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Abstract

This work deals with service provision to remote customers. Two examples are: (i) a manufacturer that has to deliver items to customers in a remote destination, and (ii) a company that provides repair and replacement service to distant clients. In both cases the remoteness of customers suggests order aggregation: a batch delivery in the first example, and a batch-visits journey in the other; the alternative is to provide individual services to customers. A key element is a contractual obligation of the company to provide service within an agreed delay-limit, and in that view the main decision problem is when to do a batch service. That decision would depend on: (random) demand-arrival patterns, the costs associated with the two service modes (batch and individual), as well as the model used to describe operating conditions.

This paper proposes and investigates several service-provision policies, with a simple enough structure to make them appealing for real-life implementation. Optimal service-provision procedures are obtained for these policies, minimizing the long-run expected cost per unit of time. The global optimal policy is also studied by means of a Markov-decision-process problem formulation, which enables us to verify properties of the optimal policy. The optimal costs of the proposed policies are compared and their relative performance is evaluated with respect to the global minimal cost (of the optimal policy) on one hand, and basic policies that employ either only batch or only individual services on the other hand. The results are also used to address the issue of the determination of a desirable delay-limit from the standpoint of the service provider. Finally, this work takes a broader view of the problem area of optimal service provision to remote customers through demand aggregation, and it discusses a range of further modelling settings of interest.

1 Introduction

In this work we investigate the issue of service provision to remote customers. Service is interpreted in a broad sense and includes delivery of production units to customers, and repair and replacement of failed items to the clients of a company that sells items such as machinery, and is subsequently responsible for their proper functioning (e.g., during warranty periods). To illustrate, consider the case of a far-east car manufacturer who sends cars to customers in Europe, or a producer of personal computers who offers customers a service contract. In a natural manner the remoteness of customers suggests order aggregation and

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batch services. The car manufacturer, for instance, receives demands for cars, and at some point makes a mass delivery by ships. The personal-computer producer receives calls for service, and at some point sends out a service truck for a journey to visit these customers.

The focus of this work is on those situations where the service provider must meet a delay-limit constraint which sets the maximum timespan between the demand arrival (for a car delivery or a computer repair in the above examples) and the moment of its fulfillment. Such constraints are normal in service contracts; here we take them to be rigid, i.e., they must not be violated. Therefore, if a customer is not served in a batch within the delay-limit, an individual service has to be done. In the examples, the car manufacturer could deliver the "critical" cars by plane, and the personal-computer producer visits the "critical" customers on a separate journey.

The main decision problem in these circumstances is *when* to do a batch service. Primarily this decision depends on the cost parameters; we assume that the variable cost for serving a customer within a batch is (much) lower than for an individual service. A batch service, however, has a fixed cost, which in most applications will be significant. The batching decision also depends on the (random) demand-arrival pattern; to describe that pattern we use a time-discretized setting with the basic time period (a day, a week, etc.) being determined on the basis of relevant operational factors. The demand quantities in these equal-length periods are assumed to be i.i.d. discrete random variables. The time-discretization also applies to the service decision process so that a batch service can be started only at period's endpoints. The service policy also depends on the specific circumstances; the particular model considered here will be described in the next section.

The paper studies several service policies, all characterized by their simple structure, which is desirable for implementation purposes. These policies are essentially control limit ones, which base the start of a batch service on pivotal quantities, namely the demand in a single period or the total number of waiting customers, which - if large enough - triggers a batch service (to avoid too many individual services). For these policies we obtain the optimal values of the control limits, and the corresponding optimal costs. To assess the deviation of these optimal costs from the minimal possible costs we also study the global optimal policy through a Markov-decision-process problem formulation. An extensive numerical analysis is carried out that calculates the optimal costs of all the above policies, and compares them, for a range of values of the costs and model parameters. To obtain a better perspective, the comparison also includes two basic policies in which service is either exclusively batch or exclusively individual.

Generally speaking, the problem considered here can be categorized as a stochastic clearing system with some structural characteristics of queues with batch service and impatient customers. A stochastic clearing system is a system fed by an exogenous stochastic input process, such that the quantity in the system builds up over time, where at a certain time instant all the quantity in the system is instantaneously cleared (see [Stidham 1974]). The two types of queueing models were investigated by different authors: see [Stidham 1977] for pioneering work on the former queueing model, as well as [Deb and Serfozo 1973] and [Avramidis and Uzsoy 1993] for related problems, and [Palm 1937] for a seminal treatment of the latter model with further results in many subsequent papers, e.g. [Baccelli et al. 1984], [Stanford 1990], and [Blanc et al. 1992]. The authors are not aware of any study of the problem described here as such (and indeed not even of a queueing model with both batch service and impatient customers).

The paper is organized as follows. In section 2 we provide a precise description of the model considered in this work, for which some preliminary general results are derived. In sections 3 and 4 we investigate several types of simple-structured control-limit policies, and derive the long-term expected cost per period. Whereas the essential mathematical treatment is given in the main text, other analyses of a more technical nature (e.g., computational algorithms) are deferred to the appendices to avoid distraction from the main flow of the paper. We devote section 5 to the study of the global optimal policy and its properties. A detailed solution is obtained for the important special case of a two-period delay-limit. In Section 6 we give a numerical evaluation of the optimal costs of the proposed policies, for a range of values of the cost and other model parameters, and compare these policies with the global optimal policy as well as with the two basic policies, which prescribe only batch services or only individual services. In section 7 we consider the determination of the delay-limit, from the point of view of the manufacturer, through its effects on the overall delivery costs incurred on one hand, and the induced delay distribution of customers (their "lead-time" in the delivery context) on the other hand. In section 8 we draw some general conclusions, and discuss future research directions with respect to model generalizations of interest.

2 The model and some general preliminary results

For concreteness we use here the delivery example. Demand for items arrives to the manufacturer according to a time-discretized stochastic process: X_n is the number of items demanded in period (day, week, etc.) n ($n = 1, 2, \ldots$). The X_n are assumed to be i.i.d. random variables with probability distribution $q_k = \Pr\{X_n = k\}$ ($k = 0, 1, 2, \ldots$). The delay-limit, i.e. the maximum timespan allowed to satisfy a demand, is D periods (excluding transportation time). The manufacturer has two delivery options: a batch delivery or an individual delivery. The cost associated with a batch of size i (i.e. including i items) is $a_B + b_B i$, where the fixed cost a_B is usually much larger than the variable cost b_B. A batch delivery can be done at the end of any period (and only there) and it is assumed that it can accommodate all waiting demand (which, by the rigidity of the delay-time constraint, must have arrived within time D backwards). The other option is individual delivery of an item at cost b_I, with $b_I >> b_B$ (note that for $b_I < b_B$ the optimal policy is to provide only individual deliveries). The model here does not assume a waiting cost so that there is no compelling incentive to deliver an item before the delay-limit D is reached (although many customers will actually get their demand before the delay-limit: see section 7 for some relevant derivations). However, once the delay-limit of a customer expires while this customer is not included in a batch delivery, an individual delivery is mandatory (and it is assumed to be feasible, e.g. with the car-delivery example, enough aircraft space can be rented for that purpose. Note here that the extent of individual services, at any given time, is normally small because any reasonable delivery policy will ensure that. Also, individual deliveries are usually done by faster transportation modes and hence there is additional flexibility in its exact timing).

The problem is to determine at which periods a batch service should be done in order to minimize the per-period long-run expected total delivery costs which includes both batch-delivery and individual delivery costs. The state vector at decision epochs, i.e. end-points of periods, on the basis of which a batch delivery decision is made, is given by $\underline{r} = (r_1, \ldots, r_D)$, where r_j ($j = 1, \ldots, D$) is the number of waiting customers that already experienced a delay of j periods. A (stationary) policy π is a set of rules that associate with possible states of the process a (stationary) decision rule of whether to do a batch or not (with the implicit addendum that any customer reaching its delay-limit must be served individually if a batch is not done then). Formally, such a policy π is a function from the set of all possible state vectors $I\!N^D$ to $\{0,1\}$, specifying for each vector $\underline{r}$ either a batch delivery (action "1"), or avoiding such a delivery (action "0"). (By making the policy dependent only on $\underline{r}$ we exclude policies which relate a batch service decision to whatever happened before time D backwards, e.g. customers already served and previous batch timings. Obviously this will not exclude the global optimal policy.) Every such policy π generates a stochastic process $\{\underline{R}^{(n)}, n = 1, 2, \ldots\}$, where $\underline{R}^{(n)} = (R_1^{(n)}, \ldots, R_D^{(n)})$ denotes the state vector at the n^{th} decision epoch. By virtue of the i.i.d. demand process $\{X_n, n = 1, 2, \ldots\}$ and the stationary decision rule, $\{\underline{R}^{(n)}, n = 1, 2, \ldots\}$ is a (time and state discrete) Markov process. (This indeed justifies adding an index π to $\underline{R}^{(n)}$, but since the different policies are dealt with separately this index is often suppressed in the notation with no risk of ambiguity.)

We shall throughout consider only values of D greater than or equal to 2 since for $D = 1$ the optimal policy is clearly to make a batch delivery at the end of any period in which the number of arriving customers exceeds $\frac{a_B}{b_I - b_B}$ and otherwise to provide each of them with an individual delivery. The cost of the policy is

$$\sum_{k=0}^{\lfloor \frac{a_B}{b_I - b_B} \rfloor} q_k \cdot k + \sum_{k=\lfloor \frac{a_B}{b_I - b_B} \rfloor + 1}^{\infty} q_k \cdot (a_B + b_B k) = \mu b_B + (b_I - b_B) \sum_{k=0}^{\lfloor \frac{a_B}{b_I - b_B} \rfloor} k q_k + a_B (1 - Q_{\lfloor \frac{a_B}{b_I - b_B} \rfloor}), \quad (1)$$

where $Q_k = \sum_{j=0}^{k} q_j = \Pr\{X_n \leq k\}$ $(k = 0, 1, \ldots)$ and $\mu := E(X_n)$ is the expected demand in a period.

Next we derive a general expression for the per-period long-term expected cost of an arbitrary policy π. To do that we first observe that whatever stationary (fixed) policy π is employed the Markov process $\{\underline{R}^{(n)}, n = 1, 2, \ldots\}$ regenerates itself at batch-delivery moments where all waiting customers are cleared and $\underline{r} = \underline{0}$. This regenerative property of the $\{\underline{R}^{(n)}\}$ process implies that for any given policy π the "statistical picture" between any two consecutive batch-delivery moments (henceforth: a cycle) is the same, and subsequently we can introduce the following notations for any policy π:

S_π := number of periods in a cycle;

Y_π := number of individual deliveries in a cycle;

Z_π := number of demands delivered by batch in a cycle;

N_π := total demand in a cycle.

By the construction of the model we have the following expressions:

$$Y_\pi = \sum_{n=1}^{S_\pi - D} X_n, \quad Z_\pi = \sum_{n=S_\pi - D + 1}^{S_\pi} X_n, \quad N_\pi = Y_\pi + Z_\pi \quad (2)$$

(where the numbering of X_n starts anew after every batch delivery).

We first observe that, since any policy π that prescribes a batch delivery in at least one state $\underline{r}$ is dominated, as can easily be seen, by a geometrically distributed random variable, we must have

$$E(S_\pi) < \infty. \quad (3)$$

For the policy that never prescribes a batch delivery, denoted by π = NB (standing for Never Batch), the long-term expected cost per period is simply

$$g_{NB} = \sum_{k=1}^{\infty} q_k \cdot b_I k = \mu b_I, \tag{4}$$

Due to the regenerativity of $\{\underline{R}^{(n)}\}$ for any given π, and (3), we have by the renewal-reward theorem (see e.g. [Tijms 1994], theorem 1.3.1) that the long term expected cost per period g_π can be expressed as

$$g_\pi = \frac{a_B + b_B E(Z_\pi) + b_I E(Y_\pi)}{E(S_\pi)}. \tag{5}$$

The event $\{S_\pi = s\}$ is completely determined by $X_1, \ldots, X_s$, since the batch-delivery decision is completely determined by $\underline{R}^{(1)}, \ldots, \underline{R}^{(s)}$ which in turn is completely determined by $X_1, \ldots, X_s$. Hence S_π is a stopping time for $\{X_n, n = 1, 2, \ldots\}$ and we have, by applying Wald's theorem, that

$$E(N_\pi) = E(Y_\pi) + E(Z_\pi) = \sum_{n=1}^{S_\pi} X_n = \mu E(S_\pi), \tag{6}$$

Combining (6) and (5) we obtain

$$g_\pi = b_B \mu + \frac{a_B + (b_I - b_B) E(Y_\pi)}{E(S_\pi)}. \tag{7}$$

From (7) we can immediately conclude that for the sake of the search for the optimal π (whether the global optimal or a local optimal one within a given subspace of policies) we can standardize costs by setting, arbitrarily, $b_I = 1$ and $b_B = 0$. This leaves us, very conveniently, with just one (standardized) cost parameter a_B for the rest of the analysis.

