an allocation (d, R) of $N_I \cup N_P$ is a core allocation if for each coalition $S \subset N_I \cup N_P$ there exists no allocation $(\tilde{d}, \tilde{R})$ for S such that each player i prefers the payoff $(\tilde{d}, \tilde{R})_i$ to the payoff $(d, R)_i$, i.e., $E(u_i((\tilde{d}, \tilde{R})_i)) > E(u_i((d, R)_i))$ for all $i \in S$. The set of all core allocations for a game $\Gamma \in IG(N_I, N_P)$ is denoted by $Core(\Gamma)$. Note that a core allocation is Pareto optimal for $N_I \cup N_P$. Hence, $(d, R) \in Core(\Gamma)$ implies that the allocation risk exchange matrix R has the structure of the Pareto optimal allocation risk exchange matrix R^* as described in Theorem 2.3 with $S = N_I \cup N_P$.

We will show that the core of an insurance game is nonempty. So, there always exists an allocation of $N_I \cup N_P$ which is stable in the sense described above. To prove this result, we make use of the results stated in *Suijs* and *Borm (1996)*.

First, we associate with each insurance game $\Gamma \in IG(N_I, N_P)$ a cooperative game $\Delta_\Gamma \in IG(N_I, N_P)$ with deterministic payoffs. This means that the payoff of a coalition in the game Δ_Γ is a real number instead of a random variable. The game Δ_Γ is called the deterministic equivalent of Γ.

Let $S \subset N_I \cup N_P$. The payoff x_S of coalition S in the game Δ_Γ is defined by

$$x_S = \max_{(d,R) \in Z(S)} \sum_{i \in S} m_i((d, R)_i). \tag{5}$$

The payoff x_S is based on Proposition 2.2, which states that an allocation is Pareto optimal for S if and only if the sum of the corresponding deterministic equivalents equals x_S. The game Δ_Γ is then described by $\Delta_\Gamma = (N_I \cup N_P, (x_S)_{S \subset N_I \cup N_P}, (u_i)_{i \in N_I \cup N_P})$. The following result is a consequence of Theorem 3.1 in *Suijs* and *Borm (1996)*.

Proposition 2.5 Let $\Gamma \in IG(N_I, N_P)$ be an insurance game and let Δ_Γ be its deterministic equivalent. Then

$Core(\Gamma) \neq \emptyset$ if and only if $Core(\Delta_\Gamma) \neq \emptyset$.

Moreover, let (d, R) be an allocation of $N_I \cup N_P$ and let $y \in \mathbb{R}^{N_I \cup N_P}$ be such that $m_i((d, R)_i) = y_i$ for all $i \in N_I \cup N_P$. Then

$(d, R) \in Core(\Gamma)$ if and only if $y \in Core(\Delta_\Gamma)$.

So, to prove nonemptiness of the core of insurance games it is sufficient to prove that the core of the corresponding deterministic equivalent is nonempty. Since $y \in Core(\Delta_\Gamma)$ if and only if $\sum_{i \in N_I \cup N_P} y_i = x_{N_I \cup N_P}$ and $\sum_{i \in S} y_i \geq x_S$ for all $S \subset N_I \cup N_P$ we can apply the Bondareva Shapley Theorem to check nonemptiness of the core. Therefore, let $\lambda : 2^{N_I \cup N_P} \to \mathbb{R}_+$ be a map assigning to each coalition $S \subset N_I \cup N_P$ a nonnegative number $\lambda(S)$. Such a map is called a balanced map if $\sum_{S \subset N_I \cup N_P : i \in S} \lambda(S) = 1$ for all $i \in N_I \cup N_P$. The Bondareva Shapley Theorem applied to insurance games then reads as follows (see *Bondareva (1963)* and *Shapley (1967)*).

Proposition 2.6 Let $\Gamma \in IG(N_I, N_P)$ and let Δ_Γ be its deterministic equivalent. Then $Core(\Delta_\Gamma) \neq \emptyset$ if and only if for all balanced maps $\lambda : 2^{N_I \cup N_P} \to \mathbb{R}_+$ it holds that

$$\sum_{S \subset N_I \cup N_P} \lambda(S) x_S \leq x_{N_I \cup N_P}.$$

Theorem 2.7 Let $\Gamma \in IG(N_I, N_P)$. Then $Core(\Gamma) \neq \emptyset$.

PROOF: First, recall that $\mathcal{K}_j = \{k \in \mathcal{K} | f_{jk} \neq 0\}$ for all $j \in N_I \cup N_P$. Then for $S \subset N_I \cup N_P$ we have for all $d \in \mathbb{R}^S$ that

$$x_S = \sum_{i \in S} m_i((d, R^*)_i)$$

$$= \sum_{i\in S_I}\sum_{j\in S_I}\sum_{k\in\mathcal{K}_j}\frac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)+$$

$$\sum_{i\in S_I}\sum_{j\in S_P}\sum_{k\in\mathcal{K}_j}\frac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)+\sum_{i\in S_P}\sum_{k\in\mathcal{K}_i}\frac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum_{h\in S_I\cup\{i\}}\frac{1}{\alpha_h}}\right)$$

$$= \sum_{i\in S_I}\sum_{j\in S_I}\sum_{k\in\mathcal{K}_j}\frac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_i}}\right)$$

