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Determinateness of Strategic Games with a Potential

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Abstract: Finite potential games are determined, i.e. have Nash-equilibria in pure strategies. In this paper we investigate the determinateness of potential games, in which one or more players have infinitely many pure strategies.

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1 Introduction

It is a well-known fact that strategic games, in which every player has only finitely many strategies, need not be determined, i.e. need not have Nash-equilibria. However, if mixed strategies are allowed, such games are determined. For two-person matrix games this was shown by von Neumann [5] and for general n-person games by Nash [4]. For two-person games, where one player has infinitely (but countably) many pure strategies, Norde and Potters [6] proved (weak) determinateness by showing that (approximate) equilibria in mixed strategies always exist. However games, in which two or more players have infinitely many pure strategies, need not be determined, even if mixed strategies are allowed. A famous example is the following $\infty \times \infty$-bimatrix game, given by Wald [9], with payoff matrices

$$\begin{pmatrix} 0 & -1 & -1 & \cdots \\ 1 & 0 & -1 & \cdots \\ 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 & \cdots \\ -1 & 0 & 1 & \cdots \\ -1 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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(i.e. the two-person game where both players choose a natural number and the player, choosing the smallest number, pays one dollar to the other player).

In [3] Monderer and Shapley introduced potential games. Such games have the nice property that they possess Nash-equilibria in pure strategies, if the strategy spaces are finite. So, for these games, mixing strategies is not necessary in order to get determinateness. However, if one or more players have infinitely many pure strategies, (approximate) equilibria in pure strategies need not exist. The main aim of this paper is to provide sufficient conditions for certain classes of potential games, which guarantee (weak) determinateness.

In section 2 some definitions, concerning potential games and the concept of weak determinateness, are provided. Moreover, a decomposition theorem for potential games is given. Weak determinateness for a class of potential games, in which one player has a large action space, is shown in section 3. In section 4 we deal with $\infty \times \infty$-bimatrix games. For these games a remarkable phenomenon occurs: there are two $\infty \times \infty$-bimatrix games, which have the same potential function, whereas one is weakly determined and the other is not. A characterization of the potentials, which are such that *any* game with this potential is weakly determined, is provided. Concluding remarks are presented in section 5.

Notation. For a strategic n-person game $< N, (X_i)_{i \in N}, (u_i)_{i \in N} >$ with player set $N = \{1, \ldots, n\}$ the set $X := \prod_{i \in N} X_i$ denotes the set of all strategy-tuples and, for every $i \in N$, the set X_{-i} is defined by $X_{-i} := \prod_{j \neq i} X_j$. For a strategy-tuple $x_{-i} \in X_{-i}$ and a strategy $x_i \in X_i$ the vector $x = (x_{-i}, x_i) \in X$ denotes the strategy-tuple, in which player i plays strategy x_i and the other players play according to x_{-i}.

2 Potential games and determinateness

A strategic game $< N, (X_i)_{i \in N}, (u_i)_{i \in N} >$ is called a *potential game* if there is a *potential* $P : X \to I\!R$ such that $\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, x_i' \in X_i$ we have

$$u_i(x_{-i}, x_i) - u_i(x_{-i}, x_i') = P(x_{-i}, x_i) - P(x_{-i}, x_i').$$

So, in potential games the change in payoff for a unilaterally deviating player is measured by the potential P.

Special classes of potential games are the class of *coordination games* and the class of *dummy games*. A game $< N, (X_i)_{i \in N}, (u_i)_{i \in N} >$ is a coordination game if there is a $u : X \to I\!R$ such that $u_i = u$ for every $i \in N$ (every player has the same payoff function, which evidently is a potential for this game). A game $< N, (X_i)_{i\in N}, (u_i)_{i\in N} >$ is a dummy game if $\forall i \in N, \forall x_{-i} \in X_{-i}, \forall x_i, x_i' \in X_i$ we have $u_i(x_{-i}, x_i) - u_i(x_{-i}, x_i') = 0$. Clearly, the zero function is a potential for dummy games. In [1] Facchini, van Megen, Borm and Tijs proved the following proposition, which states that every potential game is the sum of a coordination game and a dummy game. A similar result can be found in Slade [7].