In the next two sections we use the result in (7) for the derivation of the local optimal policy within several classes of policies whose simple structure makes them easier for practical implementation. Throughout the analysis of these policies we shall impose in advance the condition that $S_\pi \geq D$, since, as noted earlier, there is no incentive in the model described here to do a batch delivery earlier.

3 The Critical-Group policy

The first policy we now investigate is one that makes the arrival of a large demand in a period a trigger for a batch-delivery time D later, when their delay-limit expires, to make sure that this critical group of customers is not given expensive individual deliveries. It can be shown (see Appendix A for a proof) that the optimal structure for this policy, denoted by π =CG (standing for Critical-Group), is of control-limit type, i.e. the rule is: when the demand in a period is for the first time (since the last batch delivery) K or more, plan to do a batch-delivery $D-1$ periods later (the maximum timespan if we want to avoid individual deliveries for them; the use of $D-1$ rather than D is due to the nature of the time discretization).

The Critical-Group policy has a simple control-limit structure depending on just one parameter and is therefore easy to administer. The optimization procedure consists of finding the value K^* that minimizes its long term expected cost per period $g_{CG}(K)$.

To proceed with this optimization we first derive $g_{CG}(K)$, employing the general formula (7) which requires the computation of $E(S_{CG})$ and $E(Y_{CG})$ (clearly S_{CG} and Y_{CG} depend on K but for ease of notation this is suppressed).

Beginning with $E(S_{CG})$, we have by the definition of the CG-policy

$$S_{CG} = T + D - 1, \tag{8}$$

where T is the number of periods, since the last batch delivery, until a critical group arrives, i.e.

$$T = \min\{n \geq 1 : X_n \geq K\} \tag{9}$$

(where the numbering of X_n begins anew after every batch delivery). It is easily observable that the random variable T has a geometric distribution with parameter Q_{K-1} so that, by (8), the distribution of S_{CG} is given by

$$\Pr\{S_{CG} = s\} = (Q_{K-1})^{s-D}(1 - Q_{K-1}) \quad (s = D, D+1, \ldots), \tag{10}$$

and thus

$$E(S_{CG}) = D + \frac{Q_{K-1}}{1 - Q_{K-1}}. \tag{11}$$

Proceeding with $E(Y_{CG})$, we have

$$E(Y_{CG}) = E(E(Y_{CG} \mid S_{CG})), \tag{12}$$

where by the rule for individual-delivery provisions

$$E(Y_{CG} \mid S_{CG} = s) = \sum_{n=1}^{s-D} E(X_n \mid S_{CG} = s). \tag{13}$$

Now, by the mutual independence of X_n

$$\begin{aligned} E(X_n \mid S_{CG} = s) &= E(X_n \mid X_1 < K, \ldots, X_{s-D} < K, X_{s-D+1} \geq K) \\ &= E(X_n \mid X_n < K) = \frac{\sum_{k=0}^{K-1} kq_k}{Q_{K-1}} \quad (n = 1, \ldots, s-D). \end{aligned} \tag{14}$$

Substituting (14) into (13) we find

$$E(Y_{CG} \mid S_{CG}) = (S_{CG} - D)\frac{\sum_{k=0}^{K-1} kq_k}{Q_{K-1}}, \tag{15}$$

and by substitution of (15) into (12) and using (11) we obtain

$$E(Y_{CG}) = (E(S_{CG}) - D)\frac{\sum_{k=0}^{K-1} kq_k}{Q_{K-1}} = \frac{\sum_{k=0}^{K-1} kq_k}{1 - Q_{K-1}}. \tag{16}$$

Substitution of (11) and (16) into (7) (with $b_B = 0$ and $b_I = 1$) yields the expected cost in terms of K:

$$g_{CG}(K) = \frac{a_B(1 - Q_{K-1}) + \sum_{k=0}^{K-1} kq_k}{D(1 - Q_{K-1}) + Q_{K-1}}. \tag{17}$$

In searching for the optimal K^* we have the following result.

Theorem 1 *The optimal K^* is the smallest K that satisfies the inequality*

$$K + (D-1)\sum_{k=0}^{K-1}(1-Q_k) \geq a_B.$$

Moreover, $K^* \leq \lceil a_B \rceil$.

Proof.
From (17) it follows that

$$\begin{aligned}
& g_{CG}(K+1) \geq g_{CG}(K) \\
\Leftrightarrow \quad & \left(a_B(1-Q_K) + \sum_{k=0}^{K} kq_k\right)(D(1-Q_{K-1}) + Q_{K-1}) \geq \\
& \left(a_B(1-Q_{K-1}) + \sum_{k=0}^{K-1} kq_k\right)(D(1-Q_K) + Q_K) \\
\Leftrightarrow \quad & \frac{DKq_K - (D-1)\left(Q_{K-1}\sum_{k=0}^{K} kq_k - Q_K \sum_{k=0}^{K-1} kq_k\right)}{q_K} \geq a_B. \qquad (18)
\end{aligned}$$

Next, using the fact that

$$\sum_{k=0}^{K-1} kq_k = (K-1)Q_{K-1} - \sum_{k=0}^{K-2} Q_k,$$

we have

$$Q_{K-1}\sum_{k=0}^{K} kq_k - Q_K \sum_{k=0}^{K-1} kq_k = Q_{K-1}Kq_K - q_K \sum_{k=0}^{K-1} kq_k = q_K \sum_{k=0}^{K-1} Q_k. \qquad (19)$$

Substituting (19) into (18) we conclude that

$$\begin{aligned}
g_{CG}(K+1) \geq g_{CG}(K) \quad & \Leftrightarrow \quad DK - (D-1)\sum_{k=0}^{K-1} Q_k \geq a_B \\
& \Leftrightarrow \quad K + (D-1)\sum_{k=0}^{K-1}(1-Q_k) \geq a_B. \qquad (20)
\end{aligned}$$

Now define

$$f(K) \quad := \quad K + (D-1)\sum_{k=0}^{K-1}(1-Q_k).$$

Since

$$f(K) - f(K-1) = 1 + (D-1)(1-Q_{K-1}) \geq 1,$$

$f(K)$ is an increasing function of K. Moreover, $f(0) = 0 < a_B$ and $\lim_{K\to\infty} f(K) = \infty$, whence $g_{CG}(K)$ has a unique minimum characterized as the smallest value of K for which (20) holds. Finally, $f(\lceil a_B \rceil) \geq a_B$, implying that $K^* \leq \lceil a_B \rceil$. $\Box$

3.1 An extension of the Critical-Group policy: Adding delivery-timing flexibility to the policy

The main idea of the Critical-Group policy, namely ensuring that the (large) critical group is included in a batch delivery can still be captured using an extension of this policy which allows the batch delivery to be executed anywhere within the next $D-1$ periods (and not necessarily in the final one). We can thus decide that a batch delivery will be done in the first period after the arrival of the critical group in which the total waiting demand is large enough, i.e. exceeds some control limit. Indeed, the relevant total demand quantity only includes the demand prior to the arrival of the critical group (which is still waiting for delivery), since those arriving after it will in any case be included in the coming batch delivery and hence should not influence its timing. As a matter of fact we also impose an additional condition, requiring that the current critical group, at the time of the batch delivery execution, is large enough, thereby adding one more control limit. This extension of the Critical-Group policy loses some of the simplicity of the basic policy, although we still retain a (multiple) control-limit type structure, and it is of interest to check what is gained for that in terms of cost reduction. In order not to interfere with the main vein of the paper we defer the mathematical formulation of this policy and the derivation of its cost criterion to Appendix B.

4 The Total-Demand policy

Another simple and sensible policy is one that bases the batch delivery on the total waiting demand, so that when for the first time since the last batch delivery it is K or larger, a batch delivery is done. Since, as pointed out earlier, there is no incentive in this model to make a batch delivery before D periods have elapsed since the previous one, the above rule is modified accordingly so that if the total existing demand reaches K or more beforehand, the batch delivery is postponed until that moment. Defining for $n = 1, 2, \ldots$

$$L_n \quad = \quad \text{the total demand still waiting for delivery at the end of period } n$$

(where the numbering of periods begins anew after a batch delivery), the policy, denoted by $\pi = \mathrm{TD}$ (standing for Total-Demand), is thus: do a batch delivery at the smallest n such that $L_n \geq K$ and $n \geq D$.

To find the optimal K which minimizes $g_{TD}(K)$, the long run expected cost per period of the policy, we first want to derive, using (7), an expression in terms of K which requires the computation of $E(S_{TD})$ and $E(Y_{TD})$. From the above, we have

$$S_{TD} = \min\{n \geq D : L_n \geq K\}, \tag{21}$$

and since, by definition,

$$L_n = \begin{cases} \sum\limits_{m=1}^{n} X_m & (n < D); \\ \sum\limits_{m=n-D+1}^{n} X_m & (n \geq D), \end{cases}$$

we obtain

$$E(S_{TD}) \quad = \quad D + \sum_{s=D}^{\infty} \Pr\{S_{TD} > s\}$$

$$= D + \sum_{s=D}^{\infty} \Pr\{L_n < K, n = D, \ldots, s\}$$

$$= D + \sum_{s=D}^{\infty} \Pr\{\sum_{m=n}^{n+D-1} X_m < K, n = 1, \ldots, s - D + 1\}. \tag{22}$$

To compute $E(Y_{TD})$ we use an identity valid for any policy π,

$$E(Y_\pi) = E(N_\pi) - E(Z_\pi) = \mu E(S_\pi) - E(\sum_{i=S_\pi-D+1}^{S_\pi} X_i), \tag{23}$$

where the second equality follows from (3). This is useful because for the policy here $E(Z_\pi)$, being the total number of customers served in a batch, is easier to compute than $E(Y_\pi)$. We have

$$\begin{aligned} E(\sum_{i=S_\pi-D+1}^{S_\pi} X_i) &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} E(\sum_{i=n-D+1}^{n} X_i \mid S_\pi = n) \\ &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} \sum_{k=0}^{\infty} \Pr\{\sum_{i=n-D+1}^{n} X_i > k \mid S_\pi = n\} \\ &= \sum_{n=D}^{\infty} \Pr\{S_\pi = n\} \left(K + \sum_{k=K}^{\infty} \Pr\{\sum_{i=n-D+1}^{n} X_i > k \mid S_\pi = n\} \right) \\ &= K + \sum_{n=D}^{\infty} \sum_{k=K}^{\infty} \Pr\{\sum_{i=n-D+1}^{n} X_i > k, S_\pi = n\}. \end{aligned} \tag{24}$$

Since the expressions in (22) and (24) cannot be simplified further (mainly because $\{L_n\}$ is not a Markov chain), a computational scheme has been developed to enable numerical evaluations (see Appendix C).

4.1 An extension of the Total-Demand policy: Adding current critical-group restrictions to the policy

A weak point of the TD policy is that the actual batch-delivery may be done when the current critical group, i.e. the customers that arrived time D ago and thus have now reached their delay-limit, is very small. In that case it looks sensible to defer the batch delivery. That is all the more evident when at that point $r_D = 0$, i.e. the critical group is empty, and postponement of the batch delivery for at least one more period can only improve things (since no immediate individual deliveries are needed). All in all this suggests a policy that bases the batch delivery on both the total demand and the current critical-group size and in line with our general approach to keep policies simply-structured, we shall again consider a (double) control-limit type policy. Specifically, the rule is: do a batch delivery when for the first time, since the last batch and as always not less than D periods after it, $L_n \geq K_1$ and $R_D^{(n)} \geq K_2$. Clearly $K_1 \geq K_2$, and by the argument above, only $K_2 \geq 1$ needs to be considered. This extension of the TD-policy generalizes both the CG- and TD-policies, which correspond to the special cases $K_1 = 0$ and $K_2 = 0$, respectively, and hence its optimal cost must be less than either of theirs. As for the magnitude of the savings, in return for the added complexity, it turns out that in some cases this policy can make a meaningful improvement and bring the optimal cost quite close to the global minimal one (see the results of the numerical analysis in section 6). Unfortunately, as explained later, the computational algorithm for the derivation of the cost criterion cannot be made on the basis of those developed for either the CG-policy or the TD-policy and a "brute-force" type of computational approach, based on two $(D-1)$-dimensional systems of equations, is needed. Consequently, the derivation becomes computationally infeasible already for $D = 4$ (see Appendix C for details).