$$+\sum_{j\in S_P}\sum_{i\in S_I\cup\{j\}}\sum_{k\in\mathcal{K}_j}\frac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_i}}\right)$$

$$= \sum_{j\in S_I}\sum_{k\in\mathcal{K}_j}\frac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}$$

$$+\sum_{j\in S_P}\sum_{k\in\mathcal{K}_j}\frac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}, \qquad (6)$$

where the second equality follows from Theorem 2.3 and expression (4). Next, let $\lambda : 2^{N_I\cup N_P}\to\mathbb{R}_+$ be a balanced map. Then

$$\sum_{S\subset N_I\cup N_P}\lambda(S)x_S = \sum_{S\subset N_I\cup N_P}\sum_{j\in S_I}\sum_{k\in\mathcal{K}_j}\lambda(S)\frac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I}\frac{1}{\alpha_h}}$$

$$+\sum_{S\subset N_I\cup N_P}\sum_{j\in S_P}\sum_{k\in\mathcal{K}_j}\lambda(S)\frac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S_I\cup\{j\}}\frac{1}{\alpha_h}}$$

$$\leq \sum_{S\subset N_I\cup N_P}\sum_{j\in S_I}\sum_{k\in\mathcal{K}_j}\lambda(S)\frac{f_{jk}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in N_I}\frac{1}{\alpha_h}}$$

$$
\begin{aligned}
& + \sum_{S \subset N_I \cup N_P} \sum_{j \in S_P} \sum_{k \in \mathcal{K}_j} \lambda(S) \tfrac{f_{jk}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \\
= & \sum_{j \in N_I} \sum_{S \subset N_I \cup N_P : j \in S} \lambda(S) \sum_{k \in \mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I} \frac{1}{\alpha_h}} \right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I} \frac{1}{\alpha_h}} \\
& + \sum_{j \in N_P} \sum_{S \subset N_I \cup N_P : j \in S} \lambda(S) \sum_{k \in \mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \\
= & \sum_{j \in N_I} \sum_{k \in \mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I} \frac{1}{\alpha_h}} \right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I} \frac{1}{\alpha_h}} \\
& + \sum_{j \in N_P} \sum_{k \in \mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \right)^{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}} \\
= & \; x_{N_I \cup N_P}
\end{aligned}
$$

where the inequality follows from Lemma A.1 with $c = 0$ and the third equality follows from $\sum_{S \subset N_I \cup N_P : j \in S} \lambda(S) = 1$ for all $j \in N_I \cup N_P$. Applying Proposition 2.5 and Proposition 2.6 then completes the proof. □

Example 2.8 Consider the situation described in Example 2.4. In order to calculate the deterministic equivalent of this insurance game, note that since $f_{jk} = 1$ for all $k \in \mathcal{K}_j$ and all $j \in N_I$ we have

$$
\begin{aligned}
x_S = & \sum_{j \in S_I} \sum_{k \in \mathcal{K}_j} \left(\sum_{i \in S_I} \frac{1}{\alpha_i} \right) \log \left(1 - \frac{1}{\mu_k \sum_{i \in S_I} \frac{1}{\alpha_i}} \right) + \\
& \sum_{j \in S_P} \sum_{k \in \mathcal{K}_j} \left(\sum_{i \in S_I \cup \{j\}} \frac{1}{\alpha_i} \right) \log \left(1 - \frac{1}{\mu_k \sum_{i \in S_I \cup \{j\}} \frac{1}{\alpha_i}} \right)
\end{aligned}
$$

for all $S \subset N_I \cup N_P$ (cf. expression (6)). Hence, we get

$$x_{\{1\}} = 1800 \cdot 3 \log\left(1 - \tfrac{1}{5 \cdot 3}\right) + 10 \cdot 3 \log\left(1 - \tfrac{1}{0.5 \cdot 3}\right) = -405.52.$$

Similarly, one can calculate the value x_S for each coalition S. These values are presented in Table 1.

S	x_S	S	x_S	S	x_S
$\{1\}$	-405.52	$\{2,5\}$	-239.77	$\{2,3,4\}$	-490.11
$\{2\}$	-237.61	$\{3,4\}$	-311.28	$\{2,3,5\}$	-492.03
$\{3\}$	-311.08	$\{3,5\}$	-313.38	$\{2,4,5\}$	-239.97
$\{4\}$	-0.21	$\{4,5\}$	-2.98	$\{3,4,5\}$	-313.58
$\{5\}$	-2.77	$\{1,2,3\}$	-869.53	$\{1,2,3,4\}$	-869.73
$\{1,2\}$	-620.21	$\{1,2,4\}$	-620.41	$\{1,2,3,5\}$	-871.63
$\{1,3\}$	-661.65	$\{1,2,5\}$	-622.34	$\{1,2,4,5\}$	-622.14
$\{1,4\}$	-405.72	$\{1,3,4\}$	-661.85	$\{1,3,4,5\}$	-664.06
$\{1,5\}$	-407.88	$\{1,3,5\}$	-663.86	$\{2,3,4,5\}$	-492.23
$\{2,3\}$	-489.91	$\{1,4,5\}$	-408.08	$\{1,2,3,4,5\}$	-871.83
$\{2,4\}$	-237.81				

TABLE I: The deterministic equivalent Δ_Γ.

