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Publication date:
1996

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The research of G. van der Laan, A.J.J. Talman and Z. Yang is part of the VF-program "Competition and Cooperation".

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Abstract

The well known Sperner lemma states that in a simplicial subdivision of a simplex with a properly labeled boundary there is a completely labeled simplex. We present two combinatorial theorems on polytopes which generalize Sperner's lemma. Using balanced simplices, a generalized concept of completely labeled simplices, a unified existence result of balanced simplices in any simplicial subdivision of a polytope is given. This theorem implies the well-known lemmas of Sperner, Scarf, Shapley, and Garcia as well as some other results as special cases. A second theorem which imposes no restrictions on the integer labeling rule is established; this theorem implies several results of Freund.

Keywords: completely labeled simplices, balanced simplices, Sperner's lemma, integer labeling, simplicial subdivision, fixed points.

1 Introduction

The lemma of Sperner (1928) is perhaps one of the most elegant and fundamental results in combinatorial topology. It has become quite familiar in the fields of mathematical programming and economics, during the last thirty years, because of its successful use in the computation of fixed points of a continuous function, see Scarf (1967, 1973), Kuhn (1968), Eaves (1972), Merrill (1972), van der Laan and Talman (1979), and others. Todd (1976), Forster (1980) and Doup (1988) provide excellent surveys of the developments of Sperner's lemma. Sperner's lemma states that given a simplicial subdivision of the unit simplex

$$S^n = \{\, x \in R^n_+ \mid \sum_{i=1}^{n} x_i = 1 \,\},$$

where R^n_+ is the nonnegative orthant of the n-dimensional Euclidean space, and a labeling function L from the set of vertices of simplices of the simplicial subdivision into the set $\{\,1, \cdots, n\,\}$, such that $x_i = 0$ implies that $L(x) \neq i$ for any vertex $x \in S^n$, there exists a completely labeled simplex, i.e., a simplex whose vertices carry all of the labels from 1 up to n. Scarf's lemma (1967, 1973) states a similar result if $x_i = 0$ implies that

$$L(x) = \min\{\, j \mid x_j = 0 \text{ and } x_{j+1} > 0\,\},$$

for any vertex $x \in S^n$, where $l+1 = 1$ if $l = n$. Shapley (1973) generalized Sperner's lemma by using a set labeling rule instead of an integer labeling rule. Furthermore, the existence results of completely labeled simplices have been generalized to the cube and the simplotope, while also more general labeling rules have been considered, see Tucker (1946), Fan (1967), Garcia (1976), van der Laan and Talman (1981, 1982), Freund (1984, 1986), and van der Laan, Talman and Van der Heyden (1987). In Freund (1985) the lemmas of Sperner, Scarf, and Garcia on a full-dimensional simplex are extended to a full-dimensional polytope. In this paper we generalize the concept of completely labeled simplices to the concept of balanced simplices. A general condition is formulated to guarantee the existence of a balanced simplex in any simplicial subdivision of an arbitrary polytope. This leads to the first main theorem which implies most results mentioned above, including the lemmas of Sperner, Scarf, Shapley, and Garcia, as special cases. Using different labeling rules we establish the second theorem which unifies several results of Freund (1985).

In Section 2 we discuss the basic notations and concepts related to polytopes and simplicial subdivisions. In Section 3 we present and prove the main theorems on arbitrary polytopes. In Section 4 we show that the existing results on the unit simplex as well as their extensions on the simplotope can be derived from the first theorem as special cases. In Section 5 it is shown that the second theorem implies the results of Freund for completeness. In Section 6 some related results will be given.

2 Preliminaries for analysis

For a convex set $B \subset R^n$, let $bnd(B)$, $int(B)$ and $dim(B)$ denote the relative boundary, the relative interior and the dimension of B, respectively. For k a positive integer, the set of integers $\{1, \cdots, k\}$ is denoted by I_k. For given integer l, $0 \leq l \leq n$, let I be a finite set of at least $l+1$ integers. Let P be a polytope. The polytope P can be written as

$$P = \{ x \in R^n \mid {a^i}^\top x \leq \alpha_i,\ i \in I \ \text{ and } \ {d^h}^\top x = \delta_h,\ h \in I_{n-l} \}.$$

We assume throughout the paper that P is l-dimensional, none of the constraints ${a^i}^\top x \leq \alpha_i$, $i \in I$, is an implicit equality, and no constraint is redundant.

For $T \subset I$, we define

$$F(T) = \{ x \in P \mid {a^i}^\top x = \alpha_i \ \text{ for } \ i \in T \},$$

with $F(\emptyset) = P$. In case $F(T)$ is nonempty, we call $F(T)$ a face of P. If $F(T)$ is a face of P with at least one dimension less than P, we call $F(T)$ a proper face of P. If the dimension of a face $F(T)$ is zero, then $F(T)$ is a vertex of P.