5 The optimal policy

Policies of a simple structure, as the ones considered until now, are practical from the point of view of real-life implementation. Still, there is a theoretical interest in learning about the global optimal policy, its structure and costs. Also, the results for the optimal policy can be used to evaluate the deviation of the local optimal costs, i.e. the optimal costs of the policies considered above, from the global minimum cost.

We shall now study the global optimal policy using a Markov-decision-process (MDP) formulation of the problem: At every decision epoch, i.e. period endpoints, a choice has to be made whether to do a batch delivery or not, labelled actions "1" and "0", respectively. The state of the system at every decision epoch is, as argued earlier, completely characterized by the vector $\underline{r} = (r_1, \ldots, r_D)$ specifying the number of waiting customers at each delay level. As discussed earlier this implies the exclusion of policies that relate a batch delivery decision to whatever happened before time D backwards, e.g. previously served customers and past batch-delivery times, or any exogenous factor. The decision process is thus Markovian in $\underline{R}$ (see the argumentation in section 2) and we denote its state space by

$$\Omega := \{\underline{r} \mid r_i \geq 0,\ i = 1, \ldots, D\}.$$

Following the routine of MDP analysis we now define g and $v(\underline{r})$ as, respectively, the expected cost of the optimal policy and its relative values when starting the process in state $\underline{r}$. The (standardized) costs associated with the two possible actions are (we recall that, for the sake of policy optimization, we can set $b_B = 0$ and $b_I = 1$ without loss of generality): r_D for action "0" (the cost of providing individual deliveries to all customers whose delay-limit just expired), and a_B for action "1", the cost of a batch delivery. The state transitions from the present state $\underline{r} = (r_1, \ldots, r_D)$, if k $(k = 0, 1, \ldots)$ customers arrive in this coming period (the probability of which is q_k), are to state $(k, r_1, \ldots, r_{D-1})$ if action "0" is taken and to $(k, 0, \ldots, 0)$ if action "1" is taken.

Consequently, the optimality equations of the MDP are:

$$\begin{aligned} v(r_1, \ldots, r_D) \quad = \quad & \min\{r_D - g + \sum_{k=0}^{\infty} q_k v(k, r_1, \ldots, r_{D-1}), \\ & a_B - g + \sum_{k=0}^{\infty} q_k v(k, 0, \ldots, 0)\},\ \underline{r} \in \Omega. \end{aligned} \tag{25}$$

From [Ross 1983] (theorem V.2.1) it follows that for this denumerable-state MDP an optimal stationary policy π^* exists. Its numerical computation from equation (25) is however infeasible for $D \geq 4$ because of the curse of dimensionality. Still the equations can be used for $D = 2$ and $D = 3$ and indeed for the former we even obtained an (almost complete) analytical solution (see section 5.2). Moreover the optimality equations can also be utilized to verify properties of the global optimal policy, for an arbitrary D, and this last goal is the concern of the following section.

5.1 Properties of the optimal policy

We first consider the cost criterion of the expected discounted costs, with a discount factor α, and will then infer back on the cost criterion of the expected average costs, considered above.

Define for $n = 1, 2, \dots$

$$\begin{aligned} v_n(\underline{r}) &= \text{minimal } \alpha\text{-discounted costs starting in state } \underline{r} \text{ with } n \text{ transitions to go;} \\ \pi_n^*(\underline{r}) &= \text{optimal action in state } \underline{r} \text{ with } n \text{ transitions to go.} \end{aligned}$$

For ease of notation also define for $n = 1, 2, \dots$

$$\begin{aligned} h_n^1 &:= a_B + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, 0, \dots, 0); \\ h_n^0(\underline{r}) &:= r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, r_1, \dots, r_{D-1}); \\ h_n(\underline{r}) &:= h_n^1 - h_n^0(\underline{r}). \end{aligned}$$

Now the dynamic programming equations can be written as

$$v_n(\underline{r}) = \min\{h_n^0(\underline{r}), h_n^1\} \quad (n = 1, 2, \dots;\ \underline{r} \in \Omega). \tag{26}$$

We begin by verifying some properties of the relative values $v_n(\underline{r})$.

Theorem 2 *(i)* $v_n(\underline{r}) \le v_n(\underline{r}')$, $\underline{r} \le \underline{r}'$;
(ii) $v_n(\underline{r}) - v_n(\underline{r}') \le a_B$;
(iii) $v_n(\underline{r} + e_i) \le 1 + v_n(\underline{r})$;
(iv) $v_n(\underline{r} + e_i) \le v_n(\underline{r} + e_j)$, $i \le j$;
(v) $v_n(\underline{r}) \le v_n(\underline{r} + k(e_j - e_i))$, $i \le j$;
(vi) If $\underline{r}$ *and* $\underline{r}'$ *are such that* $\sum_{j=1}^{i} r_j \ge \sum_{j=1}^{i} \underline{r}'_j$ $(i = 1, \dots, D-1)$ *and* $\sum_{j=1}^{D} r_j = \sum_{j=1}^{D} \underline{r}'_j$, *then* $v_n(\underline{r}) \le v_n(\underline{r}')$.

Proof. See Appendix D.

The statements of Theorem 2 are better understood by thinking in terms of difference in expected future costs when starting from different states. Property (ii) states that this difference is bounded by a_B, and property (i) that adding customers increases the costs, but then, by property (iii), these costs cannot increase by more than $b_I = 1$ per customer. According to property (iv) the additional costs from an added customer increases with his delay, while by property (v) moving any number of customers to a higher delay level increases the costs. Finally, property (vi) states that if we have two states with the same total number of customers but with a different internal delay distribution, so that in one of them the number of customers with a delay not greater than i is smaller than in the other for all i, then the costs associated with the former state are higher.

Next we use Theorem 2 to derive some structural properties of π^*.

Theorem 3 *(i)* $\pi_n^*(r_1, \ldots, r_{D-1}, 0) = 0$*;*
(ii) If $\pi_n^*(\underline{r}) = 1$ *then* $\pi_n^*(\underline{r} + e_i) = 1$*;*
(iii) If $\pi_n^*(\underline{r} + e_i) = 1$ *then* $\pi_n^*(\underline{r} + e_j) = 1$*,* $j > i$*.*

Proof.
(i) It follows from (25) that $\pi_n^*(r_1, \ldots, r_{D-1}, 0) = 0$ if

$$\sum_{k=0}^{\infty} q_k \Big(v_{n-1}(k, r_1, \ldots, r_{D-1}) - v(k, 0, \ldots, 0) \Big) \leq a_B,$$

which is true by Theorem 2(ii).
(ii) Using Theorem 2(i) we have that

$$h_n(\underline{r} + e_i) - h_n(\underline{r}) = h_n^0(\underline{r}) - h_n^0(\underline{r} + e_i) = -I_{\{i=D\}} + \alpha \sum_{k=0}^{\infty} q_k \cdot$$
$$\cdot \Big(v_{n-1}(k, r_1, \ldots, r_{D-1}) - v_{n-1}((k, r_1, \ldots, r_{D-1}) + I_{\{i \neq D\}} e_{i+1}) \Big) \leq 0.$$

Since $\pi_n^*(\underline{r}) = 1$ implies $h_n(\underline{r}) \leq 0$, it follows that $h_n(\underline{r}+e_i) \leq h_n(\underline{r}) \leq 0$ and hence $\pi_n^*(\underline{r}+e_i) = 1$.
(iii) For $j < D$ we use Theorem 2(i) and for $j = D$ Theorem 2(iii) to obtain

$$h_n(\underline{r} + e_j) - h_n(\underline{r} + e_i) = h_n^0(\underline{r} + e_i) - h_n^0(\underline{r} + e_j) = -I_{\{j=D\}} \underline{r}'_D + \alpha \sum_{k=0}^{\infty} q_k \cdot$$
$$\cdot \Big(v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1}) - v_{n-1}((k, r_1, \ldots, x_{D-1}) + I_{\{j \neq D\}} e_{j+1}) \Big) \leq 0.$$

Since $\pi_n^*(\underline{r} + e_i) = 1$ implies $h_n(\underline{r} + e_i) \leq 0$, it follows that $h_n(\underline{r} + e_j) \leq h_n(\underline{r} + e_i) \leq 0$ and hence $\pi_n^*(\underline{r} + e_j) = 1$. □

Corollary 1 $S_{\pi^*} \geq D$*.*

Proof. Follows directly from Theorem 3(i). □

Theorem 3(i) confirms the fact that a batch service should not be started when the number of customers requiring an individual service is zero, because by waiting further the customers arriving in the next period can be included in the batch with no additional costs. Parts (ii) and (iii) of Theorem 3 state that if the optimal decision in some state is to start a batch, then a batch should also be started if, respectively, one customer is added or one customer is moved to a higher delay level.

Theorem 4 *(i)* $v_n(\underline{r} + e_i + e_j) - v_n(\underline{r} + e_i) \leq v_n(\underline{r} + e_j) - v_n(\underline{r})$*;*
(ii) $v_n(x - e_i + e_{i+k}) \leq v_n(x - e_j + e_{j+k})$*,* $i \leq j$*.*

Proof. See Appendix D.

Part (i) of Theorem 4 states that the cost of adding a customer decreases with the number of customers added, while part (ii) states that the cost of increasing the delay level of a customer by k periods is an increasing function of his initial delay level.

The above analysis is for the discounted cost criterion, but it is not difficult to verify, through limiting arguments (see e.g. [Ross 1983]), that the results of Theorems 2-4 also hold, for the corresponding quantities, when the expected average cost criterion is used.

5.2 Solution for the case $D = 2$

We now use the optimality equations (25) and some of the above properties of the optimal policy for a detailed solution for the important special case $D = 2$. In this case the state space is $\Omega = \{(i,j) \mid i,j \geq 0\}$, with i (j) denoting the number of customers that arrived one (two) period(s) ago. It follows from theorem 3 that the optimal policy has the following structure:

$$\pi^*(i,j) = \begin{cases} 0 & (j < K_i^*); \\ 1 & (j \geq K_i^*). \end{cases} \tag{27}$$

We first note the correspondence between special cases of K_i and the policies considered earlier:
(a) $K_i = K$ for all i: the CG-policy;
(b) $K_i = K - i$ for all i: the TD-policy (without the added condition that $S_\pi \geq D$);
(c) $K_i = \max\{K_1 - i, K_2\}$: the extended TD-policy (section 4.2).
We begin the analysis of the optimal policy with the following result.

Theorem 5 *(i)* $K_0^* = \lceil a_B \rceil$;
(ii) K_i^* *is non-increasing in* i;
(iii) $K_i^* - K_{i+1}^* \in \{0,1\} \quad (i = 0,1,\ldots)$.