The core of this game is then defined by

$$Core(\Delta_\Gamma) = \{y \in \mathbb{R}^5 | \sum_{j=1}^{5} y_j = -871.83,\ \forall_{S \subset N_I \cup N_P} : \sum_{i \in S} y_i \geq x_S\}.$$

Next, note that for a Pareto optimal allocation (d, R^*) of N_I we have that

$$\begin{aligned} m_1((d,R^*)_1) &= d_1 - 153.77, \\ m_2((d,R^*)_2) &= d_2 - 512.59, \\ m_3((d,R^*)_3) &= d_3 - 205.04, \\ m_4((d,R^*)_4) &= d_4 - 0.03, \\ m_5((d,R^*)_5) &= d_5 - 0.40. \end{aligned}$$

Next, take $d^0 = (-229.65, 278.33, -46.81, -0.17, -1.70)$. Then the resulting payoffs equal $m_i((d^0, R^*)_i)_{i \in \{1,2,3,4,5\}} = (-383.42, -234.26, -251.85, -0.20, -2.10)$. It is easy to check that this allocation is in the core of the deterministic equivalent Δ_Γ. Hence, $(d^0, R^*) \in Core(\Gamma)$.

So, since the core is nonempty, we know that if all players cooperate then there exist allocations such that this cooperation is stable. Moreover, from the Pareto optimality of a core allocation it follows that the allocation risk matrix is uniquely determined. A similar argument, however, does not hold for the alloction transfer payments (i.e., the premiums that have to be paid). Since the number of core allocations will mostly be infinite, the number of premiums resulting in a core allocation will also be infinite. Consequently, the insurers still have to agree on the premiums that have to be paid. A possibility is considering existing premium calculation principles and check if they result in core allocations for insurance games. This approach is elaborated in the next subsection.

2.3 The zero utility principle

Premium calculation principles indicate how to determine the premium for a certain risk. In the past, various of these principles were designed, for example, the net premium principle, the expected value principle, the standard deviation principle, the Esscher principle, and the zero utility principle (cf. *Goovaerts, De Vylder* and *Haezendonck (1984)*). In this section we focus on the zero utility principle. A premium calculation principle determines a premium $\pi_i(X)$ for individual i for bearing the risk X. The zero utility principle assigns a premium $\pi_i(X)$ to X such that the utility level of individual i, who bears the risk X, remains unchanged when the wealth w_i of this individual changes to $w_i + \pi_i(X) - X$. Since individuals are expected utility maximizers this means that the premium $\pi_i(X)$ satisfies $u_i(w_i) = E(u_i(w_i + \pi_i(X) - X))$. Note that the premium of the risk X depends on the individual who bears this risk and his wealth w_i.

Now, let us return to insurance games and utilize the zero utility principle to determine the allocation transfer payments $d \in \mathbb{R}^{N_I \cup N_P}$. At first this might seem difficult since the zero utility principle requires initial wealths w_i which do not appear in our model of insurance games. The assumption of constant absolute risk aversion, however, yields that the zero utility principle is independent of these initial wealths w_i. To see this, let

$\Gamma \in RG(N_I, N_P)$ be an insurance game. Since utility functions are exponential we can rewrite the expression $u_i(w_i) = E(u_i(w_i + \pi_i(X) - X))$ as follows

$$w_i \;=\; u_i^{-1}(E(u_i(w_i + \pi_i(X) - X))) \;=\; w_i + \pi_i(X) + u_i^{-1}(E(u_i(-X))).$$

Hence, $\pi_i(X) = -u_i^{-1}(E(u_i(-X))) = -m_i(-X)$ which indeed is independent of the wealth w_i. Given this expression we can calculate the premium individuals receive for the risk they bear. For this, recall that for the Pareto optimal allocation risk exchange matrix R^* we have

$$r^*_{ij} = \begin{cases} \frac{\frac{1}{\alpha_i}}{\sum_{h \in S_I} \frac{1}{\alpha_h}} & \text{, if } i, j \in S_I, \\ \frac{\frac{1}{\alpha_i}}{\sum_{h \in S_I \cup \{j\}} \frac{1}{\alpha_h}} & \text{, if } i \in S_I \cup \{j\} \text{ and } j \in S_P, \\ 0 & \text{, otherwise.} \end{cases}$$

Consequently, the risk that insurer i bears equals $\sum_{j \in N_I \cup N_P} r^*_{ij} X_j$. The premium he should receive for bearing this risk according to the zero utility principle equals

$$\begin{aligned}
\pi_i(\sum_{j \in N_I \cup N_P} r^*_{ij} X_j) &= \pi_i(\sum_{j \in N_I \cup N_P} \sum_{k \in \mathcal{K}} r^*_{ij} f_{jk} Y_k) \\
&= -m_i(-\sum_{j \in N_I \cup N_P} \sum_{k \in \mathcal{K}} r^*_{ij} f_{jk} Y_k) \\
&= -\sum_{j \in N_I \cup N_P} \sum_{k \in \mathcal{K}} \tfrac{1}{\alpha_i} \log\left(1 - \tfrac{1}{\mu_k} \alpha_i r^*_{ij} f_{jk}\right) \\
&= -\sum_{j \in N_I} \sum_{k \in \mathcal{K}_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I} \frac{1}{\alpha_h}}\right) \\
&\quad - \sum_{j \in N_P} \sum_{k \in \mathcal{K}_j} \tfrac{1}{\alpha_i} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}} \sum_{h \in N_I \cup \{j\}} \frac{1}{\alpha_h}}\right),
\end{aligned}$$

where the third equality follows from expression (3) with $d_i = 0$.