With respect to the polytope P, we define

$$V = \{ \, x \in R^n \,|\, x = \sum_{h \in I_{n-l}} \nu_h d^h, \ \nu_h \in R \, \},$$

with $V = \{ \, \mathbf{0} \, \}$ when $l = n$. For $T \subset I$, define

$$A(T) = \{ \, x \in R^n \,|\, x = \sum_{i \in T} \lambda_i a^i, \ \lambda_i \geq 0 \, \} + V,$$

with $A(\emptyset) = \{ \, \mathbf{0} \, \}$ when $l = n$.

Next, in general, for given an integer q, $0 \leq q \leq n$, a q-dimensional simplex or q-simplex in R^n, denoted by $\sigma(x^1, \cdots, x^{q+1})$, in short by σ, is defined as the convex hull of $q+1$ affinely independent vectors $x^1, \cdots, x^{q+1}$ in R^n. For k, $0 \leq k \leq q$, a k-simplex being the convex hull of $k+1$ vertices of $\sigma(x^1, \cdots, x^{q+1})$ is a face of σ. A finite collection $\mathcal{G}$ of l-simplices is a simplicial subdivision of the polytope P if

(a) P is the union of all simplices in $\mathcal{G}$;

(b) the intersection of any two simplices in $\mathcal{G}$ is either the empty set or a common face of both.

We denote the set of vertices of simplices of $\mathcal{G}$ by $\mathcal{G}^0$. The symbol $\mathcal{G}^+$ denotes the collection of all simplices of $\mathcal{G}$ and their faces. Moreover, every face $F(T)$ of P is simplicially subdivided by faces of simplices of $\mathcal{G}$ in $F(T)$. The simplicial subdivision of a face $F(T)$ of P induced by $\mathcal{G}$ is denoted by $\mathcal{G}(T)$, i.e.,

$$\mathcal{G}(T) = \{ \, \tau \subset F(T) \,|\, \tau = \sigma \bigcap F(T), \ \sigma \in \mathcal{G}, \ dim(\tau) = dim(F(T)) \, \}.$$

Given a subset B of P, define the carrier of B as

$$Car(B) = \{ \, j \in I \mid a^{j\top} x = \alpha_j \ \text{ for all } \ x \in B \, \}.$$

For a point $v \in P$, define $Car(v) = Car(\{ v \})$.

For some finite nonempty set J, let a collection of vectors $c^j \in R^n$, $j \in J$, be given. For a nonempty set $T \subset J$, we define

$$C(T) = Conv(\{ c^j \mid j \in T \}),$$

where $Conv(B)$ denotes the convex hull of a set B in R^n.

Finally, we assign each element of $\mathcal{G}^0$ an index from the set J. Let $L : \mathcal{G}^0 \mapsto J$ be such a labeling rule. For a q-simplex $\sigma(x^1, \cdots, x^{q+1})$ being a face of a simplex of $\mathcal{G}$, let $L(\sigma) = \{ L(x^1), \cdots, L(x^{q+1}) \}$ be the label set of σ. We are now ready to define the concept of balanced simplices.

Definition 2.1 *A q-simplex $\sigma(x^1, \cdots, x^{q+1})$ in $\mathcal{G}^+$ is balanced if $\mathbf{0} \in C(L(\sigma))$.*

If $\mathbf{0} \in C(L(\sigma))$ we also call the collection $\{ c^j \,|\, j \in L(\sigma) \}$ or $L(\sigma)$ itself balanced. A set S is balanced if and only if the system of equations $\sum_{j \in S} \mu_j c^j = \mathbf{0}$, $\sum \mu_j = 1$ has a nonnegative solution. In the next section we formulate a sufficient condition to guarantee the existence of a balanced simplex in $\mathcal{G}^+$.

3 Main integer labeling combinatorial results

Given a polytope P as defined in the previous section, a finite nonempty set J and a collection of vectors $\{ c^j \,|\, j \in J \}$ in R^n, let $\mathcal{G}$ be a simplicial subdivision of P. A sufficient condition for existence of at least one balanced simplex in $\mathcal{G}^+$ is given.

Theorem 3.1 Main Theorem I *Let $\{ c^j \,|\, j \in J \}$ be a collection of vectors in R^n with $C(J) \bigcap V = \{ \mathbf{0} \}$ and let $\mathcal{G}$ be a triangulation of the polytope P. Let $L : \mathcal{G}^0 \mapsto J$ be a labeling rule such that for every simplex σ of the induced triangulation $\mathcal{G}(T)$ of a proper face $F(T)$ of P for some $T \subset I$, the set $A(T) \bigcap C(L(\sigma))$ either is empty or contains the point $\mathbf{0}$. Then there exists a balanced simplex in $\mathcal{G}^+$.*