Proof.
(i) It follows from (25) that $\pi^*(0,j) = 1$ if $j > 0$ and $a_B < j$, implying that $K_0 = \min\{j : a_B < j\} = \lceil a_B \rceil$.
(ii) Suppose that $K_i < K_{i+1}$ for some i. Then $\pi^*(i,K_i) = 1$ and $\pi^*(i+1,K_i) = 0$, contradicting theorem 3(ii).
(iii) It follows from (i) that $K_i - K_{i+1} \geq 0$. Therefore suppose that $K_i - K_{i+1} > 1$ for some i. Then $\pi^*(i+1,K_{i+1}) = 1$ while $\pi^*(i,K_{i+1}+1) = 0$ since $K_{i+1}+1 < K_i$, contradicting Theorem 3(iii). □

Now, for a fixed policy $\pi = (K_1, K_2, \ldots)$ the expected average costs g_π and the relative values $v_\pi(\underline{r})$ can be calculated by solving the following system:

$$\begin{aligned} v_\pi(i,j) &= j - g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(k,i) \quad (j < K_i); \\ v_\pi(i,j) &= a_B - g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(k,0) \quad (j \geq K_i); \\ v_\pi(0,0) &= -g_\pi + \sum_{k=0}^{\infty} q_k v_\pi(k,0) := 0. \end{aligned} \tag{28}$$

It immediately follows from (28), by using $v_\pi(0,0) = 0$, that

$$v_\pi(i,j) = \begin{cases} j & (i = 0,\ 0 \leq j < K_0); \\ a_B & (i \geq 0,\ j \geq K_i); \\ j + v_\pi(i,0) & (i > 0,\ 0 \leq j < K_i). \end{cases} \tag{29}$$

It remains to find $v_\pi(i,0)$ for $i > 0$, and by using (28) and (29) we obtain

$$v_\pi(i,0) = -g_\pi + \sum_{k:i<K_k} q_k \Big(i + v_\pi(k,0)\Big) + \sum_{k:i \geq K_k} q_k a_B$$

$$
\begin{aligned}
&= a_B - g_\pi + \sum_{k=0}^{\delta_i} q_k \Big(i - a_B + v_\pi(k,0) \Big) \\
&= a_B - g_\pi + (i - a_B) Q_{\delta_i} + \sum_{k=0}^{\delta_i} q_k v_\pi(k,0) \quad (i \geq 0),
\end{aligned} \tag{30}
$$

where $\delta_i := \min\{j : K_j \leq i\}$, so that $\delta_i = j$ if and only if $K_j \leq i$ and $K_{j-1} > i$. Note that δ_i is well-defined by Theorem 5(ii) and strictly decreasing in i for $i \geq K_\infty$ by Theorem 5(iii). We have thus reduced the two-dimensional system (28) for $v_\pi(i,j)$ and g_π to the one-dimensional system (30) for $v_\pi(i,0)$ and g_π (although with an infinite number of equations). Equation (30) can be simplified further by observing that

$$
v_\pi(i,0) = \begin{cases} i & (0 \leq i < K_\infty); \\ a_B - g_\pi & (i \geq K_0). \end{cases} \tag{31}
$$

Using (31) in turn we can solve (30) for those i with $\delta_i \leq K_\infty$, i.e. for $i \leq \delta_{K_\infty}$, yielding

$$
v_\pi(i,0) = a_B - g_\pi + (i - a_B) Q_{\delta_i} + \sum_{k=0}^{\delta_i} k q_k \quad (\delta_{K_\infty} \leq i < K_0). \tag{32}
$$

Consequently, for policies with $\delta_{K_\infty} \leq K_\infty$, (31) and (32) together give a complete solution of (30). Moreover, substituting this solution in

$$
g_\pi = \sum_{k=0}^{\infty} q_k v_\pi(k,0) \tag{33}
$$

and solving for g_π we obtain

$$
g_\pi = \frac{a_B(1 - Q_{K_\infty - 1}) + \sum_{i=1}^{K_\infty - 1} k q_k + \sum_{i=K_\infty}^{K_0 - 1} (i - a_B) q_i Q_{\delta_i - 1} + \sum_{i=K_\infty}^{K_0 - 1} \sum_{k=0}^{\delta_i - 1} k q_k}{2 - Q_{K_\infty - 1}}. \tag{34}
$$

On the other hand, if $\delta_{K_\infty} > K_\infty$ then what remains of (30) are the equations for $K_\infty \leq i < \delta_{K_\infty}$, and these can be written as

$$
v_\pi(i,0) \;=\; a_B - g_\pi + (i - a_B) Q_{\delta_i - 1} + \sum_{k=0}^{K_\infty - 1} k q_k + \sum_{k=K_\infty}^{\delta_i - 1} q_k v_\pi(k,0). \tag{35}
$$

Finally, substituting (31) and (32) in (33), we end up with a finite system of $\delta_{K_\infty} - K_\infty + 1$ equations in the unknowns $v_\pi(i,0)$ $(i = K_\infty, \ldots, \delta_{K_\infty} - 1)$ and g_π. We use this for the construction of an efficient policy iteration algorithm, where in every iteration system (28) is solved for some policy π.

6 Numerical evaluations of global and local optimal costs and comparisons

We shall now carry out an extensive numerical analysis of the various policies considered hitherto, and compare them with respect to their respective optimal cost. The global optimal policy naturally provides in that respect a lower bound for any of the other suggested policies.

On the other extreme we have the policies which either prescribe only batches (denoted by OB) or no batches at all (denoted by NB). The cost of the latter is given by (4). As for the former, we obtain it by making the observation that due to the rigidity of the delay-limit constraint, not having the option of individual deliveries means that even a single customer will initiate a batch delivery time D later so that the OB-policy in effect corresponds to the special case of the CG-policy with control limit $K = 1$. Hence, by (17) the cost of this policy is

$$g_{OB} = \frac{a_B(1-q_0)}{D(1-q_0)+q_0}. \tag{36}$$

Indeed, the NB-policy is also a special case of the CG-policy with control limit $K \to \infty$ (which, from (17), confirms the result in (4)). The NB- and OB-policies thus provide upper bounds, as far as their costs are concerned, to the optimal costs of the suggested policies. Beyond that it is also of interest to see by *how much* the suggested policies do better than these two basic policies.

As a matter of fact there is yet another basic policy, namely a periodic one that does a batch delivery every fixed number of intervals, say L. It is however easily verified that L attains its optimum at either of the two extreme values ∞ or D, depending on whether $\frac{a_B}{D\mu}$ is larger or smaller, respectively, than $b_I - b_B$. Moreover, it can be verified that the expected cost of this policy is always inferior to either the NB policy (when $L = \infty$) or the OB-policy (when $L = D$). Hence, if we use the NB- and OB-policies as upper-bound reference bases there is no further perspective gain in including the periodic policy as well.

Table 1 presents the optimal control limits and optimal costs of the various policies considered: Critical-Group (CG) and its extension (denoted by ECG), Total-Demand (TD) and its extension (denoted by ETD), as well as the structure and the costs of the global optimal policy π^*. The distribution of X_n is assumed to be Poisson with mean λ, i.e. $q_k = e^{-\lambda}\frac{\lambda^k}{k!}$ $(k = 0, 1, \ldots)$. The results cover a range of values of λ and the (standardized) cost a_B.

λ	a_B	g_{NB}	g_{OB}	g^*_{CG}	(K^*)	g^*_{ECG}	(K_1^*, K_2^*)	g^*_{TD}	(K^*)	g^*_{ETD}	(K_1^*, K_2^*)	g_{π^*}	$(K_0^*, K_1^*, \ldots)$
1	1.5	1	0.5810	0.5810	(1)	0.5716	(2,1)	0.6138	(2)	0.5395	(2,1)	0.5395	(2,1)
1	2	1	0.7746	0.7090	(2)	0.6848	(2,1)	0.7335	(3)	0.6848	(3,1)	0.6848	$(2^2,1)$
1	2.5	1	0.9683	0.8135	(2)	0.7980	(2,1)	0.8311	(3)	0.7797	(3,1)	0.7797	(3,2,1)
3	4.5	3	2.1926	2.0250	(3)	2.0250	(3,3)	2.1398	(5)	2.0012	(5,3)	2.0012	(5,4,3)
3	6	3	2.9234	2.5031	(4)	2.4723	(4,3)	2.5862	(7)	2.4438	(7,3)	2.4438	$(6,5,4^2,3)$
3	7.5	3	3.6543	2.8084	(5)	2.7680	(5,3)	2.8169	(9)	2.7303	(8,4)	2.7275	$(8,7,6,5,4^2,3)$
5	7.5	5	3.7373	3.5364	(4)	3.5096	(5,4)	3.6650	(8)	3.4921	(8,4)	3.4921	(8,7,6,5,4)
5	10	5	4.9831	4.3661	(6)	4.3337	(6,5)	4.4838	(12)	4.2803	(11,5)	4.2803	$(10,9,8,7,6^2,5)$
5	12.5	5	6.2289	4.8334	(8)	4.7806	(8,5)	4.8323	(14)	4.7299	(13,6)	4.7288	$(13\text{-}7,6^3,5)$
10	15	10	7.4998	7.3032	(8)	7.2918	(9,8)	7.4509	(15)	7.2762	(15,8)	7.2762	(15-9,8)
10	20	10	9.9998	9.1171	(11)	9.0479	(12,10)	9.2786	(22)	8.9814	(21,10)	8.9814	$(20\text{-}13,12^2,11,10^4,9)$
10	25	10	12.4997	9.9013	(16)	9.8427	(15,10)	9.8716	(27)	9.7744	(26,11)	9.7743	$(25\text{-}17,16^2,15\text{-}13,12^2,11^3,10)$

Table 1: Numerical comparison of different policies for $D = 2$

We used the following shorthand notation for the optimal policy's critical values $(K_i^*; i = 0, 1, \ldots)$: n^m denotes a string of m n's, $n - m$ denotes the string $n, n-1, \ldots, m$ $(n > m)$ and the last number is K_∞^*.

Some conclusion of interest can be drawn from these results. Firstly, it is clear that the ETD-policy performs extremely well here: it either entirely coincides with the global optimal policy or only slightly deviates from it. In most cases here the CG-policy outperforms the TD-policy but is outperformed by the ECG-policy.

Table 2 repeats the above calculations for $D = 3$. The structure of the global optimal policy is now omitted because of its complexity, and the global minimal costs for $\lambda = 10$ are missing due to computational infeasibility.

λ	a_B	g_{NB}	g_{OB}	g^*_{CG} (K^*)	g^*_{ECG} (K^*_1, K^*_2, K^*_3)	g^*_{TD} (K^*)	g^*_{ETD} (K^*_1, K^*_2)	g_{π^*}
1	2.25	1	0.6281	0.6281 (1)	0.5944 (2,1,1)	0.6310 (3)	0.5843 (3,1)	0.5798
1	3	1	0.8375	0.7593 (2)	0.7364 (2,1,1)	0.7551 (4)	0.7270 (4,1)	0.7229
1	3.75	1	1.0469	0.8890 (2)	0.8643 (3,1,1)	0.8467 (5)	0.8339 (5,1)	0.8253
3	6.75	3	2.2114	2.0853 (3)	2.0853 (3,3,3)	2.1275 (8)	2.0589 (7,3)	2.0537
3	9	3	2.9485	2.6059 (4)	2.5638 (5,3,3)	2.5734 (11)	2.5215 (10,3)	2.5157
3	11.25	3	3.6856	2.9027 (6)	2.8520 (6,3,3)	2.8240 (13)	2.8021 (12,4)	2.7988
5	11.25	5	3.7415	3.5958 (5)	3.5725 (5,4,4)	3.6459 (13)	3.5625 (12,4)	3.5523
5	15	5	4.9887	4.5038 (6)	4.4283 (7,4.5,5)	4.4428 (17)	4.3815 (16,5)	4.3739
5	18.75	5	6.2359	4.9375 (9)	4.8786 (9,5,5)	4.8323 (20)	4.8156 (20,6)	4.8090
10	22.5	10	7.4999	7.3632 (8)	7.3499 (9,7.5,8)	7.4419 (25)	7.3437 (23,8)	
10	30	10	9.9998	9.2920 (12)	9.2061 (13,9.5,10)	9.2114 (33)	9.1251 (31,10)	
10	37.5	10	12.4998	9.9800 (18)	9.9412 (17,10,10)	9.8757 (39)	9.8672 (38,12)	

Table 2: Numerical comparison of different policies for $D = 3$

Once more we see that the ETD-policy performs well: it is always close to the global optimal policy and better than any of the other policies. The TD-policy is superior to the CG-policy (except for very low values of λ and a_B). In general, it is intuitively clear that as D gets larger the CG-policy will act more and more inferiorly because of the relative loss of significance of the size of the critical-group r_D that triggers a batch with respect to the whole state information $\underline{r}$.

It is also interesting to compare the performance of the various policies against the (standardized) cost a_B and this is done in Figures 1 and 2 (see page 30 and 31), when $D = 2$, $\lambda = 3$ and $D = 3$, $\lambda = 2$, respectively. Included here are also the "basic" policies NB (never batch) and OB (only batch), corresponding to the straight lines in the graph. Figures 3 and 4 (see page 32 and 33) again demonstrate for the above special cases of D and λ the relative savings of the (optimally administered) proposed policies in comparison with the above "basic" policies, thereby revealing the value of using "sophisticated" policies.

In Figures 1 and 3 we see concavity in a_B of the global optimal cost as well as of the optimal costs of the proposed policies. Since $\lim_{a_B \to 0} g_{\pi^*} = g_{OB}$ and $\lim_{a_B \to \infty} g_{\pi^*} = g_{NB}$, the OB-policy performs well for small values of a_B and the NB-policy for large ones. As a matter of fact the OB- and NB-policies have the same cost when $a_B = \mu(D + \frac{q_0}{1-q_0})$ with the OB-policy being superior for smaller values of a_B and the contrary for larger values of a_B.

7 On the choice of D

First of all, it is important to observe that for large values of D the costs as well as the delay characteristics can be closely approximated by increasing the basic period length, and correspondingly decreasing D. After all it is the product of the period length and D that matters for the delay representation and for large D the difference between the policy with the increased period length and the original one will be small. Indeed, it is this last argument that has enabled us to consider moderate values of D throughout the paper, without sacrificing too much the applicability of the results obtained.