Note that for these type of games the zero utility principle satisfies additivity, that is, $\pi_i(\sum_{j \in N_I \cup N_P} r^*_{ij} X_j) = \sum_{j \in N_I \cup N_P} \pi_i(r^*_{ij} X_j)$. As a consequence, we let the premium that insurer i has to pay for reinsuring the fraction r^*_{ji} of his own portfolio X_i at insurer j, equal the premium that insurer j wants to receive for bearing this risk, that is,

$$\pi_j(r^*_{ji}X_i) = -m_j(-\sum_{k\in\mathcal{K}} r^*_{ji}f_{ik}Y_k) = -\sum_{k\in\mathcal{K}_i} \tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right).$$

Then the premium insurer i receives in aggregate equals

$$\sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I}\pi_j(r^*_{ji}X_i).$$

Similarly, the premium that individual $i\in N_P$ has to pay for insuring his loss at insurer j equals the zero utility premium that this insurer wants to receive for bearing this risk. Hence, individual i pays insurer j an amount

$$\pi_j(r^*_{ji}X_i) = -\sum_{k\in\mathcal{K}_i} \tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right).$$

Because individuals are not allowed to bear (part of) the risk of any other individual/insurer he does not receive any premium. So in aggregate he receives

$$-\sum_{j\in N_I}\pi_j(r^*_{ji}X_i).$$

Since

$$\sum_{i\in N_I}\left(\sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I}\pi_j(r^*_{ji}X_i)\right) - \sum_{i\in N_P}\sum_{j\in N_I}\pi_j(r^*_{ji}X_i) \;=\; 0,$$

the zero utility principle yields an allocation transfer payments vector d^0 where

$$\begin{aligned}
d^0_i &= \sum_{j\in N_I\cup N_P}\pi_i(r^*_{ij}X_j) \;-\; \sum_{j\in N_I}\pi_j(r^*_{ji}X_i)\\
&= -\sum_{j\in N_I}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right) - \sum_{j\in N_P}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)\\
&\quad+\sum_{j\in N_I}\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right) \qquad (7)
\end{aligned}$$

for all $i \in N_I$ and

$$d_i^0 = -\sum_{j \in N_I} \pi_j(r_{ji}^* X_i) = \sum_{j \in N_I} \sum_{k \in \mathcal{K}_i} \tfrac{1}{\alpha_j} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h \in N_I \cup \{i\}} \frac{1}{\alpha_h}} \right) \qquad (8)$$

for all $i \in N_P$.

Example 2.9 Consider again the situation described in Example 2.4. Applying the zero utility principle gives for insurer 1

$$\begin{aligned} d_1^0 &= -1200 \cdot 3 \log\left(1 - \tfrac{1}{5 \cdot 17}\right) - 115 \cdot 3 \log\left(1 - \tfrac{1}{0.5 \cdot 17}\right) - 1 \cdot 3 \log\left(1 - \tfrac{2}{5 \cdot 39}\right) \\ &\quad -1 \cdot 3 \log\left(1 - \tfrac{1}{0.5 \cdot 21}\right) + 1800 \cdot 10 \log\left(1 - \tfrac{1}{5 \cdot 17}\right) + 10 \cdot 10 \log\left(1 - \tfrac{1}{0.5 \cdot 17}\right) \\ &\quad +1800 \cdot 4 \log\left(1 - \tfrac{1}{5 \cdot 17}\right) + 10 \cdot 4 \log\left(1 - \tfrac{1}{0.5 \cdot 17}\right) \\ &= 42.60 + 43.18 + 0.03 + 0.30 - 213.02 - 12.52 - 85.21 - 5.01 \\ &= -229.65. \end{aligned}$$

Similarly, we get for insurers 2 and 3 and individuals 4 and 5

$$\begin{aligned} d_2^0 &= 248.52 + 125.17 + 0.10 + 1.00 - 31.95 - 9.39 - 42.60 - 12.52 = 278.33 \\ d_3^0 &= 127.81 + 17.53 + 0.04 + 0.40 - 10.65 - 33.79 - 35.50 - 112.65 = -46.81 \\ d_4^0 &= -0.03 - 0.10 - 0.04 = -0.17 \\ d_5^0 &= -0.3 - 1.00 - 0.40 = -1.70. \end{aligned}$$

So, $d^0 = (-229.65, 278.33, -46.81, -0.17, -1.70)$. From Example 2.8 we know that the resulting allocation (d^0, R^*) is in the core of the game.

In Example 2.9 it is seen that the allocation corresponding to the zero utility principle is a core allocation. The next theorem shows that this is not a coincidence.

Theorem 2.10 Let $\Gamma \in RG(N_I, N_P)$. If d^0 is the vector of transfer payments determined by the zero utility premium calculation principle and R^* is the Pareto optimal risk exchange matrix then $(d^0, R^*) \in Core(\Gamma)$.