Proof: Let x be any point in P and let $\sigma(x^1, \cdots, x^{q+1})$ be the unique simplex in $\mathcal{G}^+$ containing x in its relative interior. There exist unique positive numbers $\gamma_1, \cdots, \gamma_{q+1}$ satisfying $\sum_{i=1}^{q+1} \gamma_i = 1$ such that

$$x = \sum_{i=1}^{q+1} \gamma_i x^i.$$

Then the function f at x is defined by

$$f(x) = \sum_{i=1}^{q+1} \gamma_i c^{i_j},$$

where $i_j = L(x^j)$, $j = 1, \cdots, q+1$, for any x in the interior of the simplex σ. Clearly, f is a continuous function from P to $C(J)$, and $f(x) = \mathbf{0}$ implies that a simplex σ containing x in its interior is a balanced simplex. Since P is compact and convex and f is continuous there exists an $x^* \in P$ such that

$$x^\top f(x^*) \leq x^{*\top} f(x^*) \text{ for all } x \in P.$$

Consequently, x^* is a solution of the linear programming problem

$$\begin{array}{lll} & \text{maximize } x^\top f(x^*) & \\ \text{subject to} & a^{i\top} x \leq \alpha_i, & i \in I \\ & d^{h\top} x = \delta_h, & h \in I_{n-l}. \end{array}$$

According to the primal-dual theory in linear programming there exist $\lambda_i^* \geq 0$ for $i \in T^*$ and $\nu_h^* \in R$ for $h \in I_{n-l}$, where $T^* = \{\, i \in I \mid a^{i\top} x^* = \alpha_i \,\}$, satisfying

$$f(x^*) = \sum_{i \in T^*} \lambda_i^* a^i + \sum_{h=1}^{n-l} \nu_h^* d^h.$$

Hence $x^* \in F(T^*)$ and $f(x^*) \in A(T^*)$. Next, let σ^* be any simplex of the induced triangulation $\mathcal{G}(T^*)$ of the face $F(T^*)$ of P containing x^*. We shall show that σ^* is balanced. Notice that since $x^* \in \sigma^*$, we have $f(x^*) \in C(L(\sigma^*))$. Hence, $f(x^*) \in A(T^*) \bigcap C(L(\sigma^*))$. Suppose that $T^* \neq \emptyset$. Then $F(T^*)$ is a proper face of P and therefore according to the boundary condition we have $\mathbf{0} \in A(T^*) \bigcap C(L(\sigma^*))$.

Consequently, σ^* is balanced. Now suppose that $T^* = \emptyset$. Then $A(T^*) = V$ and therefore $f(x^*) \in V \bigcap C(L(\sigma^*))$. Since $V \bigcap C(L(\sigma^*))$ is a nonempty subset of $V \cap C(J)$ and $V \bigcap C(J) = \{\mathbf{0}\}$, it implies that $f(x^*) = \mathbf{0}$ and σ^* is balanced. $\square$

If a labeling rule L on P satisfies the boundary condition of Theorem 3.1 for every simplex $\sigma \in \mathcal{G}(T)$, we call L a proper labeling rule. It should be noted that although we do not require a balanced simplex to be of dimension l, it holds that every simplex of $\mathcal{G}$ containing a balanced simplex as a face is also balanced. Furthermore, notice that the condition $C(J) \bigcap V = \{\mathbf{0}\}$ is satisfied if $\mathbf{0} \in C(J)$ and $C(J) \subset V^*$. Without the boundary condition in the Main Theorem I, we easily obtain the following result.

Corollary 3.2 *For a finite collection of vectors $\{c^j \mid j \in J\}$ in R^n, let $\mathcal{G}$ be a triangulation of the polytope P and let $L : \mathcal{G}^0 \mapsto J$ be a labeling rule. Then there exist $T \subset I$ and a simplex $\sigma \in \mathcal{G}(T)$ with $A(T) \cap C(L(\sigma)) \neq \emptyset$.*

In order to introduce the second theorem, we assume that with respect to an n-dimensional polytope P the vectors a^i, $i \in I$, are scaled such that P can be written as

$$P = \{x \in R^n \mid a^{i\top}x \leq 1 + a^{i\top}x^0,\ i \in I\},$$

for some arbitrarily chosen $x^0 \in int(P)$. Notice that the above operation is always possible. Let $X = Conv(\{c^j \mid j \in I\})$. Observe that if $F(T)$ is a face of P for some $T \subset I$, then the set $Conv(\{a^j \mid j \in T\})$ is a face of X, see Grunbaum (1967), pp. 47-49. For $y \in X$, define

$$\begin{aligned} E(y) \;\; = \;\; & \{(S,T) \subset J \times I \mid \textstyle\sum_{i\in S} \mu_i c^i + \sum_{j\in T} \nu_j a^j = y, \\ & \textstyle\sum_{i\in S} \mu_i + \sum_{j\in T} \nu_j = 1,\ \mu_i \geq 0,\ i \in S,\ \nu_j \geq 0,\ j \in T\}. \end{aligned}$$

Now we can present the second main result.