The delay characteristics and the optimal costs of the different batching policies have been derived for a given value of D. These results, however, can be used for a sensitivity analysis with respect to D, so that the supplier can examine the effect of the choice of D on the costs on one hand, and the service performance, as provided by the delay characteristics, on the other hand. This sensitivity analysis might be a useful input for the determination of D.

We shall not examine this issue in depth here and will suffice in illustrating the point with respect to the Critical-Group policy, which is mathematically and computationally the easiest in this regard. Thus, we shall obtain the delay distribution and its main characteristics in terms of D. Combining that with the results obtained earlier for the optimal costs of this policy, again in terms of D, will demonstrate the tradeoff between costs and service performance as far as the determination of D is concerned.

Define for any stationary policy π

$$\begin{aligned} N_\pi(i) &:= \text{number of arriving customers in a cycle incurring a delay of} \\ &\quad\ i \text{ periods} \quad (i = 1, \ldots, D); \\ N_\pi &:= \text{total number of arriving customers in a cycle.} \end{aligned}$$

The following relations hold for any π:

$$\begin{aligned} N_\pi(i) &= X_{S_\pi - i + 1} \quad (i = 1, \ldots, D-1); \\ N_\pi(D) &= \sum_{n=1}^{S_\pi - D + 1} X_n = Y_\pi + X_{S_\pi - D + 1}; \\ N_\pi &= \sum_{i=1}^{D} N_\pi(i) = \sum_{n=1}^{S_\pi} X_n. \end{aligned} \tag{37}$$

If we define W_π as the delay of an arbitrary customer under policy π, then it follows from the renewal-reward theorem that

$$w_\pi(i) := \Pr\{W_\pi = i\} = \frac{E(N_\pi(i))}{E(N_\pi)} \quad (i = 1, \ldots, D). \tag{38}$$

As an example we will now derive the implied delay distribution when a Critical-Group policy is employed. Using (11) and (37) we have that

$$\begin{aligned} E(N_{CG}(i)) &= E(X_{T_1 + D - i}) = \mu \quad (i = 1, \ldots, D-1); \\ E(N_{CG}(D)) &= \sum_{n=1}^{T_1} X_n = \mu E(T_1) = \frac{\mu}{1 - Q_{K-1}}; \\ E(N_{CG}) &= \sum_{i=1}^{D} N_{CG}(i) = \frac{\mu}{1 - Q_{K-1}} + (D-1)\mu, \end{aligned} \tag{39}$$

and substituting in (38) gives

$$\begin{aligned} w_{CG}(i) &= \frac{E(N_{CG}(i))}{E(N_{CG})} = \frac{1 - Q_{K-1}}{1 + (D-1)(1 - Q_{K-1})} \quad (i = 1, \ldots, D-1); \\ w_{CG}(D) &= \frac{E(N_{CG}(D))}{E(N_{CG})} = \frac{1}{1 + (D-1)(1 - Q_{K-1})}. \end{aligned} \tag{40}$$

So under a Critical-Group policy the implied delay of a customer is uniformly distributed over $\{1, \ldots, D-1\}$ with a different probability for a delay of D.

8 Conclusions and further research

The paper deals with an important segment in the service supplier/customer relationship, namely the actual provision of the service or delivery to the customers. The focus is on situations with customers far enough from the service provider to justify demand aggregation rather than all-out individual services. This situation would surely occur with manufacturers delivering their products internationally and across continents, as well as with companies that provide repair and replacement service to clients nation- (or region-) wide.

The main decision problem in these circumstances is the *timing* of batch services as the demand arrives. That timing would depend on the costs involved, namely batch-service costs and individual service costs, as well as on the particular model employed. Here we have considered a basic model; its assumptions can be generalized or altered to cover broader situations of interest. We shall now elaborate on some possible model generalizations and alterations.

(1) *Batch-service capacity and costs*

The model here assumes that a batch delivery can potentially be provided at any decision point, and can then accommodate all waiting demand. To keep the latter assumption valid may sometimes require several supply vehicles (ships for delivery, trucks for repair service, etc.) in which case adjustment of the "fixed" cost of a batch delivery is needed, e.g. two ships incurring fixed cost of $2a_B$ (or a bit less if administrative costs are common to both). Removing the assumption of enough batch "servers" (e.g. ships and trucks) leads to a queueing analysis of the finite-server type (as opposed to the essentially infinite-server assumption in the model here).

(2) *Variable delay-limit*

The model here assumes a constant delay-limit D. Though this is a sensible assumption with regard to most supplier-customer contracts, there are several situations of interest where this assumption does not hold. The delay-limit in the model has already been adjusted for (batch) transportation time. This would leave D as a constant only if the transportation time is a constant. When transportation time has non-negligible variability, D should be taken as a random variable. This change has an important impact on the analysis, as the introduction of variability always does; for instance, the policy characteristic of $S_\pi \geq D$ becomes void.

Variability of the delay-limit can also be induced when customers have different preset delay-limits, but are still served jointly. The distribution of D at any give period depends on the composition of the waiting customers with regard to their individual preset delay-limits.

(3) *Model variations of a more technical nature*

(i) The model here assumes no waiting time costs. While this could very well be the case when only the preset delay-limit matters, the company can still have a good reason to provide the service earlier, namely enhance its reputation. That can be incorporated into the model through a delay-dependent cost function. Another case where waiting costs become relevant arises when delays are not entirely considered rigid, and the service provider may occasionally be ready to assume contractual penalties and not serve within the delay-limit.

(ii) The demand process here is assumed stable in the sense that the X_n are identically distributed. In those instances where this is not a realistic assumption (e.g., when a seasonality factor exists) appropriate modifications of this assumption are required.

(iii) Individual service may also be grouped into "mini-batches", so a fixed cost a_I becomes relevant (while the variable cost b_I is likely to be reduced).

(4) *Broader contexts*

Beyond the modelling variations considered above, it may be necessary to broaden the problem formulation framework and consider the delivery issue within the general production context; for instance, when enough stock for delivery is not always available at period end-points (a tacit assumption in the model here). In that case the delivery problem cannot be considered in isolation, so an integrated approach is needed. Still, the analysis and results here can then be utilized as building blocks within that broader study.

Appendix A: The CG-policy

In this appendix we show the optimality of a control-limit type policy within the class of Critical-Group policies. Define for $n = 1, 2, \ldots$

$$\begin{aligned} T_n &:= \text{number of periods since the last batch service at the end of period } n; \\ U_n &:= \begin{cases} T_n & (T_n < D); \\ R_D^{(n)} & (T_n \geq D). \end{cases} \end{aligned}$$

Then $\{U_n, n = 1, 2, \ldots\}$ is a stochastic process on the state space

$$\{i' \mid i' = 1', \ldots, D-1'\} \cup \{i \mid i = 0, 1, \ldots\}.$$

Finding the optimal policy here boils down to solving the following optimality equations:

$$\begin{aligned} v(i') &= -g + v(i+1') \quad (i = 1, \ldots, D-2); \\ v(D-1') &= -g + \sum_{k=0}^{\infty} q_k v(k); \\ v(i) &= \min\{a_B - g + v(1'), i - g + \sum_{k=0}^{\infty} q_k v(k)\} \quad (i = 0, 1, \ldots). \end{aligned} \tag{A1}$$

It is easily seen that

$$v(1') = -(D-2)g + v(D-1') = -(D-1)g + \sum_{k=0}^{\infty} q_k v(k),$$

so that (A1) reduces to

$$v(i) = \min\{a_B - Dg + \sum_{k=0}^{\infty} q_k v(k), i - g + \sum_{k=0}^{\infty} q_k v(k)\} \quad (i = 0, 1, \ldots). \tag{A2}$$

From (A2) it follows that a batch service is started if $i > a_B - (D-1)g$, proving the control-limit structure.

Finally, we note that for a fixed CG-policy with control limit K the optimality equations are given by

$$v_{CG}(i) = \begin{cases} i - g_{CG} + \sum_{k=0}^{\infty} q_k v_{CG}(k) & (i < K); \\ a_B - Dg_{CG} + \sum_{k=0}^{\infty} q_k v_{CG}(k) & (i \geq K), \end{cases}$$

which upon setting $v_{CG}(0) = 0$ reduces to

$$v_{CG}(i) = \begin{cases} i & (i < K); \\ a_B - (D-1)g_{CG} & (i \geq K). \end{cases}$$

Next we find g_{CG} from

$$g_{CG} = \sum_{k=0}^{\infty} q_k v_{CG}(k) = \sum_{k=0}^{K-1} kq_k + (a_B - (D-1)g_{CG})(1 - Q_{K-1}),$$

yielding

$$g_{CG} = \frac{a_B(1 - Q_{K-1}) + \sum_{k=0}^{K-1} kq_k}{Q_{K-1} + D(1 - Q_{K-1})},$$

in accordance with (17).

Appendix B: The ECG-policy

Under the ECG-policy a batch is executed at time $T_1 + T_2$ since the last batch, where

$$\begin{aligned} T_1 &:= \min\{n = 1, 2, \ldots : X_n \geq K_1\}; \\ T_2 &:= \min\{D-1, \min\{n = 0, 1, \ldots, D-2 : \frac{\sum_{i=2}^{D-n} R_i^{(T_1)}}{D-n-1} \geq K_2 \wedge R_{D-n}^{(T_1)} \geq K_3\}\}. \end{aligned}$$

In words, one first waits for a group of at least K_1 customers, and then a batch is executed the first period in which the mean number of individual services per period to be avoided is at least K_2 and the size of the *current* critical group is at least K_3. Otherwise the batch delivery is done, like the CG-policy, $D-1$ periods later. The ECG-policy uses three control parameters: K_1 and K_3 which are integers, and K_2 which need not be integer. The search for the optimal values of these parameters requires the computation of its expected cost $g_{ECG}(K_1, K_2, K_3)$, which in turn requires the computation of $E(S_{ECG})$ and $E(Y_{ECG})$. With this in mind we define

$$\begin{aligned} U_1 &:= \sum_{n=1}^{T_1 - D} X_n; \\ U_2 &:= \sum_{n=T_1-D+1}^{T_1+T_2-D} X_n = \sum_{n=D-T_2+1}^{D} R_n^{(T_1)}, \end{aligned}$$

and clearly

$$\begin{aligned} E(S_{ECG}) &= E(T_1) + E(T_2); \\ E(Y_{ECG}) &= E(U_1) + E(U_2). \end{aligned}$$

We now observe, using earlier arguments, that T_1 and T_2 are independent. Moreover, since T_1 has a geometric distribution $\mathrm{G}(Q_{K_1-1})$, it follows that

$$E(T_1) = \frac{1}{1 - Q_{K_1-1}} \tag{A3}$$

and

$$
\begin{aligned}
E(U_1) &= \sum_{n=D+1}^{\infty} \Pr\{T_1 = n\} \sum_{i=1}^{n-D} E\{X_i \mid X_i \le K_1\} \\
&= \sum_{n=D+1}^{\infty} Q_{K_1-1}^{n-1}(1-Q_{K_1-1})(n-D)\frac{\sum_{k=0}^{K_1-1} kq_k}{Q_{K_1-1}} \\
&= \frac{Q_{K_1-1}^{D-1}}{1-Q_{K_1-1}} \sum_{k=0}^{K_1-1} kq_k. \qquad \text{(A4)}
\end{aligned}
$$

Next we need expressions for $E(T_2)$ and $E(U_2)$, and for that we need the joint distribution of

$$(R_2^{(T_1)}, \ldots, R_D^{(T_1)}) = (X_{T_1-1}, \ldots, X_{T_1-D+1}).$$

Note that $E(T_2)$ and $E(U_2)$ do not depend on $R_1^{(T_1)} = X_{T_1}$ since the triggering group of customers arriving in period T_1 is always included in the batch. First observe that $T_1 = n < D$ implies that $R_n^{(T_1)} = \cdots = R_D^{(T_1)} = 0$. Therefore we have to distinguish between the case where $r_D > 0$ and the case where $r_m = 0$ and $r_i = 0$ $(i = m+1, \ldots, D)$ for some $m < D$. For $r_D > 0$ we have

$$
\begin{aligned}
& \Pr\{R_2^{(T_1)} = r_2, \ldots, R_D^{(T_1)} = r_D\} \\
&= \Pr\{X_{T_1-D+1} = r_D, \ldots, X_{T_1-1} = r_2\} \\
&= \sum_{n=D}^{\infty} \Pr\{T_1 = n\} \Pr\{X_{n-D+1} = r_D, \ldots, X_{n-1} = r_2 \mid X_1 < K_1, \ldots, X_{n-1} < K_1, X_n \ge K_1\} \\
&= \sum_{n=D}^{\infty} \Pr\{T_1 = n\} \Pr\{X_{n-D+1} = r_D \mid X_{n-D+1} < K\} \cdots \Pr\{X_{n-1} = r_2 \mid X_{n-1} < K\} \\
&= \sum_{n=D}^{\infty} Q_{K_1-1}^{n-1}(1-Q_{K_1-1})\frac{q_{r_D}\cdots q_{r_2}}{Q_{K_1-1}^{D-1}} \\
&= \sum_{n=D}^{\infty} Q_{K_1-1}^{n-D}(1-Q_{K_1-1})\prod_{i=2}^{D} q_{r_i} \\
&= \prod_{i=2}^{D} q_{r_i} \quad (0 \le r_i \le K-1, i = 2, \ldots, D-1; 1 \le r_D \le K-1). \qquad \text{(A5)}
\end{aligned}
$$