PROOF: By Proposition 2.5 it is suffices to show that $(m_i((d^0,R^*)_i))_{i\in N_I\cup N_P} \in Core(\Delta_\Gamma)$. Hence, we must show that $\sum_{i\in S} m_i((d^0,R^*)_i) \geq x_S$ for all $S \subset N_I \cup N_P$. Since for $i \in N_I$ it holds that

$$\begin{aligned}
&m_i((d^0,R^*)_i) = \\
&= -\sum_{j\in N_I}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right)-\sum_{j\in N_P}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)\\
&\quad+\sum_{j\in N_I}\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right)\\
&\quad+\sum_{j\in N_I}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right)+\sum_{j\in N_P}\sum_{k\in\mathcal{K}_j}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{jk}}\sum\limits_{h\in N_I\cup\{j\}}\frac{1}{\alpha_h}}\right)\\
&= \sum_{j\in N_I}\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right)\\
&= \sum_{k\in\mathcal{K}_i}\tfrac{f_{ik}}{\mu_k}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I}\frac{1}{\alpha_h}}
\end{aligned}$$

and for $i \in N_P$ that

$$\begin{aligned}
&m_i((d^0,R^*)_i) = \\
&= \sum_{j\in N_I}\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right)+\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_i}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right)\\
&= \sum_{j\in N_I\cup\{i\}}\sum_{k\in\mathcal{K}_i}\tfrac{1}{\alpha_j}\log\left(1-\frac{1}{\frac{\mu_k}{f_{ik}}\sum\limits_{h\in N_I\cup\{i\}}\frac{1}{\alpha_h}}\right)
\end{aligned}$$

$$= \sum_{k\in\mathcal{K}_i} \tfrac{f_{ik}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}$$

we have for $S \subset N_I \cup N_P$ that

$$\begin{aligned}
\sum_{i\in S} m_i((d^0, R^*)_i) &= \sum_{i\in S_I}\sum_{k\in\mathcal{K}_i} \tfrac{f_{ik}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I} \frac{1}{\alpha_h}} \\
&\quad + \sum_{i\in S_P}\sum_{k\in\mathcal{K}_i} \tfrac{f_{ik}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in N_I\cup\{i\}} \frac{1}{\alpha_h}} \\
&\geq \sum_{i\in S_I}\sum_{k\in\mathcal{K}_i} \tfrac{f_{ik}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in S_I} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in S_I} \frac{1}{\alpha_h}} \\
&\quad + \sum_{i\in S_P}\sum_{k\in\mathcal{K}_i} \tfrac{f_{ik}}{\mu_k} \log \left(1 - \frac{1}{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in S_I\cup\{i\}} \frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{ik}} \sum\limits_{h\in S_I\cup\{i\}} \frac{1}{\alpha_h}} = x_S,
\end{aligned}$$

where the inequality follows from Lemma A.1 with $c = 0$. □

Example 2.11 Consider the insurance game introduced in Example 2.4. Now, let us take a closer look at the changes in insurer 1's utility when the allocation (d^0, R^*) is realized. In the initial situation insurer 1 bears the risk X_1 of his own insurance portfolio. The deterministic equivalent of X_1 equals

$$m_1(X_1) = 1800 \cdot 3 \log\left(1 - \tfrac{1}{5\cdot 3}\right) + 10 \cdot 3 \log\left(1 - \tfrac{1}{0.5\cdot 3}\right) = -405.52.$$

To allocate the total risk in a Pareto optimal way, insurer 1 bears the fraction $r^*_{12} = \frac{3}{17}$ of the risk X_2 of insurer 2. For this risk he receives a premium $\pi_1(\frac{3}{17}X_2)$ determined by the zero utility principle. From the definition of the zero utility calculation principle it follows that

$m_1(X_1 + \frac{3}{17}X_2 - \pi_1(\frac{3}{17}X_2)) = -405.52$. So insurer 1's welfare does not change when he insures a part of the risk of insurer 2. A similar argument holds when he insures a part of the risks of the other players. Hence

$$m_1(X_1 - \tfrac{3}{17}X_2 + \pi_1(\tfrac{3}{17}X_2) - \tfrac{3}{17}X_3 + \pi_1(\tfrac{3}{17}X_3) - \tfrac{6}{39}X_4 + \pi_1(\tfrac{6}{39}X_4) - \tfrac{3}{21}X_5 + \pi_1(\tfrac{3}{21}X_5)) = -405.52.$$

The increase in insurer 1's welfare arises only from the risks $\frac{10}{17}X_1$ and $\frac{4}{17}X_1$ he transfers to insurers 2 and 3, respectively. Indeed,

$$m_1(\tfrac{3}{17} - \pi_2(\tfrac{10}{17}X_1) - \pi_3(\tfrac{4}{17}X_1) + X_1 - \tfrac{3}{17}X_2 + \pi_1(\tfrac{3}{17}X_2) - \tfrac{3}{17}X_3 + \pi_1(\tfrac{3}{17}X_3) - \tfrac{6}{39}X_4 + \pi_1(\tfrac{6}{39}X_4) - \tfrac{3}{21}X_5 + \pi_1(\tfrac{3}{21}X_5)) = m_1((d^0, R^*)_1) = -229.65 > -405.52.$$

The situation described in the example above is subsistent in the definition of the zero utility principle. This means that the welfare of an insurer always remains the same when he bears the risk of someone else in exchange for the zero utility principle based premium. An increase in welfare only arises when he transfers (a part of) his own risk to someone else. Consequently, the insurers' welfare does not increase when individuals insure their losses. Hence, the insurers' incentives to insure the individuals' losses is low. To increase these incentives it may be better to utilize other premium calculation principles. One could, for example, consider subadditive premiums. In the next section we give another reason why it could be desirable that insurance companies employ subadditive premiums.