Theorem 3.3 Main Theorem II *Let the polytope P be as just described. For a nonempty finite set J, let $\{c^j \mid j \in J\}$ be a collection of vectors in R^n. Let*

$\mathcal{G}$ be a triangulation of the n-dimensional polytope P and let $L : \mathcal{G}^0 \mapsto J$ be a labeling rule. Then for each $y^0 \in int(X)$, there exists a simplex $\sigma \in \mathcal{G}^+$ such that $(L(\sigma), Car(\sigma)) \in E(y^0)$.

Proof: Let x be any point in P and let $\sigma(x^1, \cdots, x^{q+1})$ be the unique simplex in $\mathcal{G}^+$ containing x in its relative interior. There exist unique positive numbers γ_1, $\cdots$, γ_{q+1} satisfying $\sum_{i=1}^{q+1} \gamma_i = 1$ such that $x = \sum_{i=1}^{q+1} \gamma_i x^i$. Define a correspondence $\xi : P \Rightarrow R^n$ by

$$\xi(x) = Conv(\{\, y^0 - c^j \mid j = L(x^i) \ \text{ if } \ \gamma_i = \max_h \gamma_h \,\}).$$

The polytope

$$Q = \{\, x \in R^n \ \mid \ a^{i\top} x \leq 2 + a^{i\top} x^0, \, i \in I \,\},$$

contains P in its interior. Let x be a point in Q but not in P. Then let $\lambda_x \in (0, 1)$ be such that $x^0 + \lambda_x(x - x^0) \in bnd(P)$. Since $x^0 \in int(P)$, such a λ_x exists. Let $p(x) = x^0 + \lambda_x(x - x^0)$. Notice that $\{\, i \in I \ \mid \ a^{i\top} x = 2 + a^{i\top} x^0 \,\} = Car(p(x))$ for any $x \in bnd(Q)$ because $\lambda_x = 1/2$. Now we define a correspondence $\psi : Q \Rightarrow R^n$ by

$$\begin{aligned} \psi(x) &= \xi(x), \, x \in int(P) \\ \psi(x) &= Conv(\xi(x) \cup \{\, y^0 - a^i \ \mid \ i \in Car(x)\}), \, x \in bnd(P) \\ \psi(x) &= Conv(\{\, y^0 - a^i \ \mid i \in Car(p(x)) \,\}), \, x \in Q \setminus P. \end{aligned}$$

The correspondence ψ is upper semi-continuous, nonempty-valued, convex-valued and compact-valued. For a compact convex set Y containing $\bigcup_{x \in Q} \psi(x)$, let ϕ : $Y \Rightarrow Q$ be a correspondence, defined by

$$\phi(y) = \{\, x \in Q \mid z^\top y \leq x^\top y \ \text{ for all } \ z \in Q \,\}.$$

The correspondence ϕ is upper semi-continuous, nonempty-valued, convex-valued and compact-valued. Hence $\psi \times \phi : Y \times Q \Rightarrow Y \times Q$, defined by $(\psi \times \phi)(y, x) = \psi(x) \times$

$\phi(y)$, is upper semi-continuous, nonempty-valued, convex-valued, and compact-valued. Therefore, according to Kakutani's fixed point theorem there exists a pair of vectors $(y^*, x^*) \in Y \times Q$ such that $y^* \in \psi(x^*)$ and $x^* \in \phi(y^*)$. The latter implies that

$$z^\top y^* \leq x^{*\top} y^* \text{ for all } z \in Q.$$

Consequently, x^* is a solution of the linear programming problem

$$\begin{array}{ll} & \text{maximize } z^\top y^* \\ \text{subject to} & a^{i\top} z \leq 2 + a^{i\top} x^0, \quad i \in I. \end{array}$$

According to the primal-dual theory in linear programming there exist $\lambda_i^* \geq 0$ for $i \in T^*$ where $T^* = \{\, i \in I \,|\, a^{i\top} x^* = 2 + a^{i\top} x^0 \,\}$, such that $y^* = \sum_{i \in T^*} \lambda_i^* a^i$. Recall that $T^* = Car(p(x^*))$ when $T^* \neq \emptyset$.

We need to consider the following cases.

(i) In case $T^* \neq \emptyset$ and hence $x^* \in bnd(Q)$, there exist nonnegative numbers μ_j, $j \in T^*$, summing to one such that

$$\sum_{i \in T^*} \lambda_i^* a^i = \sum_{j \in T^*} \mu_j (y^0 - a^j).$$

Hence, $\sum_{i \in T^*} \lambda_i^* + \sum_{j \in T^*} \mu_j \geq 1$ and $y^0 = \sum_{j \in T^*} (\mu_j + \lambda_j^*) a^j$. This is a contradiction with the fact that $y^0 \in int(X)$ and $F(T^*)$ is a face of P. Indeed, if $\sum_{i \in T^*} \lambda_i^* + \sum_{j \in T^*} \mu_j > 1$, then y^0 lies outside X, and if this total sum is equal to 1, y^0 lies on the boundary of X.