More generally, we have for $m = 1, \ldots, D$

$$
\begin{aligned}
& \Pr\{R_2^{(T_1)} = r_2, \ldots, R_m^{(T_1)} = r_m, R_{m+1}^{(T_1)} = 0, \ldots, R_D^{(T_1)} = 0\} \\
&= \sum_{n=m}^{D-1} Q_{K_1-1}^{n-1}(1-Q_{K_1-1})\frac{q_0^{n-m} q_{r_m}\cdots q_{r_2}}{Q_{K_1-1}^{n-1}} + \sum_{n=D}^{\infty} Q_{K_1-1}^{n-1}(1-Q_{K_1-1})\frac{q_0^{D-m} q_{r_m}\cdots q_{r_2}}{Q_{K_1-1}^{D-1}} \\
&= (1-Q_{K_1-1})\left(\prod_{i=2}^{m} q_{r_i}\right)\sum_{n=m}^{D-1} q_0^{n-m} + q_0^{D-m}\prod_{i=2}^{m} q_{r_i} \\
&= \prod_{i=2}^{m} q_{r_i}\left(q_0^{D-m} + (1-Q_{K_1-1})\frac{1-q_0^{D-m}}{1-q_0}\right) \\
& \quad (0 \le r_i \le K-1, i = 2, \ldots, m-1; 1 \le r_m \le K-1). \qquad \text{(A6)}
\end{aligned}
$$

Note that for $m = D$ (A6) reduces to (A5). Since conditional on $\underline{R}^{(T_1)}$, T_2 and U_2 are just deterministic functions of $(r_2, \ldots, r_D)$, we can write

$$\begin{aligned} T_2(r_2,\ldots,r_D) &:= \min\{D-1, \min\{n \le D-2 : \frac{\sum_{i=2}^{D-n} r_i}{D-n-1} \ge K_2 \wedge r_{D-n} \ge K_3\}\}; \\ U_2(r_2,\ldots,r_D) &:= \sum_{i=D-T_2(r_2,\ldots,r_D)+1}^{D} r_i. \end{aligned}$$

Conditioning on $\underline{R}^{(T_1)}$ then yields

$$E(T_2) = \sum_{r_2,\ldots,r_D<K_1} \Pr\{R_2^{(T_1)} = r_2, \ldots, R_D^{(T_1)} = r_D\} T_2(r_2,\ldots,r_D); \tag{A7}$$

$$E(U_2) = \sum_{r_2,\ldots,r_D<K_1} \Pr\{R_2^{(T_1)} = r_2, \ldots, R_D^{(T_1)} = r_D\} U_2(r_2,\ldots,r_D). \tag{A8}$$

Appendix C: The TD-policy

To develop a computational scheme for $E(S_{TD})$ we introduce the functions

$$\begin{aligned} F_n(k_1,\ldots,k_n) &:= \Pr\{\sum_{j=1}^{i} X_j \le k_i, i = 1,\ldots,n\} \quad (n = 1,\ldots,D-1); \\ F_n(k_1,\ldots,k_{D-1}) &:= \Pr\{\sum_{j=1}^{i} X_j \le k_i, i = 1,\ldots,D-1; \sum_{j=i-D+1}^{i} X_i < K, i = D,\ldots,n\} \ (n \ge D). \end{aligned}$$

By conditioning on X_n we then obtain the following recursive relations:

$$\begin{aligned} F_1(k_1) &= Q_{k_1}; \\ F_n(k_1,\ldots,k_n) &= \sum_{k=0}^{k_1} q_k F_{n-1}(k_2-k,\ldots,k_n-k) \quad (n = 2,\ldots,D-1); \\ F_n(k_1,\ldots,k_{D-1}) &= \sum_{k=0}^{k_1} q_k F_{n-1}(k_2-k,\ldots,k_{D-1}-k, K-1-k) \quad (n \ge D). \end{aligned} \tag{A9}$$

Define $P_n := \Pr\{S_{TD} > n\}$ $(n \ge D)$, then by conditioning on $(X_1,\ldots,X_{D-1})$ we find

$$P_n = \sum_{k_1+\cdots+k_{D-1}<K} q_{k_1}\cdots q_{k_{D-1}} F_{n-D+1}(K-1-\sum_{i=1}^{D-1} k_i, \ldots, K-1-\sum_{i=n-D+1}^{D-1} k_i) \tag{A10}$$

for $n = D,\ldots,2D-2$, and

$$P_n = \sum_{k_1+\cdots+k_{D-1}<K} q_{k_1}\cdots q_{k_{D-1}} F_{n-D+1}(K-1-\sum_{i=1}^{D-1} k_i, \ldots, K-1-k_{D-1}) \tag{A11}$$

for $n \ge 2D-1$. Finally, by (22) we have

$$E(S_{TD}) = D + \sum_{n=D}^{\infty} P_n, \tag{A12}$$

and since $\lim_{n\to\infty} P_n = 0$ we can truncate this infinite sum when P_n is sufficiently small.

Computational remark. Numerical analysis reveals that, asymptotically, P_n constitutes a geometric series, or $\lim\limits_{n\to\infty} \frac{P_n}{P_{n-1}} = C$ $(0 < C < 1)$, so that $E(S_{TD})$ in (A12) can be approximated as

$$D + \sum_{n=D}^{N} P_n + \frac{P_{N-1}P_N}{P_{N-1} - P_N} \quad (N \geq D+1). \tag{A13}$$

It turns out that this approximation performs very good already for small values of N.

For a computational scheme for $E(Y_{TD})$, we continue from (24) and define

$$y_n \quad := \quad \sum_{k=K}^{\infty} \Pr\{\sum_{i=n-D+1}^{n} X_i > k, S_{TD} = n\}.$$

For $n = D$ we have

$$\begin{aligned} y_D \quad &= \quad \sum_{k=K}^{\infty} \Pr\{\sum_{i=1}^{D} X_i > k\} \\ &= \quad E\{\sum_{i=1}^{D} X_i\} - \sum_{k=0}^{K-1} \Pr\{\sum_{i=1}^{D} X_i > k\} \\ &= \quad D\mu - \sum_{i=1}^{D}(1 - Q_k^{D*}), \end{aligned} \tag{A14}$$

while for $n > D$ we have

$$\begin{aligned} y_n \quad &= \quad \sum_{k=K}^{\infty} \Pr\{\sum_{i=n-D+1}^{n} X_i > k, S_{TD} = n\} \\ &= \quad \sum_{k=K}^{\infty} \Pr\{\sum_{i=1}^{D} X_i < K, \ldots, \sum_{i=n-D}^{n-1} X_i < K, \sum_{i=n-D+1}^{n} X_i > k\} \\ &= \quad \sum_{k=K}^{\infty} \sum_{\sum_{j=n-D+1}^{n-1} i_j < K} q_{i_{n-D+1}} \cdots q_{i_{n-1}} \Pr\{X_n > k - \sum_{j=n-D+1}^{n-1} i_j\} \cdot \\ &\cdot \Pr\{X_{n-D} < K - \sum_{j=n-D+1}^{n-1} i_j, \ldots, \sum_{i=n-2D+2}^{n-D} X_i < K - i_{n-D+1}, \sum_{i=n-2D+1}^{n-D} X_i < K, \ldots, \sum_{i=1}^{D} X_i < K\} \\ &= \quad \sum_{\sum_{j=n-D+1}^{n-1} i_j < K} q_{i_{n-D+1}} \cdots q_{i_{n-1}} \Big(\mu - \sum_{k=0}^{K-1-\sum_{j=n-D+1}^{n-1} i_j} (1 - Q_k)\Big) \cdot \\ &\qquad \cdot F_{n-D+2}(K - 1 - \sum_{j=n-D+1}^{n-1} i_j, \ldots, K - 1 - i_{n-D+1}), \end{aligned} \tag{A15}$$

where we compute $F_n(\cdot)$ recursively as in (A9). Note that for $n = D$ (A15) does not reduce to (A14). Combining (23) and (24) we obtain

$$E(Y_{TD}) = \mu E(S_{TD}) - K - \sum_{n=D}^{\infty} y_n.$$

Since $\lim_{n\to\infty} y_n = 0$, we truncate the infinite sum when y_n is sufficiently small.

Although the computation time of the numerical schemes for $E(S_{TD})$ and $E(Y_{TD})$ increases exponentially with D and K, for $D \leq 5$ it remains within the order of seconds.

As an alternative to the above probabilistic procedure, $E(S_{TD})$ and $E(Y_{TD})$ can also be computed using a "brute-force" approach. The computation time for this approach also increases exponentially with D, but considerably faster than for the probabilistic procedure. We will describe the brute-force approach for the TD- as well as the ETD-policy. To this end, we compute the first entrance times and "costs" for the Markov chain $\{\underline{R}^{(n)}\}$ induced by the NB-policy into the sets $\{\underline{r} : \sum_{i=1}^{D} r_i \geq K\}$ and $\{\underline{r} : \sum_{i=1}^{D} r_i \geq K_1 \wedge r_D \geq K_2\}$, respectively.

We start with the TD-policy. Define

$S_{TD}(\underline{r}_{-D})$:= expected number of periods until the next batch service when presently there are r_i customers with delay level i $(i = 1, \ldots, D-1)$, given that there is no immediate batch service;

$Y_{TD}(\underline{r}_{-D})$:= expected number of individual services until the next batch when presently there are r_i customers with delay level i $(i = 1, \ldots, D-1)$, given that there is no immediate batch service and excluding possible immediate individual services,

where $\underline{r}_{-D} := (r_1, \ldots, r_{D-1})$. Note that these quantities are defined in such a way that they are independent of r_D, thereby reducing the dimension of the state space from D to $D-1$. Conditioning on the number of arriving customers in the next period we obtain the following two finite systems of equations for $\sum_{i=1}^{D-1} r_i < K$:

$$S_{TD}(r_1, \ldots, r_{D-1}) = 1 + \sum_{k=0}^{K-1-\sum_{i=1}^{D-1} r_i} q_k S_{TD}(k, r_1, \ldots, r_{D-2}); \tag{A16}$$

$$Y_{TD}(r_1, \ldots, r_{D-1}) = \sum_{k=0}^{K-1-\sum_{i=1}^{D-1} r_i} q_k \Big(r_{D-1} + Y_{TD}(k, r_1, \ldots, r_{D-2}) \Big). \tag{A17}$$

Finally, incorporating the stipulation that $S_{TD} \geq D$, we can compute

$$E(S_{TD}) = D + \sum_{\substack{k_1, \ldots, k_D: \\ \sum_{i=1}^{D} k_i < K}} q_{k_1} \cdots q_{k_D} S_{TD}(k_D, \ldots, k_2);$$

$$E(Y_{TD}) = \sum_{\substack{k_1, \ldots, k_D: \\ \sum_{i=1}^{D} k_i < K}} q_{k_1} \cdots q_{k_D} \Big(k_1 + Y_{TD}(k_D, \ldots, k_2) \Big) = Y_{TD}(0, \ldots, 0).$$

Setting up a finite system of equations for the ETD-policy is more complicated. We exploit the fact that batch services are now limited to periods with $r_D \geq K_2$, i.e. we use the Markov chain $\{\underline{R}^{(n)}\}$ induced by the NB-policy embedded on $\{(r_1, \ldots, r_D) : r_D \geq K_2\}$. Define

$S_{ETD}(\underline{r}_{-D})$:= expected number of periods until the next batch service when presently there are r_i customers with delay level i $(i = 1, \ldots, D-1)$, given that there is no immediate batch service and $r_D \geq K_2$;

$Y_{ETD}(\underline{r}_{-D})$:= expected number of individual services until the next batch when presently there are r_i customers with delay level i $(i = 1, \ldots, D-1)$, given that there is no immediate batch service and $r_D \geq K_2$, and excluding possible immediate individual services.