3 Subbaditivity for collective insurances

In the insurance games defined in the previous section individual persons are not allowed to cooperate; they cannot redistribute the risk amongst themselves. Looking at the individuals' behaviour in everyday life, this is a justified assumption. People who want to insure themselves against certain risks do so by contacting insurance companies, pension funds etc. We show, however, that when this restriction is abandoned then the mere fact that risk exchanges could take place between individuals implies that insurance companies have incentives to employ subadditive premiums. Whether or not such risk exchanges actually do take place is not important. As a consequence, collective insurances become cheaper for the individuals.

Let N_P be the set of individuals. A premium calculation principle π is called subadditive if for all subsets $S, T \subset N_P$ with $S \cap T = \emptyset$ it holds that $\pi(X_S) + \pi(X_T) \geq \pi(X_{S \cup T})$. Here, X_S denotes the total loss of the coalition S. So, it is attractive for the individuals to take a collective insurance, since this reduces the total premium they have to pay.

Next, consider a game with player set N_P only where the individuals are allowed to redistribute their risks. This situation can be described by an insurance game $\Gamma \in IG(N_P, \emptyset)$. So, the individuals N_P can now insure their losses among each other. Thus, $\Gamma = (N_P, (X_S)_{S \subset N_P}, (u_i)_{i \in N_P})$. Then we can associate with Γ the deterministic equivalent $\Delta_\Gamma = (N_P, (x_S)_{S \subset N_P}, (u_i)_{i \in N_P})$, with

$$x_S = \max_{(d,R) \in Z(S)} \sum_{i \in S} m_i((d, R)_i),$$

for all $S \subset N_P$. Note that this maximum is attained for Pareto optimal allocations (d, R^*) for coalition S. For this game, the value x_S can be interpreted as the maximum premium coalition S wants to pay for the insurance of the total risk X_S. To see this, suppose that the coalition S can insure the loss X_S for a premium $\pi(X_S)$ that exceeds the valuation of the risk X_S, that is, $-\pi(X_S) < x_S$. Then for each allocation $y \in \mathbb{R}^S$ of the premium $-\pi(X_S)$ there exists an allocation $(\tilde{d}, R^*) \in Z(S)$ such that $E(u_i((\tilde{d}, R^*)_i)) > u_i(y_i)$ for all $i \in S$. Indeed, let $(d, R^*) \in Z(S)$ be such that $\sum_{i \in S} m_i((d, R^*)_i) = x_S$. Define

$$\tilde{d}_i = d_i - m_i((d, R^*)_i) + y_i + \tfrac{1}{|S|} \left(x_S + \pi(X_S) \right),$$

for all $i \in S$. Then by the linearity of m_i in $\tilde{d}_i$ (cf. expression (3)) we have for all $i \in S$ that

$$m_i((\tilde{d}, R^*)_i) = y_i + \tfrac{1}{|S|} \left(x_S + \pi(X_S) \right) > y_i.$$

Hence, the members of S prefer the allocation $(\tilde{d}, R^*)$ of X_S to an insurance of X_S and paying the premium $\pi(X_S)$. Consequently, they will not pay more for the insurance of the risk X_S than the amount $-x_S$. The next theorem shows that this maximum premium $-x_S$ is subadditive, i.e., $-x_S - x_T \geq -x_{S\cup T}$, or equivalently, $x_S + x_T \leq x_{S\cup T}$, for all disjoint subcoalitions S and T of N_P.

Theorem 3.1 Let $S, T \subset N_P$ such that $S \cap T = \emptyset$. Then

$$x_S + x_T \ \leq \ x_{S\cup T}.$$

PROOF: Define for all $S \subset N_P$, all $j \in N_P$, and all $k \in \mathcal{K}$

$$a_{jk}(S) = \tfrac{\mu_k}{f_{jk}} \sum_{h\in S} \tfrac{1}{\alpha_h}.$$

Recall from expression (6) that

$$x_S = \sum_{j\in S}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log\left(1 - \frac{1}{\frac{\mu_k}{f_{jk}}\sum_{h\in S}\frac{1}{\alpha_h}}\right)^{\frac{\mu_k}{f_{jk}}\sum_{h\in S}\frac{1}{\alpha_h}} = \sum_{j\in S}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k} \log\left(1 - \frac{1}{a_{jk}(S)}\right)^{a_{jk}(S)},$$

for all $S \subset N_P$. Now, take $S, T \subset N_P$ such that $S \cap T = \emptyset$. We have to show that $x_S + x_T \leq x_{S\cup T}$.