(ii) In case $T^* = \emptyset$, the following three subcases need to be checked. (ii-1) In case $x^* \in int(P)$, we have that $y^* \in \psi(x^*)$. Moreover, there exists a unique simplex σ with vertices $w^1, \cdots, w^{t+1}$ containing x^* in its interior. It implies that there exist nonnegative numbers μ_j, $j \in L(\sigma)$, with sum equal to one such that $\sum_{j \in L(\sigma)} \mu_j (y^0 - c^j) = \mathbf{0}$. So, $y^0 \in Conv(\{\, c^j \mid j \in L(\sigma) \})$ and $(L(\sigma), Car(\sigma)) \in E(y^0)$. (ii-2) In case $x^* \in bnd(P)$, we have that $y^* \in \psi(x^*)$ and that there exists a simplex σ containing x^* in its interior. In other words, we have

$$\sum_{i\in L(\sigma)} \mu_i(y^0 - c^i) + \sum_{j\in S^*} \nu_j(y^0 - a^j) = \mathbf{0}$$

for some nonnegative numbers μ_i, $i \in L(\sigma)$, ν_j, $j \in S^*$, with $\sum_{i\in L(\sigma)} \mu_i + \sum_{j\in S^*} \nu_j = 1$. Clearly, $Car(\sigma) = S^*$. Hence,

$$y^0 = \sum_{i\in L(\sigma)} \mu_i c^i + \sum_{j\in Car(\sigma)} \nu_j a^j,$$

and $(L(\sigma), Car(\sigma)) \in E(y^0)$. (ii-3) If x^* lies in the interior of Q but not in P, we have that $y^* \in \psi(x^*)$. It holds that $\sum_{i\in Car(p(x^*))} \mu_i(y^0 - a^i) = \mathbf{0}$ for some nonnegative numbers μ_i with $\sum_{i\in Car(p(x^*))} \mu_i = 1$. So, $y^0 = \sum_{i\in Car(p(x^*))} \mu_i a^i$. This contradicts the fact that $y^0 \in int(X)$ and $F(Car(p(x^*)))$ is a face of P. $\square$

4 Applications to the unit simplex

In this section we apply Theorem 3.1 to obtain several well-known results on the $(n-1)$-dimensional unit simplex S^n. For $h \in I_n$, S^n_h denotes the facet $S^n_h = \{\, x \in S^n \mid x_h = 0 \,\}$, and for a proper subset $T \subset I_n$, $S^n(T) = \bigcap_{h\in T} S^n_h$. Furthermore, for $S \subset I_n$, let the n-vector m^S be defined by $\sum_{i\in S} \frac{1}{|S|} e^i$, where $|S|$ denotes the number of elements in S and e^i is the i-th unit vector in R^n. Observe that $m^S = e^i$ if $S = \{\, i \,\}$. For ease of notation we write $m^{I_n} = m$. Now, take $l = n-1$, $d^1 = m$, $\delta_1 = 1/n$, $I = I_n$, $a^i = m - e^i$ and $\alpha_i = 1/n$ for $i \in I_n$. Observe that $a^i \in V^*$ for all $i \in I_n$. For $S \subset I_n$, define $A'(S) = \{\, x \in R^n \mid x = \sum_{i\in S} \alpha_i a^i,\ \alpha_i \geq 0,\ i \in S \,\}$. Now, the unit simplex S^n can be rewritten in the framework of this paper as

$$S^n = \{\, x \in R^n \mid a^{i\top} x \leq \alpha_i, i \in I_n \ \text{ and } \ d^{1\top} x = \delta_1 \,\}.$$

We first prove Sperner's lemma.

Theorem 4.1 Sperner's Lemma *Let $\mathcal{G}$ be a triangulation of S^n and let $L : \mathcal{G}^0 \mapsto I_n$ be a labeling rule where $x_i = 0$ implies $L(x) \neq i$ for any vertex $x \in S^n$. Then there exists a completely labeled simplex of $\mathcal{G}$, i.e., a simplex $\sigma \in \mathcal{G}$ such that $L(\sigma) = I_n$.*

Proof: Take $J = I_n$ and for $j \in J$, set $c^j = a^j$. Clearly, $\mathbf{0} \in C(J)$ and $C(J) \subset V^*$. Therefore we have $C(J) \bigcap V = \{\mathbf{0}\}$. Notice that $\mathbf{0} \in C(S)$ if and only if $S = J$ and hence a balanced simplex must be full-dimensional and its vertices bear all labels 1 up to n. To show the existence of a balanced simplex we still have to show that the boundary condition of Theorem 3.1 is satisfied by every simplex in a proper face $S^n(T)$ of S^n. So, let $\sigma \in \mathcal{G}(T)$ for some nonempty $T \subset I_n$. Then $L(\sigma) \bigcap T = \emptyset$ since for every vertex x of σ we have $x_i = 0$ for every $i \in T$ and hence $L(x) \notin T$. Since the vectors a^i, $i \in S$, are linearly independent for any proper subset S of J we must have that $A'(L(\sigma)) \bigcap A(T) = \{\mathbf{0}\}$ and hence $C(L(\sigma)) \bigcap A(T) = \emptyset$. This completes the proof. $\square$

The next lemma is due to Scarf (1967).