For a given state $\underline{r}_{-D}$ with $\sum_{i=1}^{D-1} r_i < K_1$ let j be the smallest integer for which $r_{D-j} \geq K_2$. For states with $j \leq D-1$, conditioning on the number of customers in the next j periods yields

$$S_{ETD}(\underline{r}_{-D}) = j + \sum_{\substack{k_1,\ldots,k_j: \\ \sum_{i=1}^{j} k_i < K_1 - \sum_{i=1}^{D-j} r_i}} q_{k_1} \cdots q_{k_j} S_{ETD}(k_j, \ldots, k_1, r_1, \ldots, r_{D-j-1}); \tag{A18}$$

$$Y_{ETD}(\underline{r}_{-D}) = \sum_{i=D-j+1}^{D-1} r_i + \sum_{\substack{k_1,\ldots,k_j: \\ \sum_{i=1}^{j} k_i < K_1 - \sum_{i=1}^{D-j} r_i}} q_{k_1} \cdots q_{k_j} \Big(r_{D-j} + Y_{ETD}(k_j, \ldots, k_1, r_1, \ldots, r_{D-j-1}) \Big). \tag{A19}$$

On the other hand, for states $\underline{r}_{-D}$ with $\sum_{i=1}^{D-1} r_i < K_1$ and $r_i < K_2$ for all $i = 1, \ldots, D-1$, conditioning on the number of customers until the first period with $R_D^{(n)} \geq K_2$ yields

$$S_{ETD}(\underline{r}_{-D}) = \frac{1}{1 - Q_{K_2-1}} + D - 1 + \sum_{\substack{k_1,\ldots,k_{D-1}: \\ \sum_{i=1}^{D-1} k_i < K_1}} q_{k_1} \cdots q_{k_{D-1}} S_{ETD}(k_{D-1}, \ldots, k_1); \tag{A20}$$

$$Y_{ETD}(\underline{r}_{-D}) = \sum_{i=1}^{D-1} r_i + \frac{\sum_{k=0}^{K_2-1} k q_k}{1 - Q_{K_2-1}} + \sum_{k_1=K_2}^{K_1-1} \sum_{\substack{k_2,\ldots,k_D: \\ \sum_{i=2}^{D} k_i < K_1 - k_1}} q_{k_1} \cdots q_{k_D} \Big(k_1 + Y_{ETD}(k_{D-1}, \ldots, k_1) \Big). \tag{A21}$$

Finally, we obtain the required quantities:

$$E(S_{ETD}) = S_{ETD}(0, \ldots, 0);$$
$$E(Y_{ETD}) = Y_{ETD}(0, \ldots, 0).$$

Appendix D: The global optimal policy

In this appendix we provide the proofs of Theorems 2 and 4.

Proof of theorem 2.
(i) Obviously,

$$v_1(\underline{r}) = \min\{r_D, a_B\} \leq \min\{\underline{r}'_D, a_B\} = v_1(\underline{r}').$$

Next, using the induction hypothesis,

$$\begin{aligned} v_n(\underline{r}) &= \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, r_1, \ldots, r_{D-1}), h_n^1\} \\ &\leq \min\{\underline{r}'_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, \underline{r}'_1, \ldots, \underline{r}'_{D-1}), h_n^1\} \\ &= v_n(\underline{r}'). \end{aligned}$$

(ii) Using (i) we have

$$\begin{aligned} v_n(\underline{r}) - v_n(\underline{r}') &= \min\{h_0^n(\underline{r}), h_1^n\} - \min\{h_0^n(\underline{r}'), h_1^n\} \\ &\leq \max\{0, h_1^n - h_0^n(\underline{r}')\} \\ &= \max\{0, a_B - \underline{r}'_D - \alpha \sum_{k=0}^{\infty} q_k \Big(v_{n-1}(k, \underline{r}'_1, \ldots, \underline{r}'_{D-1}) - v_{n-1}(k, 0, \ldots, 0)\Big) \\ &\leq a_B \end{aligned}$$

(iii) For $i < D$ using the induction hypothesis gives

$$\begin{aligned} v_n(\underline{r} + e_i) &= \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1}), h_n^1\} \\ &\leq \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k (1 + v_{n-1}(k, r_1, \ldots, r_{D-1})), h_n^1\} \\ &= \min\{r_D + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, r_1, \ldots, r_{D-1}), h_n^1\} \\ &\leq 1 + v_n(\underline{r}). \end{aligned}$$

For $i = D$ we have

$$\begin{aligned} v_n(\underline{r} + e_D) &= \min\{r_D + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, r_1, \ldots, r_{D-1}), h_n^1\} \\ &\leq 1 + v_n(\underline{r}). \end{aligned}$$

(iv) For $j < D$ using the induction hypothesis gives

$$\begin{aligned} v_n(\underline{r} + e_i) &= \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1}), h_n^1\} \\ &\leq \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{j+1}), h_n^1\} \\ &= v_n(\underline{r} + e_j). \end{aligned}$$

For $i < j = D$ we use Theorem 2(iii) to obtain

$$\begin{aligned} v_n(\underline{r}+e_i) &= \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1}), h_n^1\} \\ &\leq \min\{r_D + \alpha \sum_{k=0}^{\infty} q_k (1 + v_{n-1}(k, r_1, \ldots, r_{D-1})), h_n^1\} \\ &= \min\{r_D + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}(k, r_1, \ldots, r_{D-1}), h_n^1\} \\ &\leq v_n(\underline{r}+e_D). \end{aligned}$$

(v) For $k = 1$ this is equivalent to (iii). Next suppose that (iv) holds for $k-1$, then using Theorem 2(iii) gives

$$v_n(\underline{r} + k(e_j - e_i)) \geq v_n(\underline{r} + (k-1)(e_j - e_i)) \geq v_n(\underline{r}).$$

(vi) Repeated application of Theorem 2(v) while using the fact that $\sum_{j=1}^{i} r_j \geq \sum_{j=1}^{i} \underline{r}'_j$ $(i = 1, \ldots, D-1)$ yields

$$\begin{aligned} v_n(\underline{r}) &\leq v_n(\underline{r}'_1, r_1 + r_2 - \underline{r}'_1, r_3, \ldots, r_D) \\ &\leq v_n(\underline{r}'_1, \ldots, \underline{r}'_{i-1}, \sum_{j=1}^{i} r_j - \sum_{j=1}^{i-1} \underline{r}'_j, r_{i+1}, \ldots, r_D) \quad (i = 3, \ldots, D-1) \\ &\leq v_n(\underline{r}'_1, \ldots, \underline{r}'_{D-1}, \sum_{j=1}^{D} r_j - \sum_{j=1}^{D-1} \underline{r}'_j) \\ &= v_n(\underline{r}'), \end{aligned}$$

where the last equality follows from $\sum_{j=1}^{D} r_j = \sum_{j=1}^{D} \underline{r}'_j$. □

Proof of Theorem 4.
(i) First suppose that $\pi_n^*(\underline{r}+e_i) = 1$. Then $\pi_n^*(\underline{r}+e_i+e_j) = 1$ by Theorem 3(ii), and hence

$$v_n(\underline{r}+e_i+e_j) - v_n(\underline{r}+e_i) = h_n^1 - h_n^1 = 0 \leq v_n(\underline{r}+e_j) - v_n(\underline{r}),$$

where the last inequality follows from Theorem 2(i).

Next suppose that $\pi_n^*(\underline{r}+e_i) = 0$ and $\pi_n^*(\underline{r}+e_j) = 1$ (by Theorem 3(iii) this is only possible if $j > i$). Then again $\pi_n^*(\underline{r}+e_i+e_j) = 1$ by Theorem 3(ii), and hence

$$v_n(\underline{r}+e_i+e_j) - v_n(\underline{r}+e_i) = h_n^1 - v_n(\underline{r}+e_i) \leq h_n^1 - v_n(\underline{r}) = v_n(\underline{r}+e_j) - v_n(\underline{r}),$$

again using Theorem 2(i).

Finally, suppose that $\pi_n^*(\underline{r}+e_i) = \pi_n^*(\underline{r}+e_j) = 0$. If $i, j < D$ then

$$\begin{aligned} & v_n(\underline{r}+e_i+e_j) - v_n(\underline{r}+e_i) \\ \leq\; & h_n^0(\underline{r}+e_i+e_j) - h_n^0(\underline{r}+e_i) \\ =\; & \alpha \sum_{k=0}^{\infty} q_k \Big(v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1} + e_{j+1}) - v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{i+1}) \Big) \end{aligned}$$

$$\begin{aligned}
&\le \alpha \sum_{k=0}^{\infty} q_k \Big(v_{n-1}((k, r_1, \ldots, r_{D-1}) + e_{j+1}) - v_{n-1}(k, r_1, \ldots, r_{D-1}) \Big) \\
&= h_n^0(\underline{r} + e_j) - h_n^0(\underline{r}) \\
&= v_n(\underline{r} + e_j) - v_n(\underline{r}),
\end{aligned}$$

where the second inequality follows from the induction hypothesis. In the last equality note that $v_n(\underline{r}) = h_n^0(\underline{r})$ since $\pi_n^*(\underline{r}) = 0$ by theorem 3(ii). If $i = D$ then

$$\begin{aligned}
v_n(\underline{r} + e_D + e_j) - v_n(\underline{r} + e_D) &\le h_n^0(\underline{r} + e_D + e_j) - h_n^0(\underline{r} + e_D) \\
&= h_n^0(\underline{r} + e_j) - h_n^0(\underline{r}) = v_n(\underline{r} + e_j) - v_n(\underline{r}),
\end{aligned}$$

while if $j = D$ then

$$\begin{aligned}
v_n(\underline{r} + e_i + e_D) - v_n(\underline{r} + e_i) &\le h_n^0(\underline{r} + e_i + e_D) - h_n^0(\underline{r} + e_i) = 1 \\
&= h_n^0(\underline{r} + e_D) - h_n^0(\underline{r}) = v_n(\underline{r} + e_D) - v_n(\underline{r}).
\end{aligned}$$

(ii) First, if $\pi_n^*(\underline{r} - e_j + e_{j+k}) = 1$ then obviously

$$v_n(\underline{r} - e_i + e_{i+k}) \le h_n^1 = v_n(\underline{r} - e_j + e_{j+k}).$$

Next, if $\pi_n^*(\underline{r} - e_j + e_{j+k}) = 0$ and $i + k < j + k \neq D$ then the result immediately follows from the induction hypothesis. Finally, suppose that $\pi_n^*(\underline{r} - e_j + e_{j+k}) = 0$ and $j + k = D$. Then we also need the fact that

$$\begin{aligned}
v_{n-1}((k, r_1, \ldots, r_{D-1}) - e_i + e_{i+k}) &\le v_{n-1}((k, r_1, \ldots, r_{D-1}) - e_j + e_D) \\
&\le 1 + v_n((k, r_1, \ldots, r_{D-1}) - e_j), \qquad \text{(A22)}
\end{aligned}$$

where the first inequality follows from the induction hypothesis and the second from Theorem 2(iii). Using (A22) it follows that

$$\begin{aligned}
v_n(\underline{r} - e_i + e_{i+k}) &\le r_D + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) - e_{i+1} + e_{i+k+1}) \\
&\le r_D + 1 + \alpha \sum_{k=0}^{\infty} q_k v_{n-1}((k, r_1, \ldots, r_{D-1}) - e_{j+1}) \\
&= v_n(\underline{r} - e_j + e_{j+k}).
\end{aligned}$$