Proposition 1 *A game* $\Gamma =< N, (X_i)_{i\in N}, (u_i)_{i\in N} >$ *is a potential game iff*

$$u_i = c_i + d_i \text{ for every } i \in N,$$

where the functions $c_i, d_i, i \in N$, *are such that* $< N, (X_i)_{i\in N}, (c_i)_{i\in N} >$ *is a coordination game and* $< N, (X_i)_{i\in N}, (d_i)_{i\in N} >$ *is a dummy game.*

Proof. For the if-part it is sufficient to observe that the sum of two potential games is again a potential game. For the only-if part we see that the functions $c_i = P, d_i = u_i - P, i \in N$, where P is a potential of Γ, satisfy the required properties. ∎

Let $\varepsilon > 0, k \in I\!R$. A strategy $x_i \in X_i$ of player i is called an ε-*best response* to $x_{-i} \in X_{-i}$ if

$$u_i(x_{-i}, x_i) \geq \sup_{x_i' \in X_i} u_i(x_{-i}, x_i') - \varepsilon$$

and a k-*guaranteeing response* to x_{-i} if

$$u_i(x_{-i}, x_i) \geq k.$$

If x_i is an ε-best response or a k-guaranteeing response (or both) to x_{-i} then x_i is called an (ε, k)-*best response* to x_{-i}. Note that the set of ε-best responses to x_{-i} as well as the set of k-guaranteeing responses to x_{-i} may be empty, but that the set of (ε, k)-best responses is always non-empty. A strategy-tuple $x = (x_1, \ldots, x_n) \in X$ is called an ε-*equilibrium* of Γ if x_i is an ε-best response to $x_{-i} := (x_j)_{j\in N, j\neq i}$ for every $i \in N$, a k-*equilibrium* if x_i is a k-guaranteeing response to x_{-i} for every $i \in N$, and an (ε, k)-*equilibrium* if x_i is an (ε, k)-best response to x_{-i} for every $i \in N$. So in an (ε, k)-equilibrium every player is reasonably satisfied, since he either receives a (large) amount of at least k or he can gain no more than ε (a small amount) by deviating unilaterally. The set of ε-equilibria, k-equilibria, and (ε, k)-equilibria of Γ will be denoted by $E^{\varepsilon}(\Gamma)$, $E^{k}(\Gamma)$, and $E^{(\varepsilon,k)}(\Gamma)$ respectively. Clearly, $E^{\varepsilon}(\Gamma) \subset E^{(\varepsilon,k)}(\Gamma)$ and $E^{k}(\Gamma) \subset E^{(\varepsilon,k)}(\Gamma)$. The game Γ is called *weakly determined* if

$$E^{(\varepsilon,k)}(\Gamma) \neq \emptyset$$

for every $\varepsilon > 0, k \in I\!R$. For two-person games, this definition of weak determinateness, coincides with the one given by Lucchetti, Patrone and Tijs in [2].

The following theorem shows that any potential game with an upper bounded potential is weakly determined. In fact, such games have ε-equilibria for every $\varepsilon > 0$.

Theorem 1 *Let $\Gamma =< N, (X_i)_{i\in N}, (u_i)_{i\in N} >$ be a potential game with an upper bounded potential P. Then Γ is weakly determined.*

Proof. Let $\varepsilon > 0$. Choose $\hat{x} \in X$ such that $P(\hat{x}) > \sup_{x\in X} P(x) - \varepsilon$. Then $\hat{x} \in E^{\varepsilon}(\Gamma)$. ∎

The following example shows that potential games, having a potential which is not upper bounded, need not be weakly determined.

Example. Let Γ be the $\infty \times \infty$-bimatrix game (i.e. a game with two players and strategy space $I\!N$ for both players) with payoff-functions given by

$$u_1(i,j) = \begin{cases} 0 & \text{if } j = i \\ -1 & \text{if } j = i+1 \\ -2j+1 & \text{elsewhere} \end{cases}$$

and

$$u_2(i,j) = \begin{cases} -1 & \text{if } j = i \\ 0 & \text{if } j = i+1 \\ -2i & \text{elsewhere} \end{cases} .$$

This game is a potential game with potential

$$P(i,j) = \begin{cases} 2i-1 & \text{if } j = i \\ 2i & \text{if } j = i+1 \\ 0 & \text{elsewhere} \end{cases} .$$

Note that the pay-off functions u_1 and u_2 are upper bounded but that the potential P is not upper bounded. Let $\varepsilon \in (0,1), k > 0$. If $(i_0, j_0) \in$

$E^{(\varepsilon,k)}(\Gamma)$ then j_0 is an ε-best response to i_0 since $u_2(i_0, j_0) \leq 0$ and vice versa. Therefore $j_0 = i_0 + 1$ and $i_0 = j_0$ which yields a contradiction. So $E^{(\varepsilon,k)}(\Gamma) = \emptyset$ for every $\varepsilon \in (0,1)$ and $k > 0$ and hence Γ is not weakly determined.