Theorem 4.2 Scarf's Lemma *Let $\mathcal{G}$ be a triangulation of S^n and let $L : \mathcal{G}^0 \mapsto I_n$ be a labeling rule satisfying that $x_j = 0$ implies $L(x) = \min\{i \,|\, x_i = 0$ and $x_{i+1} > 0\}$ for any vertex $x \in bnd(S^n)$, where $l + 1 = 1$ if $l = n$. Then there exists a completely labeled simplex of $\mathcal{G}$.*

Proof: Let $J = I_n$ and $c^j = -a^j$ for all $j \in J$. Again, $C(J) \subset V^*$ and $\mathbf{0} \in C(S)$ if and only if $S = J$. Hence a balanced simplex is full-dimensional and must carry all labels. We still have to prove that the boundary conditions of Theorem 3.1 are fulfilled for every simplex $\sigma \in \mathcal{G}(T)$ in any proper face $S^n(T)$. Suppose that $A(T) \bigcap C(L(\sigma)) \neq \emptyset$ for some nonempty subset T of J and some $\sigma \in \mathcal{G}(T)$. Then there exist nonnegative λ_i for $i \in T$, a real number ν_1, and nonnegative μ_j for $j \in S$ where $S = L(\sigma)$ such that $\sum_{i \in T} \lambda_i a^i + \nu_1 m = \sum_{j \in S} \mu_j c^j$ and $\sum_{j \in S} \mu_j = 1$. Since

$c^j = -a^j$ for all $j \in J$, this yields

$$\sum_{i \in T} \lambda_i a^i + \sum_{j \in S} \mu_j a^j = -\nu_1 m.$$

Since $m^\top a^i = 0$ for all $i \in S \bigcup T$, it implies that $\nu_1 = 0$. It means that the vectors a^j, $j \in S \bigcup T$, are linearly dependent. Hence, $S \bigcup T = I_n$. Let $x^1, \cdots, x^{q+1}$ be the vertices of σ. Suppose that for some $j \in I_n$ it holds that $x^i_j > 0$ for all $i = 1, \cdots, q+1$. Then $L(x^i) \neq j$ for all $i = 1, \cdots, q+1$ and so $j \notin S$. Moreover, $j \notin T$. This contradicts the fact that $T \bigcup S = I_n$. Consequently, for every $j \in I_n$ there is at least one $i \in \{1, \cdots, q+1\}$ satisfying $x^i_j = 0$. Since $T \neq I_n$ there is an $h \in I_n$ such that $h \notin T$, $h+1 \in T$. Because $\sigma \in \mathcal{G}(T)$ there is an i with $x^i_h > 0$. Moreover, $h \notin S$ because of the fact that no vertex x^i can carry label h if $x^i_{h+1} = 0$. Hence, $h \notin T \bigcup S$, yielding a contradiction. Therefore, there exists a balanced simplex σ in $\mathcal{G}$ which must then be completely labeled. $\square$

Notice that the properness condition in Scarf's lemma can be relaxed slightly. It is sufficient to require that $A(T) \bigcap C(L(\sigma)) = \emptyset$ for every simplex σ of $\mathcal{G}(T)$.

The third result was established in Shapley (1973). In this theorem the vertices of a triangulation of S^n are labeled with nonempty subsets of the set I_n. To prove Shapley's lemma, we need the concept of balancedness of sets. Let $\mathcal{N}$ be the collection of all nonempty subsets of the set I_n. A collection $\{B_1, \cdots, B_k\}$ of k elements of $\mathcal{N}$ is called balanced if the system of equations

$$\sum_{j=1}^{k} \lambda_j m^{B_j} = m$$

has a nonnegative solution.

Theorem 4.3 Shapley's Lemma *Let $\mathcal{G}$ be a triangulation of S^n and let $L : \mathcal{G}^0 \mapsto \mathcal{N}$ be a labeling rule such that $L(x) \subset \{i \mid x_i > 0\}$ for any vertex $x \in S^n$. Then there exists at least one face $\sigma(x^1, \cdots, x^{q+1})$ of a simplex of $\mathcal{G}$ such that the collection $\{L(x^1), \cdots, L(x^{q+1})\}$ is balanced.*