$$\begin{aligned}
& x_{T\cup S} - x_S - x_T \ = \\
&= \sum_{j\in(T\cup S)}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(T\cup S)})^{a_{jk}(T\cup S)} - \sum_{j\in S}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(S)})^{a_{jk}(S)} \\
&\quad - \sum_{j\in T}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(T)})^{a_{jk}(T)} \\
&= \sum_{j\in S}\sum_{k\in\mathcal{K}_j}\left(\tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(T\cup S)})^{a_{jk}(T\cup S)} - \tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(S)})^{a_{jk}(S)}\right) \ + \\
&\quad \sum_{j\in T}\sum_{k\in\mathcal{K}_j}\left(\tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(T\cup S)})^{a_{jk}(T\cup S)} - \tfrac{f_{jk}}{\mu_k}\log(1 - \tfrac{1}{a_{jk}(T)})^{a_{jk}(T)}\right) \\
&= \sum_{j\in S}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log\left(\left(1 - \tfrac{1}{a_{jk}(T\cup S)}\right)^{a_{jk}(T\cup S)}\left(\tfrac{a_{jk}(S)}{a_{jk}(S)-1}\right)^{a_{jk}(S)}\right) \ + \\
&\quad \sum_{j\in T}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log\left(\left(1 - \tfrac{1}{a_{jk}(T\cup S)}\right)^{a_{jk}(T\cup S)}\left(\tfrac{a_{jk}(T)}{a_{jk}(T)-1}\right)^{a_{jk}(T)}\right) \\
&= \sum_{j\in S}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log\left(\left(1 - \tfrac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)}\left(\tfrac{a_{jk}(S)}{a_{jk}(S)-1}\right)^{a_{jk}(S)}\right) \ + \\
&\quad \sum_{j\in T}\sum_{k\in\mathcal{K}_j} \tfrac{f_{jk}}{\mu_k}\log\left(\left(1 - \tfrac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)}\left(\tfrac{a_{jk}(T)}{a_{jk}(T)-1}\right)^{a_{jk}(T)}\right) \ \geq 0,
\end{aligned}$$

where the second and the fourth equality follow from $S \cap T = \emptyset$ and the inequaltiy follows from Lemma A.2 which says that

$$\left(1 - \frac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)} \left(\frac{a_{jk}(S)}{a_{jk}(S)-1}\right)^{a_{jk}(S)} \geq 1,$$

and

$$\left(1 - \frac{1}{a_{jk}(T)+a_{jk}(S)}\right)^{a_{jk}(T)+a_{jk}(S)} \left(\frac{a_{jk}(T)}{a_{jk}(T)-1}\right)^{a_{jk}(T)} \geq 1. \qquad \square$$

Recall that insurers do not benefit from insuring the risks of the individuals when utilizing the additive zero utility principle; this premium calculation principle yields the lowest premium for which insurers still want to exchange risks with the individuals (cf. Example 2.11). So, from a social point of view, it might be best to adopt a middle course and look for premiums where both insurers and individuals benefit from the insurance transaction. Interesting questions then remaining are: are these premiums additive or subadditive and do they yield core allocations?

4 Concluding Remarks

In this paper (re)insurance problems are modelled as cooperative games with stochastic payoffs. In fact, we defined a game that dealt with both the insurance and the reinsurance problem simultaneously. We showed that there is only one allocation risk exchange matrix yielding a Pareto optimal distribition of the losses and that a core allocation results when insurance premiums are calculated according to the zero utility principle. Moreover, we explained why subadditive premium calculation principles might be attractive to use for insurance companies.

An issue only briefly mentioned in this paper concerns the insurers' behaviour. What if an insurer is risk neutral or risk loving instead of risk averse? Thus, there is at least one insurer whose utility function is linear or of the form $u_i(t) = \beta_i e^{-\alpha_i t}$ ($t \in \mathbb{R}$) with $\beta_i > 0$, $\alpha_i < 0$. Although the proofs are not provided here, most of the results presented in this paper still hold for these situations. This means that the corresponding games have nonempty cores and that the zero utility principle still yields a core allocation. The result that does change is the Pareto optimal allocation of the risk. The allocations that are Pareto optimal when all insurers are risk averse are not Pareto optimal anymore when one or more insurers happen to be risk loving. In fact, they are the worst possible allocations of the risk one can think of. In that case, allocating all the risk to the most risk loving insurer is Pareto optimal. This would actually mean that only one insurance company is needed, since other insurance companies will ultimately reinsure their complete portfolios at this most risk loving insurer.

We conclude with some topics for further research. Maybe most interesting is considering problems in 'life'-insurance instead of 'non life'-insurance, which was the subject of this research. Other topics concern the premium calculation principles: are there other premium calculation principles than the zero utility principle that result in core allocations? Or, the other way around, can game theory lead to new premium calculation principles?

Appendix

Lemma A.1 Let

$$f(x) = \left(1 - \frac{1}{x+c}\right)^{x+c}$$

for $x > 1$ and $c \geq 0$. Then f is a non decreasing function in x.

PROOF: The result follows from

$$\begin{aligned}
\frac{df(x)}{dx} &= \left(1 - \frac{1}{x+c}\right)^{x+c} \left(\frac{1}{x+c-1} + \log\left(1 - \frac{1}{x+c}\right)\right) \\
&= \left(1 - \frac{1}{x+c}\right)^{x+c} \left(\frac{1}{x+c-1} + \log\left(\frac{x+c-1}{x+c}\right)\right) \\
&\geq \left(1 - \frac{1}{x+c}\right)^{x+c} \left(\frac{1}{x+c-1} + 1 - \frac{x+c}{x+c-1}\right) \\
&= \left(1 - \frac{1}{x+c}\right)^{x+c} \left(\frac{1}{x+c-1} - \frac{1}{x+c-1}\right) = 0,
\end{aligned}$$

where the inequality follows from $\log(x) \geq 1 - \frac{1}{x}$ for $x > 1$. □

Lemma A.2 Let

$$f(x) = \left(\frac{x}{x-1}\right)^{x} \left(1 - \frac{1}{x+c}\right)^{x+c}$$

for $x > 1$ and $c \geq 0$. Then f is a non increasing function with $f(x) \geq 1$ for all $x > 1$.