3 Potential games with one player having a large action space

In this section we prove the weak determinateness of a special class of potential games. These games may be characterized by the fact that there is only one player having a large action space.

Theorem 2 *Let $\Gamma =< N, (X_i)_{i\in N}, (u_i)_{i\in N} >$ be a potential game with potential P. Suppose that $X_1, \ldots, X_{n-1}$ are compact topological spaces. Suppose moreover that $x_{-n} \mapsto u_i(x_{-n}, x_n)$ is continuous for all $i \in N\backslash\{n\}$, $x_n \in X_n$ and that $x_{-n} \mapsto u_n(x_{-n}, x_n)$ is lower semi-continuous for every $x_n \in X_n$. Then Γ is weakly determined.*

Proof. According to theorem 1 it is sufficient to concentrate on a potential P, which is not upper bounded. Let $x_n \in X_n$ and choose $(y_1, \ldots, y_{n-1}) \in X_{-n}$ arbitrarily. Then

$$\begin{array}{rcl} P(x_1, \ldots, x_n) & = & u_1(x_1, x_2 \ldots, x_n) - u_1(y_1, x_2 \ldots, x_n) \\ & + & u_2(y_1, x_2 \ldots, x_n) - u_2(y_1, y_2 \ldots, x_n) \\ & & \vdots \\ & + & u_{n-1}(y_1, y_2 \ldots, x_{n-1}, x_n) - u_{n-1}(y_1, y_2 \ldots, y_{n-1}, x_n) \\ & + & P(y_1, \ldots, y_{n-1}, x_n) \end{array}$$

for every $(x_1, \ldots, x_{n-1}) \in X_{-n}$, which shows that $x_{-n} \mapsto P(x_{-n}, x_n)$ is continuous. Let $k \in I\!R$ and define $d_n := u_n - P$. Then $x_{-n} \mapsto d_n(x_{-n}, x_n)$ is lower semi-continuous for every $x_n \in X_n$. Since, moreover, d_n is constant in the n-th coordinate, we may define $l := \min_{x\in X} d_n(x)$. Choose $x' = (x'_1, \ldots, x'_n) \in X$ such that $P(x') \geq k - l$. Since $x_{-n} \mapsto P(x_{-n}, x'_n)$ is continuous and X_{-n} compact we may choose $x''_{-n} \in X_{-n}$ such that $P(x''_{-n}, x'_n) = \max_{x_{-n}\in X_{-n}} P(x_{-n}, x'_n)$. Let $\hat{x} := (x''_{-n}, x'_n)$. Since every player $i \in N, i \neq n$ cannot improve at all upon $\hat{x}$ and $u_n(\hat{x}) = P(\hat{x}) + d_n(\hat{x}) \geq P(x') + l \geq k$ we have $\hat{x} \in E^{(\varepsilon,k)}(\Gamma)$ for every $\varepsilon > 0$. ■

If we consider potential games, in which all but one player have a finite action space, and if we equip these finite spaces with the discrete topology, the following result is an immediate consequence of theorem 2.

Corollary 1 *Let* $\Gamma =< N, (X_i)_{i\in N}, (u_i)_{i\in N} >$ *be a potential game with potential* P*. Suppose* $X_1, \ldots, X_{n-1}$ *are finite sets. Then* Γ *is weakly determined.*

Compare the above results with theorems 4.1 and 4.2 in [8], where Tijs proved, under some mild assumptions, the weak determinateness in mixed strategies of games, in which all but one player have finite and compact metric action spaces, respectively.

4 Potential games with two players having a large action space

In this section we consider potential games where two players have a large (but countable) action space. In the following examples a remarkable phenomenon occurs: two games are presented, which have the same potential function, whereas one is weakly determined and the other is not.

Example. Let Γ_1 be the $\infty \times \infty$-bimatrix game with payoff-functions given by $u_1(i,j) = u_2(i,j) = i + j,\ i,j \in I\!N$. Clearly, this coordination game is a potential game with potential $P(i,j) = i + j,\ i,j \in I\!N$, and $(k,1) \in E^k(\Gamma)$ for every $k \in I\!N$. So Γ_1 is weakly determined.