Proof: Let $J = \mathcal{N}$ and $c^S = m - m^S$ for all $S \in \mathcal{N}$. Clearly, $C(J) \subset V^*$ and $\mathbf{0} \in C(J)$. We next prove that the condition of Theorem 3.1 is satisfied by every simplex $\sigma(x^1, \cdots, x^{q+1})$ of $\mathcal{G}(T)$ for any nonempty subset T of I_n. Since $\sigma \in \mathcal{G}(T)$, we must have $x^i_j = 0$ for every $j \in T$, and hence according to the boundary condition $L(x^i) \bigcap T = \emptyset$ for all $i = 1, \cdots, q+1$. Let $B_i = L(x^i)$ for $i = 1, \cdots, q+1$ and $S = \bigcup_{i=1}^{q+1} B_i$. Then also $S \bigcap T = \emptyset$. Since the vectors a^i, $i \in K$, are linearly independent for each proper subset K of I_n we have that $A'(S) \bigcap A(T) = \{\mathbf{0}\}$. For every $i \in \{1, \cdots, q+1\}$ we have $L(x^i) \subset S$ and c^{B_i} is a convex combination of the vectors a^j, $j \in B_i$. Hence, $C(L(\sigma)) \subset A'(S)$. Moreover, since for every $i \in \{1, \cdots, q+1\}$ we have $c^{B_i}_j > 0$ for any $j \in T$, it implies that $\mathbf{0} \notin C(L(\sigma))$. Consequently, $C(L(\sigma)) \bigcap A(T) = \emptyset$ and hence the boundary condition is satisfied. This guarantees the existence of a balanced simplex according to Theorem 3.1. □

The next result is due to Garcia (1976). In this lemma no restriction is imposed on the labeling rule.

Theorem 4.4 Garcia's Lemma *Let $\mathcal{G}$ be a triangulation of S^n and let $L : \mathcal{G}^0 \mapsto I_n$ be a labeling rule. Then there exists a simplex $\sigma \in \mathcal{G}^+$ such that $L(\sigma) \cup Car(\sigma) = I_n$.*

Proof: Let $J = I_n$ and let $c^j = -a^j$ for each $j \in J$. According to Corollary 3.2, there exists a simplex $\sigma \in \mathcal{G}(T)$ for some proper subset T of I_n such that $A(T) \cap C(L(\sigma)) \neq \emptyset$. Hence, the system of equations

$$\sum_{i \in T} \mu_i a^i + \beta m + \sum_{j \in L(\sigma)} \nu_j a^j = \mathbf{0}$$
$$\mu_i \geq 0,\ i \in T$$
$$\beta \in R$$
$$\sum_{j \in L(\sigma)} \nu_j = 1$$
$$\nu_j \geq 0,\ j \in L(\sigma)$$

has a solution. Clearly the above system has a solution only if $T \cup L(\sigma) = I_n$. Moreover, $T = Car(\sigma)$. Hence $Car(\sigma) \cup L(\sigma) = I_n$. □

We remark that Sperner's lemma (see [16]), Scarf's lemma (see [8]), and Garcia's lemma (see [8, 17]) have been generalized to the Cartesian product of unit simplices. These generalizations can also be derived from Theorem 3.1. We leave them to the interested reader.

5 Applications on polytopes

In this section we shall apply the main theorems to prove existing results on polytopes. First, let us consider the n-dimensional unit cube $C^n = \{\, x \in R^n \,|\, 0 \leq x_i \leq 1, i \in I_n \,\}$. The following lemmas are due to Freund (1984, 1986) and van der Laan and Talman (1981) and are easily derived from Theorem 3.1.

Lemma 5.1

Let $\mathcal{G}$ be a triangulation of C^n and let $L : \mathcal{G}^0 \mapsto I_n \bigcup -I_n$ be a labeling rule satisfying that $x_i = 1$ implies $L(x) \neq i$ and $x_i = 0$ implies $L(x) \neq -i$ for any $x \in \mathcal{G}^0$. Then there exists a complementary 1-simplex $\sigma \in \mathcal{G}^+$, i.e., a 1-simplex σ such that $L(\sigma) = \{\, k, -k \,\}$ for some $k \in I_n$.

Lemma 5.2

Let $\mathcal{G}$ be a triangulation of C^n and let $L : \mathcal{G}^0 \mapsto I_n \bigcup -I_n$ be a labeling rule where $L(x) = i$ implies $x_i = 1$ and $L(x) = -i$ implies $x_i = 0$ for any $x \in \mathcal{G}^0 \cap bnd(C^n)$. Then there exists a complementary 1-simplex $\sigma \in \mathcal{G}^+$.

Now we discuss several results of Freund (1985). Consider a full-dimensional polytope

$$P = \{\, x \in R^n \mid a^{i\top} x \leq 1,\ i \in I \,\}$$

with $|I| \geq n+1$. Since P is bounded, the point $\mathbf{0}$ lies in the convex hull of the vectors a^j, $j \in I$. Also, $V = \{\mathbf{0}\}$. Let X denote the convex hull of the vectors a^j, $j \in I$. Observe that the set X is a full-dimensional subset of R^n where if $F(T)$ is a face of P then $Conv(\{a^i \mid i \in T\})$ is a face of X. For $y \in X$, define $D(y) = \{\, T \subset I \mid y \in Conv(\{a^j \mid j \in T\})\,\}$. So, $D(y)$ is the collection of all sets T satisfying that $y \in Conv(\{a^j \mid j \in T\})$. A labeling rule $L : \mathcal{G}^0 \mapsto I$ is called dual proper if $L(v) \in Car(v)$ for all $v \in bnd(P)$ and $v \in \mathcal{G}^0$. A triangulation $\mathcal{G}$ of P is bridgeless if for each $\sigma \in \mathcal{G}$, the intersection of all faces of P that meet σ is nonempty. The following result is a generalization of Garcia (1976) on a full-dimensional simplex to a full-dimensional polytope, which is due to Freund (1985).