PROOF: Since $\lim_{x\to\infty} f(x) = e^{-1}e^{1} = 1$ it is sufficient to prove that f is non increasing in x. This follows from

$$\begin{aligned}
\frac{df(x)}{dx} &= \left(\frac{x}{x-1}\right)^{x} \left(1 - \frac{1}{x+c}\right)^{x+c} * \\
&\quad \left(\frac{-1}{x-1} + \log\left(1 + \frac{1}{x-1}\right) + \frac{1}{x+c-1} + \log\left(1 - \frac{1}{x+c}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{-1}{x-1} + \log\left(1+\frac{1}{x-1}\right) + \frac{1}{x+c-1} + \log\left(1-\frac{1}{x+c}\right) &= \\
&= \frac{-1}{x-1} + \log\left(1+\frac{1}{x-1}\right) + \frac{1}{x+c-1} + \log\left(\frac{x+c-1}{x+c}\right) \\
&= \frac{x-1-(x+c-1)}{(x-1)(x+c-1)} + \log\left(\frac{x(x+c-1)}{(x-1)(x+c)}\right) \\
&= \frac{-c}{(x-1)(x+c-1)} + \log\left(\frac{x^2+cx-x}{x^2+cx-x-c}\right) \\
&= \frac{-c}{(x-1)(x+c-1)} + \log\left(1+\frac{c}{x^2+cx-x-c}\right) \\
&\leq \frac{-c}{(x-1)(x+c-1)} + \frac{c}{x^2+cx-x-c} \\
&= \frac{-c}{(x-1)(x+c-1)} + \frac{c}{(x-1)(x+c)} \leq 0,
\end{aligned}$$

where the first inequality follows from $\log(1+x) \leq x$ and the second inequality follows from $x > 1$ and $c \geq 0$. □

PROOF OF EXPRESSION (3): Let $i \in S$. Then

$$\begin{aligned}
m_i((d,R)_i) &= u_i^{-1}\left(E(u_i(d_i - \sum_{j\in S} r_{ij}X_j))\right) \\
&= -\tfrac{1}{\alpha_i}\log\left(\tfrac{1}{\beta_i}E(\beta_i e^{-\alpha_i(d_i-\sum_{j\in S} r_{ij}X_j)}))\right) \\
&= -\tfrac{1}{\alpha_i}\log\left(E\left(e^{-\alpha_i d_i}e^{\alpha_i\sum_{j\in S}\sum_{k\in\mathcal{K}} r_{ij}f_{jk}Y_k}\right)\right) \\
&= -\tfrac{1}{\alpha_i}\log\left(e^{-\alpha_i d_i}\prod_{j\in S}\prod_{k\in\mathcal{K}} E(e^{\alpha_i r_{ij}f_{jk}Y_k})\right) \\
&= -\tfrac{1}{\alpha_i}\log(e^{-\alpha_i d_i}) - \tfrac{1}{\alpha_i}\sum_{j\in S}\sum_{k\in\mathcal{K}}\log\left(E(e^{\alpha_i r_{ij}f_{jk}Y_k})\right) \\
&= d_i - \tfrac{1}{\alpha_i}\sum_{j\in S}\sum_{k\in\mathcal{K}}\log\left(E(e^{\alpha_i r_{ij}f_{jk}Y_k})\right) \\
&= d_i - \tfrac{1}{\alpha_i}\sum_{j\in S}\sum_{k\in\mathcal{K}}\log\left(\int_0^\infty \mu_k e^{-t(\mu_k-\alpha_i r_{ij}f_{jk})}dt\right),
\end{aligned}$$

where the fourth equality follows from the independence of the random losses Y_k, $(k \in \mathcal{K})$. Since we implicitly assumed that the expected utility exists, we must have that $\mu_k - \alpha_i r_{ij} f_{jk} > 0$ for all $j \in S$ and all $k \in \mathcal{K}$. Then

$$\begin{aligned} m_i((d,R)_i) &= d_i - \tfrac{1}{\alpha_i} \sum_{j\in S} \sum_{k\in\mathcal{K}} \log\left(\frac{\mu_k}{\mu_k - \alpha_i r_{ij} f_{jk}}\right) \\ &= d_i + \sum_{j\in S} \sum_{k\in\mathcal{K}} \tfrac{-1}{\alpha_i} \log\left(\frac{1}{1 - \frac{1}{\mu_k}\alpha_i r_{ij} f_{jk}}\right) \\ &= d_i + \sum_{j\in S} \sum_{k\in\mathcal{K}} \tfrac{1}{\alpha_i} \log\left(1 - \tfrac{1}{\mu_k}\alpha_i r_{ij} f_{jk}\right). \end{aligned}$$

Using $r_{ij} = 0$ for all $i \in S_P$ and all $j \in S$ with $i \neq j$ gives the desired result. □

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