Example. Let Γ_2 be the (zero-sum) $\infty \times \infty$-bimatrix game with payoff-functions given by $u_1(i,j) = i-j,\ u_2(i,j) = j-i,\ i,j \in I\!N$ (the player choosing the highest natural number wins the difference with the other player, compare with the Wald-example in the introduction). Clearly, this game is a potential game with potential $P(i,j) = i+j,\ i,j \in I\!N$ which is not weakly determined.

Theorem 1 states that potential games with an upper bounded potential are weakly determined. The assumption, that the potential P should be upper bounded, can be weakened a little bit. In fact, it is sufficient to assume that the corresponding coordination game, with payoff-function P for every player, has an ε-equilibrium for every $\varepsilon > 0$. In other words, if a coordination game has ε-equilibria for every $\varepsilon > 0$ this property is inherited by every potential game with the same potential. The above examples showed however, that for a coordination game which has k-equilibria for every k, this property need not be inherited by every potential game with the same potential. A natural question now is the following: which potential functions P are such that *every* game with potential P is weakly determined? For $\infty \times \infty$-bimatrix games we give a complete answer to this question. In order to do so we need the following definition.

Definition. A (potential) function $P : I\!N \times I\!N \rightarrow I\!R$ is *determined* if for every $\varepsilon > 0$ there is an $i \in I\!N$ with

$$\sup_{j \in N : i \in B_1^\varepsilon(j)} P(i,j) \geq \sup_{j \in N} P(i,j) - \varepsilon, \tag{1}$$

or there is a $j \in I\!N$ with

$$\sup_{i \in N : j \in B_2^\varepsilon(i)} P(i,j) \geq \sup_{i \in N} P(i,j) - \varepsilon, \tag{2}$$

or both, where $B_1^\varepsilon(j)$ denotes the set of ε-best responses of player 1 to strategy j and $B_2^\varepsilon(i)$ the set of ε-best responses of player 2 to strategy i (here we use the convention that $\sup S = -\infty$ if $S = \emptyset$ and $\sup S = +\infty$ if S is not upper bounded).

If player 1 plays a strategy $i \in I\!N$, satisfying (1), then it is not a great disadvantage for player 2, if he is restricted to choose a strategy j for which strategy i is an ε-best response.

Example. Let $P : I\!N \times I\!N \rightarrow I\!R$ be defined by $P(i,j) := i - \frac{1}{j+1}$, $i,j \in I\!N$ and let $\varepsilon > 0$. For every $j \in I\!N$ we have $B_1^\varepsilon(j) = \emptyset$ and for every $i \in I\!N$ we have $B_2^\varepsilon(i) = \{j \in I\!N : j \geq \varepsilon^{-1} - 1\}$. Therefore, there is no $i \in I\!N$ which satisfies (1), but any $j \in I\!N$ with $j \geq \varepsilon^{-1} - 1$ satisfies (2). So P is determined.

Example. Let $P : I\!N \times I\!N \rightarrow I\!R$ be defined by $P(i,j) := i - \frac{i}{j+1}$, $i,j \in I\!N$ and let $\varepsilon > 0$. For every $j \in I\!N$ we have, again, $B_1^\varepsilon(j) = \emptyset$ and for every $i \in I\!N$ we get $B_2^\varepsilon(i) = \{j \in I\!N : j \geq i\varepsilon^{-1} - 1\}$. Clearly, there is no $i \in I\!N$ satisfying (1). For every $j \in I\!N$ we have $\{i \in I\!N : j \in B_2^\varepsilon(i)\} = \{i \in I\!N : i \leq \varepsilon(j+1)\}$, which is a finite (maybe empty) set and therefore,

$$\sup_{i \in N : j \in B_2^\varepsilon(i)} P(i,j) \in [-\infty, +\infty).$$

On the other hand,

$$\sup_{i \in N} P(i,j) = +\infty$$

for every $j \in I\!N$. So, there is no $j \in I\!N$, which satisfies (2), and hence P is not determined.

Note that the potential in the example of section 2 is not determined either. The following theorem characterizes all potentials P which are such that every $\infty \times \infty$-bimatrix game with potential P is weakly determined.

Theorem 3 *Let $P : I\!N \times I\!N \to I\!R$ be a (potential) function.*

a) If P is determined then every potential $\infty \times \infty$-bimatrix game with potential P is weakly determined.

b) If P is not determined there is a potential game $\infty \times \infty$-bimatrix game with potential P which is not weakly determined.