Theorem 5.3 Freund's Theorem I

Let $\mathcal{G}$ be a triangulation of the n-dimensional polytope P described above and let $L : \mathcal{G}^0 \mapsto I$ be a labeling rule. Then for each $y \in int(X)$, there exists a simplex σ in $\mathcal{G}^+$ such that $Car(\sigma) \bigcup L(\sigma) \in D(y)$.

Proof: Applying Theorem 3.3 with $J = I$ and $c^j = a^j$ for all $j \in J$, we obtain the conclusion. □

The next theorem easily follows.

Theorem 5.4 Freund's Theorem II

Let $\mathcal{G}$ be a bridgeless triangulation of the n-dimensional polytope P just described and let $L : \mathcal{G}^0 \mapsto I$ be a dual proper labeling rule. Then for each $y \in int(X)$ there exists a simplex σ in $\mathcal{G}^+$ such that $L(\sigma) \in D(y)$.

This result extends Scarf's lemma on a full-dimensional simplex. To introduce the next result, for each $y \in X$ let

$$\begin{aligned} V(y) \;=\; & \{(S,T) \subset I \times I \mid \textstyle\sum_{i \in T} \mu_i a^i - \sum_{j \in S} \nu_j a^j = y, \\ & \textstyle\sum_{i \in T} \mu_i + \sum_{j \in S} \nu_j = 1, \mu_i \geq 0,\, i \in T,\, \nu_j \geq 0,\, j \in S\}. \end{aligned}$$

We conclude the section with the following result which is a generalization of Sperner's lemma on a full-dimensional simplex.

Theorem 5.5 Freund's Theorem III

Let $\mathcal{G}$ be a triangulation of the n-dimensional polytope P described above and let $L : \mathcal{G}^0 \mapsto I$ be a labeling rule. Then for each $y \in int(X)$, there exists a simplex σ in $\mathcal{G}^+$ such that $(L(\sigma), Car(\sigma)) \in V(y)$.

Proof: Applying Theorem 3.3 with $J = I$ and $c^j = -a^j$ for all $j \in J$, we obtain the conclusion. □

6 Some related results

In this section we give several related results. Let $T^n = \{x \in R^n \mid -1 \leq x_i \leq 1, i \in I_n\}$. The first result is due to Knaster, Kuratowski and Mazurkiewicz (1929) and known as the KKM lemma.

Lemma 6.1 KKM Lemma *Let $\{C^1, \cdots, C^n\}$ be a collection of closed subsets of the unit simplex S^n such that*

(a) $S^n = \cup_{i=1}^n C^i$;

(b) $S^n(T) \subset \cup_{j \notin T} C^j$ for any nonempty proper subset T of I_n.

Then $\cap_{i=1}^n C^i \neq \emptyset$.

The lemma can be derived from the Sperner lemma by using a limit argument. The following lemma is due to Tucker (1946). A constructive proof can be found in Freund and Todd (1981).

Lemma 6.2 Tucker's Lemma *Let $\mathcal{T}$ be a symmetric triangulation of T^n with respect to $\mathbf{0}$ and let $L : \mathcal{T}^0 \mapsto I_n \cup -I_n$ be a labeling rule satisfying that for any*

$x \in \mathcal{T}^0$ on the boundary of T^n, it holds that $L(x) = -L(-x)$. Then there exists a complementary 1-simplex $\sigma \in \mathcal{T}^+$, i.e., a 1-simplex σ such that $L(\sigma) = \{ k, -k \}$ for some $k \in I_n$.

The next result is the well-known Borsuk-Ulam theorem.

Theorem 6.3 Borsuk-Ulam Theorem *Let $f : T^n \mapsto R^n$ be a continuous function satisfying that for each $x \in bd(T^n)$, it holds that $f(x) = -f(-x)$. Then there exists a zero point $x^* \in T^n$, i.e., $f(x^*) = \mathbf{0}$.*

Note that the above results are equivalent. Now we conclude the section with a lemma of Bapat on the unit simplex S^n which generalizes Sperner's lemma. A constructive proof can be found in Bapat (1989).

Lemma 6.4 Bapat's Lemma *Let $\mathcal{T}$ be a triangulation of S^n and for each $i \in I_n$, let $L^i : \mathcal{T}^0 \mapsto I_n$ be a labeling rule satisfying the conditions of Sperner's Lemma. Then there exist at least a simplex $\sigma \in \mathcal{T}$ with vertices $x^1, \cdots, x^n$, and a permutation $\pi = (\pi(1), \cdots, \pi(n))$ of $(1, \cdots, n)$ such that $\{L^{\pi(i)}(x^i) \mid i \in I_n\}$ is equal to I_n.*

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