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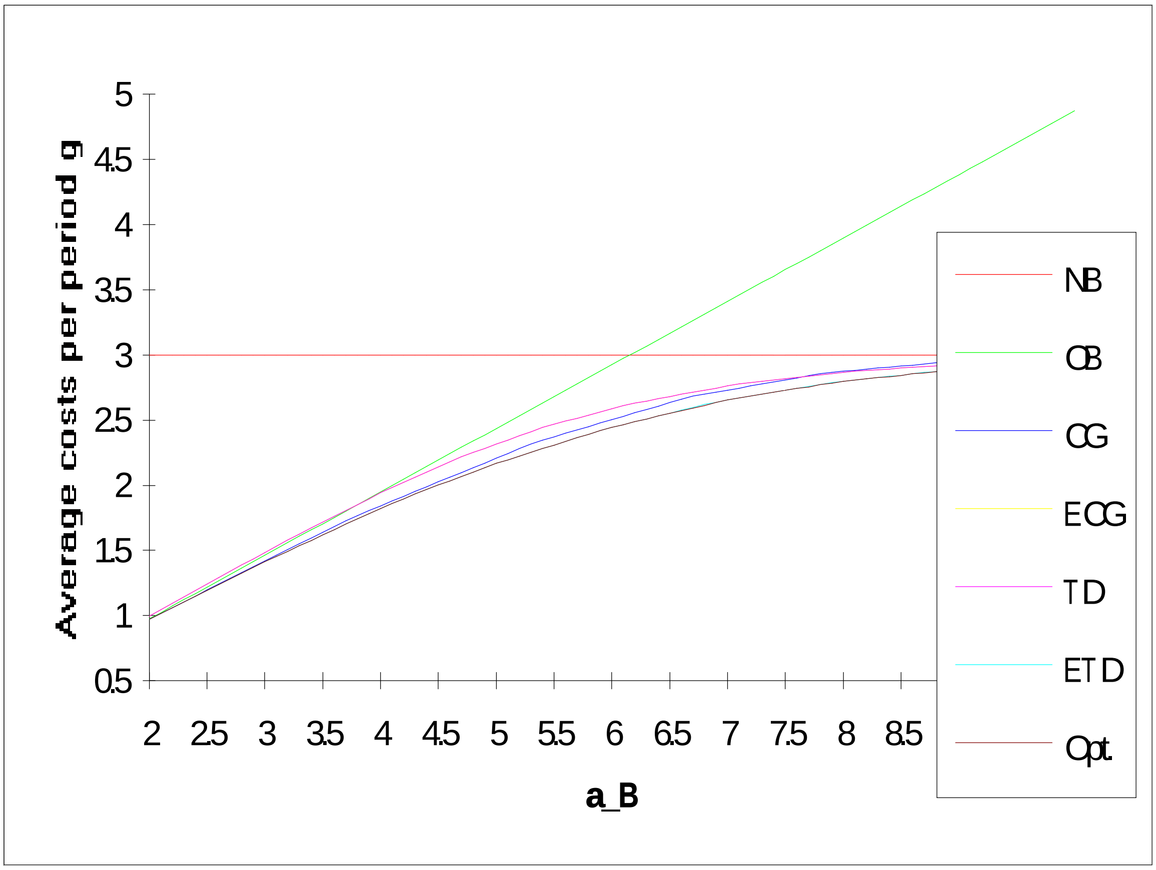

Figure 1: Average costs per period (D=2, X=Poisson(3))

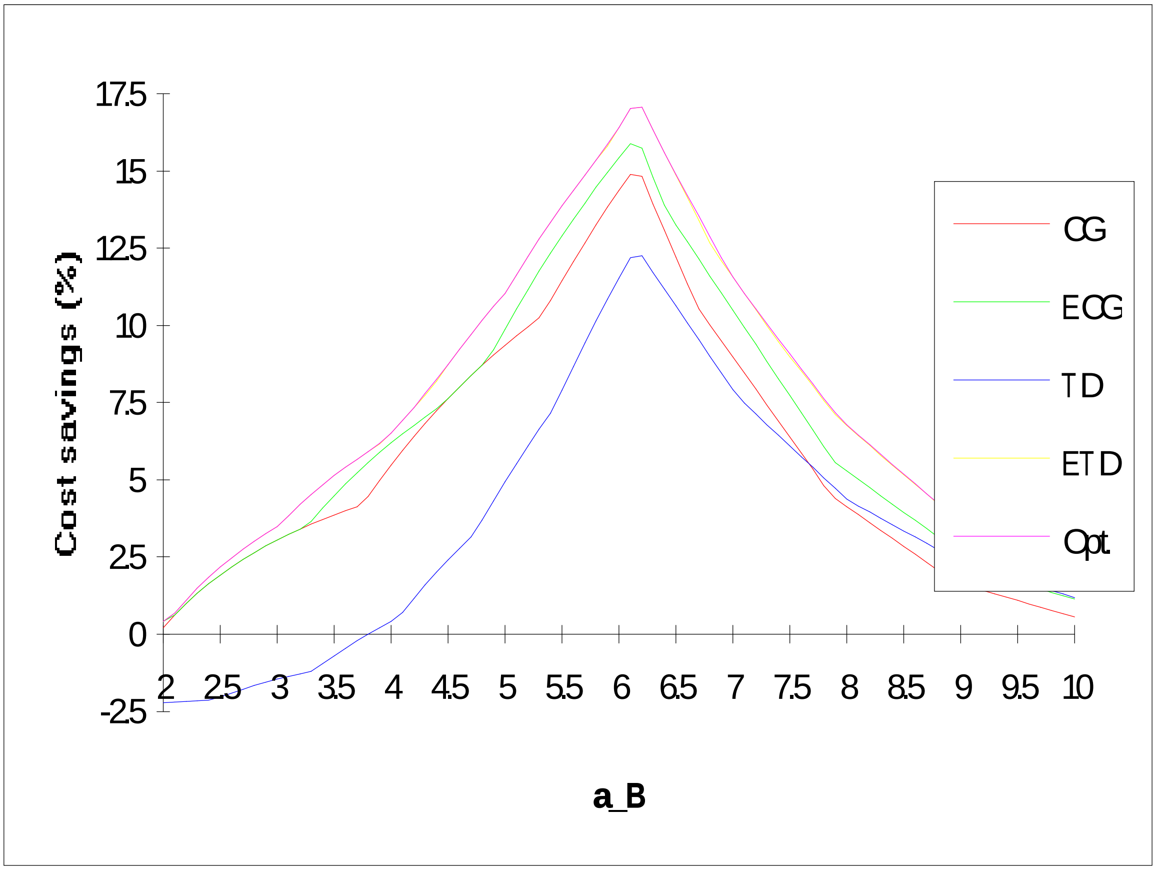

Figure 2: Percentage cost savings (D=2, X=Poisson(3))

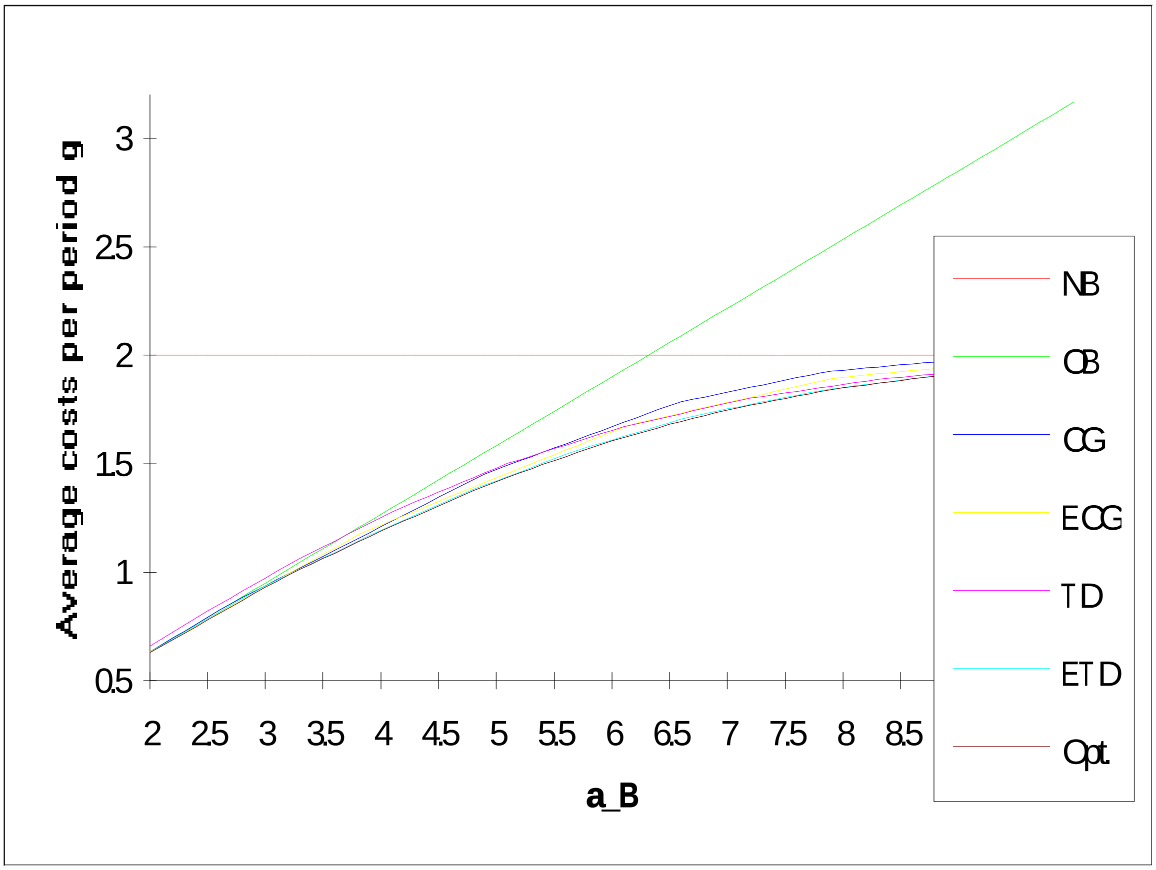

Figure 3: Average costs per period (D=3, X=Poisson(2))

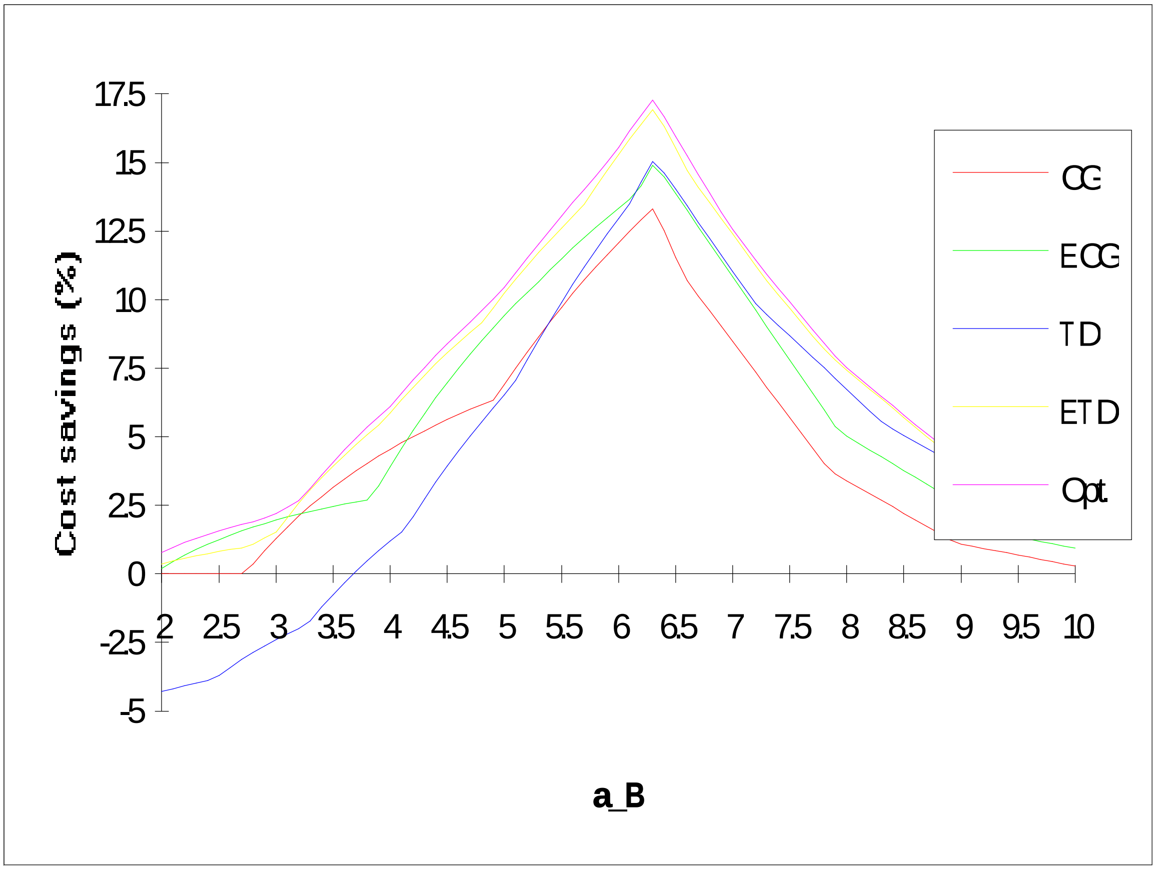

Figure 4: Percentage cost savings (D=3, X=Poisson(2))