Proof. a) Let Γ be a potential $\infty \times \infty$-bimatrix game with determined potential P and let $\varepsilon > 0, k \in I\!R$. Suppose there is an $i \in I\!N$ such that

$$\sup_{j' \in N : i \in B_1^{\varepsilon/2}(j')} P(i, j') \geq \sup_{j' \in N} P(i, j') - \varepsilon/2.$$

If $\sup_{j' \in N} P(i, j') = +\infty$ then $\sup_{j' \in N : i \in B_1^{\varepsilon/2}(j')} P(i, j') = +\infty$ and therefore $\sup_{j' \in N : i \in B_1^{\varepsilon/2}(j')} u_2(i, j') = +\infty$. Now choose j such that $i \in B_1^{\varepsilon/2}(j)$ and $u_2(i, j) \geq k$. Then $(i, j) \in E^{(\varepsilon, k)}(\Gamma)$. If $\sup_{j' \in N} P(i, j') \in I\!R$ then choose j such that $i \in B_1^{\varepsilon/2}(j)$ and $P(i, j) \geq \sup_{j' \in N} P(i, j') - \varepsilon$. Then $(i, j) \in E^{(\varepsilon, k)}(\Gamma)$. If there is a $j \in I\!N$ such that

$$\sup_{i' \in N : j \in B_2^{\varepsilon/2}(i')} P(i', j) \geq \sup_{i' \in N} P(i', j) - \varepsilon/2,$$

the proof is analogous.
b) Suppose P is not determined. Then there is an $\varepsilon_0 > 0$ such that

$$\sup_{j \in N : i \in B_1^{\varepsilon_0}(j)} P(i, j) < \sup_{j \in N} P(i, j) - \varepsilon_0$$

for every $i \in I\!N$ and

$$\sup_{i \in N : j \in B_2^{\varepsilon_0}(i)} P(i, j) < \sup_{i \in N} P(i, j) - \varepsilon_0$$

for every $j \in I\!N$. Now define the dummy payoff-functions $d_1, d_2 : I\!N \times I\!N \to I\!R$ in the following way:

$$\begin{aligned} d_1(i,j) &:= -\max\{P(1,j),\ldots,P(j,j), \sup_{i' \in N : j \in B_2^{\varepsilon_0}(i')} P(i',j)\}, \\ d_2(i,j) &:= -\max\{P(i,1),\ldots,P(i,i), \sup_{j' \in N : i \in B_1^{\varepsilon_0}(j')} P(i,j')\}. \end{aligned}$$

Note that d_1 is constant in i and that d_2 is constant in j. Let Γ be the $\infty \times \infty$-bimatrix game with payoff-functions $u_1 := P + d_1, u_2 := P + d_2$. Since $u_1(i,j) \leq 0$ for $i \leq j$ and $u_2(i,j) \leq 0$ for $i \geq j$ this game has no k-equilibria for $k > 0$. If $\varepsilon \in (0, \varepsilon_0]$ and strategy i of player 1 is an ε-best (and therefore ε_0-best) response to strategy j of player 2 then j is not an ε_0-best (and therefore not an ε-best) response to i and vice versa. So the game Γ has no ε-equilibria for every $\varepsilon \in (0, \varepsilon_0]$ either. Moreover, if $\varepsilon \in (0, \varepsilon_0]$ and strategy i of player 1 is an ε-best (and therefore ε_0-best) response to strategy j of player 2, then, since $u_2(i,j) \leq 0$, j is not a k-guaranteeing response to i and vice versa. Therefore, $E^{(\varepsilon,k)}(\Gamma) = \emptyset$ for every $\varepsilon \in (0, \varepsilon_0], k > 0$ which implies that Γ is not weakly determined. ■

5 Concluding remarks

There is a parallel between (weak) determinateness in mixed strategies for general games and (weak) determinateness in pure strategies for potential games. In fact, finite games are determined in mixed strategies, whereas finite potential games are determined in pure strategies. For games, in which one player has an infinite (countable) action space, weak determinateness is known to be true if the number of players is two. The existence of approximate equilibria for such games with more than two players is still an open problem. Potential games, in which one player has an infinite (countable) action space, are always weakly determined. For games with two (or more) players having a large action space the example of Wald in the introduction and the Wald-like potential game in section 4 are games which are not weakly determined in mixed and pure strategies respectively. So, roughly speaking, (weak) determinateness (in mixed strategies for general games and in pure strategies for potential games) is true for games in which at most one player has an infinite (countable) action space.

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